A Mean Field Game Analysis of Fire Sales and Systemic Risk under Regulatory Capital Constraints

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Introduction and Context

- Contagious interactions between financial institutions play an important role in a financial crisis. ⇒ large literature on systemic risk
- Most papers focuses on direct contagion and banking networks; see e.g. Eisenberg and Noe [2001], Elsinger et al. [2006], Rogers and Veraart [2013], Glasserman and Young [2016]
- Indirect or price-mediated contagion caused by rapid deleveraging (fire sales) important as well, see policy papers such as Hanson et al. [2011] or Basel Committee on Banking Supervision [2014]

  "at the height of the crisis, financial markets forced the banking sector to reduce its leverage in a manner that amplified downward pressures on asset prices. This de-leveraging process exacerbated the feedback loop between losses, falling bank capital and shrinking credit availability."

Risk capital constraints

- Price-mediated contagion might be reinforced by regulatory capital constraints such as Basel II or III rules.
  - Under these rules a bank's risk capital (equity) should exceed a multiple of its risk weighted assets
  - Banks typically deleverage when their position approaches capital constraints (eg. after a shock), as raising new capital is expensive
  - Forced liquidation by regulators
- Interesting twist between micro- and macroprudential regulation
- Little analysis in formal models

Contributions and research question

- We analyze price-mediated contagion in the context of a Mean Field Game (MFG) model for a large banking system.
  - Banks invest into a bank-specific tradable asset in order to maximize the expected value of their equity.
  - Contagion. The drift of the tradable assets is affected by changes in the average asset holding of banks caused by trading or liquidation, leading to a game
- We introduce a stylized form of the Basel II/III regulatory capital constraints and study numerically the impact of capital constraints on financial stability using PDE approach
- First mathematical results on existence and approximate Nash equilibria in a model with smoothed contagion as in Hambly and Sojmark [2019] or Burzoni and Campi [2023].

Literature

- PDE part is based on Frey and Traxler [2026]
- Papers on systemic risk and price-mediated contagion: Braouezec and Wagalath [2019], Feinstein [2020],Cont and Wagalath [2016]
- MFGs and systemic risk: Carmona et al. [2015]
- Mean field models with default externality: Nadtochiy and Shkolnikov [2019], Hambly et al. [2019], Hambly and Sojmark [2019], Cuchiero et al. [2023], . . .
- Mean field games with absorption (default or liquidation) Campi and Fischer [2018], Burzoni and Campi [2023].
- The mathematical analysis uses results from probabilistic approach to MFGs

The model

- Fix horizon date T, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and consider a continuum of banks.
- Each bank invests into a non-tradable asset A, a tradable risky asset S and in cash D. The inventory Q (the position in S) can be adjusted only gradually via *trading rate* $\boldsymbol{\nu} = (\nu_t)_{0 \le t \le T}$. Moreover, there are transaction costs of size $\kappa \nu_t^2$. This gives

$$\mathrm{d}A_t = \sigma_A \mathrm{d}W_t^A \tag{1}$$

$$\mathrm{d}Q_t = \nu_t \mathrm{d}t + \sigma_Q \mathrm{d}W_t^Q \tag{2}$$

$$\mathrm{d}S_t = \mu_{\text{ex}} + \alpha \bar{c}_t \mathrm{d}t + \sigma_S \mathrm{d}W_t \tag{3}$$

$$\mathrm{d}D_t = -\nu_t (S_t + \kappa \nu_t) \mathrm{d}t \tag{4}$$

  for $\alpha, \sigma_A, \sigma_S, \sigma_Q > 0$ and 3-dim BM $\mathbf{W} = (W^Q, W^A, W^S)$.
- Denote by $X_t^{\nu} = A_t + S_t Q_t^{\nu} + D_t^{\nu}$ the book equity of a generic bank and let $\mathbf{X}_t = (Q_t, X_t)$

Contagion

- Brownian motions for different banks are independent; strategic interaction is introduced via the contagion term $\bar{c}_t$.
- We assume that $\bar{c}_t$ is given by rate of change in average number of risky asset held by the banking sector. (formal definition later)
- Note that drift of S decreases if banking sector reduces overall inventory level; this might be due to demand- or information effects.

Risk capital constraints

- Under the Basel rules the risk capital of a bank must exceed a multiple of its *risk weighted assets*.
- We introduce a stylized version where risk capital at time t is given by X_t and the risk weighted assets by $\gamma|Q_t| + \tilde{C}$. Hence we have the condition $X_t > \beta|Q_t| + C$ for some $C, \beta > 0$.
- $\mathcal{A} := \{\mathbf{x} = (q, x) \in \mathbb{R} \times \mathbb{R}^+ : x > \beta|q| + C\}$ denotes acceptable positions; boundary is $\partial\mathcal{A}$.
- A bank is liquidated by the regulator at $\tau_{\mathcal{A}} = \inf\{t \geq 0, \mathbf{X}_t \notin \mathcal{A}\}$, residual value for bank owners is zero.

Optimization problem of a bank and HJB

- We get following dynamics for $\mathbf{X} = (Q_t, X_t)_{0 \le t \le T}$

$$\mathrm{d}Q_t = \nu_t \mathrm{d}t + \sigma_Q \mathrm{d}W_t^Q,$$
$$\mathrm{d}X_t = \left(Q_t(\mu_{\text{ex}} + \alpha \bar{c}_t) - \kappa \nu_t^2\right) \mathrm{d}t + \sigma_A \mathrm{d}W_t^A + Q_t \sigma_S \mathrm{d}W_t^S$$

- Bank wants to maximize terminal equity value $\mathbb{E}\left(X_T^{\boldsymbol{\nu}} 1_{\{\tau > T\}}\right)$ over $\boldsymbol{\nu}$ (liquidation $\Rightarrow$ equity value is 0.)
- Assume that bank takes some evolution $t \mapsto \mu(t)$ of the contagion term as given. Standard arguments give HJB equation for value function u

$$0 = \partial_t u + q(\mu_{\text{ex}} + \alpha \mu(t)) \partial_x u + \frac{1}{2} \sigma_Q^2 \partial_q^2 u + \frac{1}{2} \left(\sigma_A^2 + \sigma_S^2 q^2\right) \partial_x^2 u + \sup_{\nu} \left\{\nu \partial_q u - \kappa \nu^2 \partial_x u\right\}, \quad u(T, q, x) = x, \tag{5}$$

together with the terminal condition $u(t, q, x) = 0$ on ∂A.

- Optimal trading rate $\nu^*(t, q, x) = \frac{\partial_q u(t,q,x)}{2\kappa \partial_x u(t,q,x)}$.

Why mean-field game models?

- Note that for consistency we must have $\mu(t) = \bar{c}_t$.
- Suppose we are dealing with a system of N homogeneous banks (N large). If we ignore liquidation we get

$$\bar{c}_t = \frac{1}{N} \sum_{i=1}^{N} \nu^*(t, Q_{t,i}, X_{t,i}) = \int_{\mathbb{R}^2} \nu^*(t, q, x) \mu_t^N(\mathrm{d}q, \mathrm{d}x)$$

  with $\mu_t^N(\mathrm{d}q, \mathrm{d}x) = \frac{1}{N} \sum_{i=1}^{N} \delta_{(Q_{t,i}, X_{t,i})}(\mathrm{d}q, \mathrm{d}x)$
- It follows that the value function of bank i depends on the state $(\mathbf{X}_{t,1}, \ldots, \mathbf{X}_{t,N})$ of all banks in the system.
- N moderately large $\Rightarrow$ impossible
- Way out: for N large the contribution of each individual bank j to $\bar{c}$ is very small (banks are almost independent.) Hence μ_t^N should converge to a deterministic measure m_t (the mean field)

Equilibrium of the mean field game

Definition. A strategy ν^* and a deterministic measure flow $\hat{m} = (\hat{m}_t)_{0\leq t\leq T}$ are an equilibrium of the MFG if

- ν^* solves the optimization problem of the bank assuming that the drift of $\mathbf{X}$ is
$$\mu(t) = \bar{c}_t(\hat{m}) = \partial_t \langle \hat{m}_t, q\rangle$$
where for generic m $\langle m_t, q\rangle := \int_{\mathbb{R}} q\, m_t(\mathrm{d}q, \mathrm{d}x))$ is the average inventory level.
- m_t is the distribution of the state process $\mathbf{X}$ given that the bank uses the strategy ν^*

Comments.

- Economic viewpoint: Nash equilibrium
- Mathematical challenge: fixed point problem

Benchmark case without capital constraints

As a benchmark we determine an explicit solution for the case without capital constraints (see also Cardaliaguet and LeHalle [2018])

- Assumption $u^{\text{unreg}}(t,q,x) = x + v(t,q)$. $\Rightarrow$ $v(T,q) = 0$ and

$$0 = \partial_t v + q(\alpha\mu(t) + \mu_{\text{ex}}) + \frac{1}{2}\sigma_Q^2 \partial_q^2 v + \sup_{\nu} \left\{ \nu \partial_q v - \kappa \nu^2 \right\},$$

- Next, we assume that $v(t,q) = h_0(t) + h_1(t)q$. This gives

$$h_1' = -\alpha\mu(t) - \mu_{\text{ex}}, \quad h_0' = -\frac{h_1^2}{4\kappa}, \quad h_0(T) = h_1(T) = 0.$$

Moreover, optimal trading rate is $\nu^*(t,q) = \frac{1}{2\kappa}\partial_q v = \frac{1}{2\kappa} h_1(t)$.
(q-dependence for linear quadratic terminal condition)

Contagion term and equilibrium

- Without constraints system is described in terms of distribution of Q.
- The forward equation for $m_t(\mathrm{d}q)$ gives

$$\bar{c}_t = \partial_t \langle m_t, q \rangle = \langle m_t, \mathcal{L}_Q q \rangle = \langle m_t, \nu^*(t, \cdot) \rangle,$$

  i.e. contagion term equals average trading rate of the banks. This gives interpretation of the model as MFG of controls.
- We get $\bar{c}_t = \frac{1}{2\kappa} h_1(t)$. The equilibrium condition $\mu(t) = \bar{c}_t$ yields

$$h_1' = -\frac{\alpha}{2\kappa} h_1 - \mu_{\text{ex}}, \quad h_1(T) = 0\,;$$

  given h_1, the functions h_0, v and u^{unreg} are easily computed.

The system with capital constraints - PDE approach

- With capital constraints ν^* will depend on $\mathbf{x} = (q, x)$. Hence we consider HJB (5) with boundary condition $u \equiv 0$ on $\partial\mathcal{A}$.
- Pre-liquidation distribution of $\mathbf{X}$ is described by flow m with

$$\langle m_t, f \rangle = \mathbb{E}[f(\mathbf{X}_t) 1_{\{\tau_{\mathcal{A}} > t\}}], \quad 0 \leq t \leq T. \tag{6}$$

  In abstract form the forward equation for $m_t(\mathrm{d}q, \mathrm{d}x)$ is

$$\partial_t \langle m_t, f \rangle = \langle m_t, \mathcal{L}_{\mathbf{X}} f \rangle, \quad \text{for } f \text{ with } f = 0 \text{ on } \partial\mathcal{A}.$$

  Partial integration and boundary condition $m(t, q, x) \equiv 0$ on $\partial\mathcal{A}$ give equation for density $m(t, q, x)$.
- HJB and forward equation are coupled via $\bar{c}_t = \partial_t \langle m_t, q \rangle$.
  $\bar{c}_t$ reflects two effects: average trading rate $\langle m_t, \nu^*(t, \cdot) \rangle$ and average liquidation rate as banks reach $\partial\mathcal{A}$

Discussion of PDE approach ctd

- No formal result on existence of solutions to the nonlinear forward backward PDE system.
- Challenges:
  - *Liquidation cascade* might lead to systemic risk event where large part of the system is liquidated at once (Nadtochiy and Shkolnikov [2019])
  - existence of a fixed point

  But see last part for results on smoothed version of the game.
- We used numerical methods based on Picard iteration $m^0 \to u^1 \to m^1 \dots$ and finite differences to study properties of the PDE system.
- In the unregulated case theoretical and numerical values coincide, with capital constraints method converges if α and κ^{-1} are not too large, otherwise blowup, corresponding to a liquidation cascade.

Value function and optimal strategy: fixed q

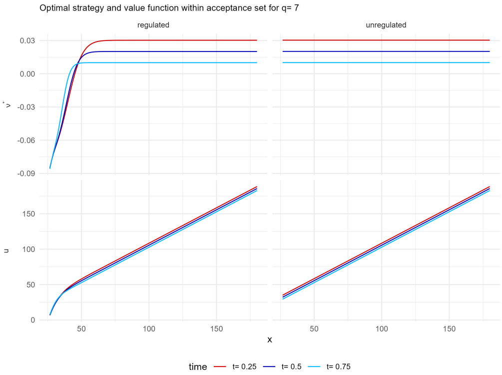

Figure: Left case with capital constraints, right unregulated case

Value function and optimal strategy: fixed x

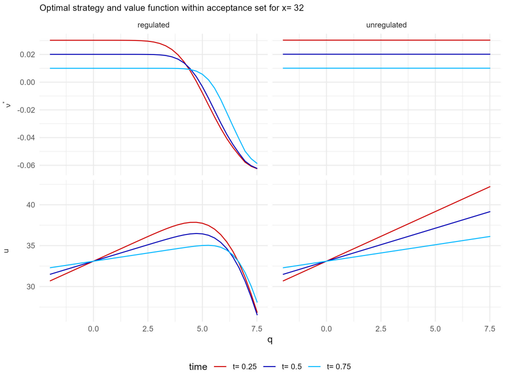

Figure: Left case with capital constraints, right unregulated case

Pre-liquidation density

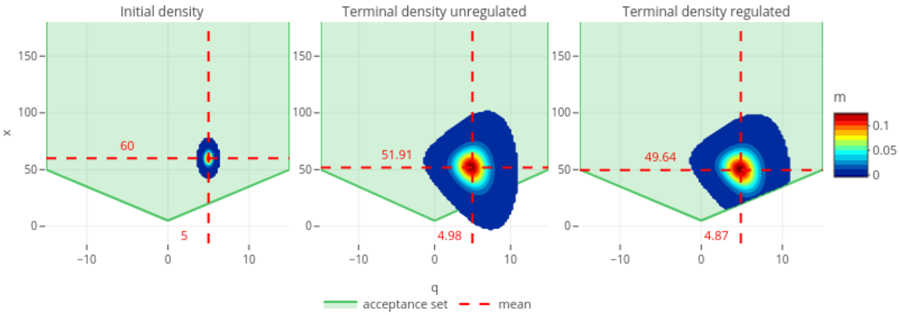

Figure: Contour plots of the pre-liquidation density at start and terminal time for the unregulated and regulated case. Acceptance region $\mathcal{A}$ in green.

Evolution of the banking system

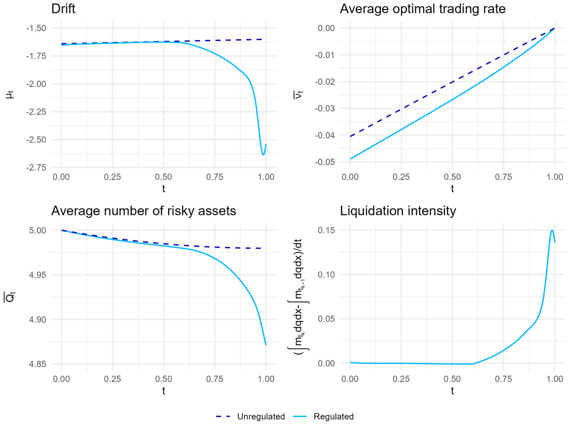

Figure: Evolution of banking system in regulated (light blue) and unregulated (dark blue) case. Note the spike in liquidations at $t \approx 0.9$

Evolution of the banking system: well-capitalized banks

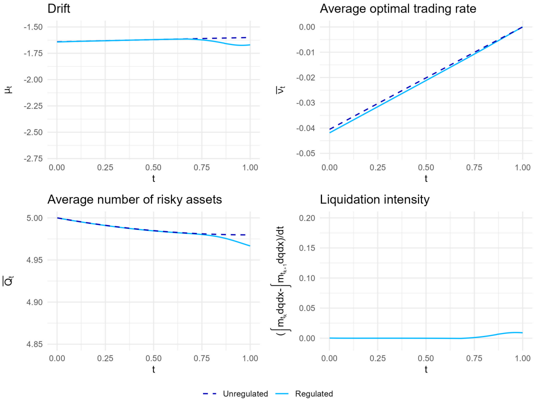

Figure: Evolution of banking system in regulated and unregulated case for (on average) well capitalized banks. In that case the system is stable, supporting claims for higher bank equity as in Admati and Hellwig [2013] or Hanson et al. [2011].

Mathematical analysis with smoothed contagion

- Next we discuss : mathematical analysis of a modified model with smoothed contagion.
- We use arguments from Burzoni and Campi [2023], and Campi et al. [2021],
- These papers use arguments from the probabilistic weak formulation of MFGs, see Carmona and Lacker [2015] and Lacker [2018]

Modified model.

State process. We model transaction costs via running cost. Hence we have

$$\mathrm{d}Q_t = \nu_t \mathrm{d}t + \sigma_Q \mathrm{d}W_t^Q,$$
$$\mathrm{d}X_t = Q_t(\mu_{\text{ex}} + \alpha \bar{c}_t)\mathrm{d}t + \sigma_A \mathrm{d}W_t^A + Q_t \sigma_S \mathrm{d}W_t^S.$$

Define $\sigma_X(q) = (\sigma_A^2 + q^2\sigma_S^2)^{\frac{1}{2}}$, $b(\mathbf{x}, c, \nu) = \begin{pmatrix} \nu \\ q(\mu_{\text{ex}} + \alpha c) \end{pmatrix}$ and let $\Sigma(q) = \text{diag}(\sigma_Q, \sigma_X(q))$. Then

$$\mathrm{d}\mathbf{X}_t = b(\mathbf{X}_t, \bar{c}_t, \nu_t)\mathrm{d}t + \Sigma(Q_t)\mathrm{d}\mathbf{W}_t \tag{7}$$

for BM $\mathbf{W} = (W^Q, W^X)'$ with $W_t^X = \int_0^t \frac{1}{\sigma_X(Q_s)} \mathrm{d}(\sigma_A \mathrm{d}W_s^A + Q_s \sigma_S \mathrm{d}W_s^S)$.

Measure flow. We let $\langle m_t, f \rangle = \mathbb{E}[f(\mathbf{X}_t) 1_{\{\tau_{\mathcal{A}} > t\}}]$, and denote by $m = (m_t)_{0 \le t \le T}$ the corresponding flow.

Smoothed contagion (Hambly and Sojmark [2019])

Consider a smooth kernel $k : [0, T] \to \mathbb{R}^+$ with support $[0, \epsilon]$ and $\int_0^\epsilon k(u)\mathrm{d}u = 1$. Fix a large $\bar{q}$ and let $\mathfrak{q}(q, x) = q \wedge \bar{q} \vee (-\bar{q})$. Then we define the time-averaged inventory level $\mathcal{I}_t$ by

$$\mathcal{I}_t := \int_{t-\epsilon}^{t} k(t-s)\langle m_s, \mathfrak{q}\rangle \mathrm{d}s.$$

$\mathcal{I}_t$ is absolutely continuous (even if $t \mapsto \langle m_t, \mathfrak{q}\rangle$ is not). We define the time-smoothed contagion term $\bar{c}_t = \bar{c}_t(m)$ by

$$\bar{c}_t := \partial_t \mathcal{I}_t = k(0)\langle m_t, \mathfrak{q}\rangle + \int_{t-\epsilon}^{t} k'(t-s)\langle m_s, \mathfrak{q}\rangle \mathrm{d}s$$

Note that $\bar{c}_t$ is bounded independently of m.

Optimization problem and equilibrium

Admissible strategies $\boldsymbol{\nu}$ is admissible if it is is adapted to $\mathbb{F}^{\mathbf{W}}$ and if $|\nu_t| \leq \bar{\nu}$, that is we assume a compact control space

Goal of a bank. Maximize

$$\mathbb{E}\Big[G(\mathbf{X}^{\boldsymbol{\nu}}_{\tau_{\mathcal{A}}}) - \int_0^{\tau_{\mathcal{A}}} \kappa \nu_s^2 \mathrm{d}s\Big] \tag{8}$$

over admissible $\boldsymbol{\nu}$, where $G(\mathbf{x}) = 0$ on $\partial\mathcal{A}$ and $G(q, x) \approx x$ on $\mathcal{A}$.

Definition (MFG equilibrium). A probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ supporting a BM $\mathbf{W}$, a measure flow m, and an admissible control $\hat{\boldsymbol{\nu}}$ are an equilibrium of the MFG if

- Given m (and hence $\bar{c}_t$), $\hat{\boldsymbol{\nu}}$ is optimal for the problem with state process $\mathbf{X}$ from (7) and objective (8)
- $m_t(\cdot) = \mathbb{P}\big(X_t \in \cdot \cap \tau_A > t\big)$.

Optimization for a given measure flow

Lemma 1. There is some M such that $\left\|\Sigma(q)^{-1}b(\mathbf{x}, c, \nu)\right\|^2 \leq M$ for all $\mathbf{x}$ and all $|c| \leq \bar{C}, |\nu| \leq \bar{\nu}$.

Weak formulation. Consider some $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ supporting a driftless state process $\mathrm{d}\mathbf{X}_t = \Sigma(\mathbf{X}_t)\mathrm{d}\mathbf{W}_t$. Fix a measure flow m and a strategy $\boldsymbol{\nu}$ and let

$$U_t = U_t^{m,\boldsymbol{\nu}} = \int_0^t \Sigma^{-1}(Q_s)b(\mathbf{X}_s, \bar{c}_s(m), \nu_s)\mathrm{d}\mathbf{W}_s$$

Lemma 1 $\to \mathcal{E}(U)$ is a martingale. Define $\mathbb{P}^{m,\boldsymbol{\nu}}$ by $\frac{\mathrm{d}\mathbb{P}^{m,\boldsymbol{\nu}}}{\mathrm{d}\mathbb{P}} = \mathcal{E}(U^{m,\boldsymbol{\nu}})_T$.
Girsanov $\Rightarrow$ under $\mathbb{P}^{m,\boldsymbol{\nu}}$, $\mathbf{X}$ has drift $b(\mathbf{X}_t, \bar{c}_t(m), \nu_t)$.

Weak objective. (for given m) Find strategy $\boldsymbol{\nu}$ that maximizes

$$\boldsymbol{\nu} \mapsto J(\boldsymbol{\nu}) = \mathbb{E}^{m,\boldsymbol{\nu}}\left[G(\mathbf{X}_{\tau_{\mathcal{A}}}) - \int_0^{\tau_{\mathcal{A}}} \kappa\nu_s^2\mathrm{d}s\right]$$

Optimization for given m ctd

Define the Hamiltonian

$$\begin{aligned} H(t, \mathbf{x}, m, \mathbf{z}, \nu) &= -\kappa\nu^2 + \mathbf{z}'\Sigma(q)^{-1} b(\mathbf{x}, \bar{c}_t(m), \nu) \\ &= -\kappa\nu^2 + \frac{z_1 \nu}{\sigma_Q} + \frac{z_2 q}{\sigma_X(q)}(\mu_{\text{ex}} + \alpha \bar{c}_t(m)) \end{aligned}$$

Denote by $\hat{\nu}(\mathbf{z}) = \frac{z_1 \sigma_Q^{-1}}{2\kappa} \vee (-\bar{\nu}) \wedge \bar{\nu}$ the unique maximizer of H wrt ν and let $\hat{H}(t, \mathbf{x}, m, \mathbf{z}) = H(t, \mathbf{x}, m, \mathbf{z}, \hat{\nu})$.

Proposition 2. Suppose $\mathbf{X}$ solves under $\mathbb{P}$ the SDE $\mathrm{d}\mathbf{X}_t = \Sigma(Q_t)\mathrm{d}\mathbf{W}_t$. Given a flow m, denote by $\hat{Y}, \hat{\mathbf{Z}}$ the unique solution of the BSDE

$$\mathrm{d}Y_t = \hat{H}(t, \mathbf{X}_t, m, \mathbf{Z}_t)\mathrm{d}t + \mathbf{Z}_t \mathrm{d}\mathbf{W}_t, \quad Y_{\tau_\mathcal{A}} = G(\mathbf{X}_{\tau_\mathcal{A}})$$

Then the control $\hat{\boldsymbol{\nu}} = (\hat{\nu}(\mathbf{Z}_t))_t$ is optimal and $J(\hat{\boldsymbol{\nu}}) = \mathbb{E}[Y_0]$.

Existence of an equilibrium

By Lemma 1 and standard estimates for stochastic exponentials, for any measure flow m and any admissible $\boldsymbol{\nu}$, the measure $\mathbb{P}^{m,\boldsymbol{\nu}}$ belongs to

$$\mathcal{E} = \left\{ \mathbb{P}' \in \mathcal{M}_1(\Omega, \mathcal{F}, \mathbb{P}) \colon \mathbb{E}\left[\left(\frac{\mathrm{d}\mathbb{P}'}{\mathrm{d}\mathbb{P}} \right)^2 \right] \leq e^{MT} \right\}$$

$\mathcal{E}$ is convex and moreover compact for the so-called τ-topology . Consider now the following map $\Phi \colon \mathcal{E} \to \mathcal{E}$:

$$\mathbb{P}' \in \mathcal{E} \overset{see (6)}{\longrightarrow} m(\mathbb{P}') \overset{\text{Prop2}}{\longrightarrow} \mathbb{P}^{m(\mathbb{P}'),\hat{\boldsymbol{\nu}}} \in \mathcal{E},$$

that is $\Phi(\mathbb{P}') = \mathbb{P}^{m(\mathbb{P}'),\hat{\boldsymbol{\nu}}}$. By definition a fixed point of Φ is an equilibrium.

Theorem 3. There is a MFG equilibrium.

This follows from Schauder's fixed point theorem and the fact that Φ is continuous in the τ topology.

Propagation of chaos and approximate Nash equilibria

In probability and statistical physics 'Propagation of Chaos' does not refer to

- the state of a typical teenager's room or
- the Mitternachtsquadrille at the WU Ball

Propagation of chaos and approximate Nash equilibria

In probability and statistical physics 'Propagation of Chaos' does not refer to

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but to the limiting behavior of the distribution of exchangeable interacting particle systems as the number N of particles gets large

Setup

Consider N banks with state process $\mathbf{X}_i^N$, $1 \leq i \leq N$ and

$$\mathrm{d}\mathbf{X}_{t,i}^N = b(\mathbf{X}_{t,i}^N, \bar{c}_t(\mu^N), \hat{\nu}_t)\mathrm{d}t + \Sigma(Q_{t,i}^N)\mathrm{d}\mathbf{W}_{t,i},$$

where

- $\hat{\nu}$ is the optimal strategy from the MFG equilibrium.
- $\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{X}_i^N}$ is the empirical measure of the trajectories $\mathbf{X}_1, \ldots \mathbf{X}_N$ on $\mathcal{C} := \mathcal{C}^0([0, T], \mathbb{R}^2)$,
- $\mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\mathbf{X}_{t,i}^N} 1_{\{\tau(\mathbf{X}_i^N)>t\}}$
- and finally $\bar{c}_t(\mu^N) = k(0)\langle \mu_t^N, \mathfrak{q}\rangle + \int_{t-\epsilon}^t k'(t-s)\langle \mu_s^N, \mathfrak{q}\rangle \mathrm{d}s$

Note that the $\mathbf{X}_i^N$ interact via the contagion term and that the measure μ^N is random.

A limit result

Denote by $\hat{\mathbb{P}}$ the equilibrium measure of the MFG. The following result, based on Lacker [2018], shows that μ^N converges to $\hat{P}$ for $N \to \infty$.

Theorem. For every open neighborhood U of $\hat{\mathbb{P}}$ in $\mathcal{M}^1(\mathcal{C})$ one has $\lim_{N\to\infty} \mathbb{P}\big(\mu^N \notin U\big) = 0$.

Implications

- for $u\colon \mathbb{R}^2 \to \mathbb{R}$ bounded continuous it holds that

$$\lim_{N\to\infty} \mathbb{P}\Big(|\langle \mu_t^N, u\rangle - \langle m_t(\hat{\mathbb{P}}), u\rangle| > \delta\Big) = 0$$

- The law of the components $\mathbf{X}_i^N$ converges in the τ-topology to $\hat{P}$.

Using these results one can show that the optimal strategy from the MFG induces an approximate Nash equilibrium for the system with finitely many banks (work in progress)

Outlook and next steps

- Reduce regularity assumptions
- Numerics adapted to weak approach
- Capital injections
- Two groups of banks Large or systemically important banks and others
- Common noise (difficult)

A. Admati and M. Hellwig. *The Bankers' New Clothes: What's Wrong with Banking and What to Do about It*. Princeton University Press, Princeton, NJ, 2013.

Basel Committee on Banking Supervision. Basel III leverage ratio framework and disclosure requirements, January 2014. Bank of International Settlements.

Yann Braouezec and Lakshithe Wagalath. Strategic fire-sales and price-mediated contagion in the banking system. *European Journal of Operational Research*, 274(3):1180–1197, 2019. doi: https://doi.org/10.1016/j.ejor.2018.11.012.

Matteo Burzoni and Luciano Campi. Mean field games with absorption and common noise with a model of bank run. *Stochastic Processes and Their Applications*, 164:206–241, October 2023. doi: 10.1016/j.spa.2023.07.007.

Luciano Campi and Markus Fischer. N-player games and mean field games with absorption. *Annals of Applied Probability*, 28(4):2188–2242, 2018. doi: 10.1214/17-AAP1354.

Luciano Campi, Maddalena Ghio, and Giulia Livieri. *N*-Player games and mean-field games with smooth dependence on past absorptions. *Annales de l'Institut Henri Poincaré B: Probability and Statistics*, 57(4): 1901–1939, 2021. doi: 10.1214/20-AIHP1138.

P. Cardaliaguet and C. A. LeHalle. Mean field game of controls and an application to trade crowding. *Mathematics and Financial Economics*, 12:335–363, 2018.

René Carmona and Daniel Lacker. A probabilistic weak formulation of mean field games and applications. *Annals of Applied Probability*, 25 (3):1189–1231, 2015. doi: 10.1214/14-AAP1020.

René Carmona, Jean-Pierre Fouque, and Li-Hsien Sun. Mean-field games and systemic risk. *Communications in Mathematical Sciences*, 13(4): 911 – 933, 2015. doi: 10.4310/CMS.2015.v13.n4.a4.

R. Cont and L. Wagalath. Fire sales forensics: Measuring endogenous risk. *Mathematical Finance*, 26(4):835–866, 2016. doi: https://doi.org/10.1111/mafi.12071.

Christa Cuchiero, Stefan Rigger, and Sara Svaluto-Ferro. Propagation of minimality in the supercooled Stefan problem. *The Annals of Applied Probability*, 33(2):1588 – 1618, 2023. doi: 10.1214/22-AAP1850.

Larry Eisenberg and Thomas H. Noe. Systemic risk in financial systems. *Management Science*, 47(2):236–249, 2001. ISSN 00251909, 15265501.

H. Elsinger, A. Lehar, and M. Summer. Risk assessment for banking systems. *Management Science*, 52:1301–1314, 2006.

Zachary Feinstein. Capital regulation under price impacts and dynamic financial contagion. *European Journal of Operational Research*, 281(2): 449–463, 2020. doi: https://doi.org/10.1016/j.ejor.2019.08.044.

Rüdiger Frey and Theresa Traxler. Playing with fire? A mean-field game analysis of fire sales and systemic risk under regulatory capital constraints. *Canadian Journal of Statistics*, 2026. doi: 10.1002/cjs.70038.

Paul Glasserman and H. Peyton Young. Contagion in financial networks. *Journal of Economic Literature*, 54(3):779–831, 2016. doi: 10.1257/jel.20151228.

B. Hambly and A. Sojmark. An SPDE model for systemic risk with endogenous contagion. *Finance and Stochastics*, 23:535–594, 2019. doi: https://doi.org/10.1007/s00780-019-00396-1.

Ben Hambly, Sean Ledger, and Andreas Sojmark. A McKean–Vlasov equation with positive feedback and blow-ups. *The Annals of Applied Probability*, 29(4):pp. 2338–2373, 2019.

S.G. Hanson, A.K Kashyap, and J.C. Stein. A macroprudential approach to financial regulation. *Journal of Economic Perspectives*, 25(1):3–28, 2011.

Daniel Lacker. On a strong form of propagation of chaos for McKean-Vlasov equations. *Electronic Communications in Probability*, 23:1 – 11, 2018. doi: 10.1214/18-ECP150. URL https://doi.org/10.1214/18-ECP150.

S. Nadtochiy and M. Shkolnikov. Particle systems with singular interaction through hitting times: Application in systemic risk modeling. *The Annals of Applied Probability*, pages 89–129, 2019.

L.C.G. Rogers and L.A. Veraart. Failure and rescue in an interbank network. *Management Science*, 59:882–898, 2013.