Bounds for Expected Occupation Densities and Applications

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Outline

- Occupation Measures and Local Times
- Expected Occupation bounds
- Applications

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Occupation Measure

Definition (Occupation Measure)

Let $(X_t)_{t\geq 0}$ be a continuous sample paths stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. For $t \geq 0$, define the *occupation measure* by

$$\Lambda_t(A) := \int_0^t \mathbf{1}_A(X_s)\, ds, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

$$\mu_t(A) := \int_0^t \mathbf{1}_A(X_s)\, d[X]_s, \quad A \in \mathcal{B}(\mathbb{R}^d)$$

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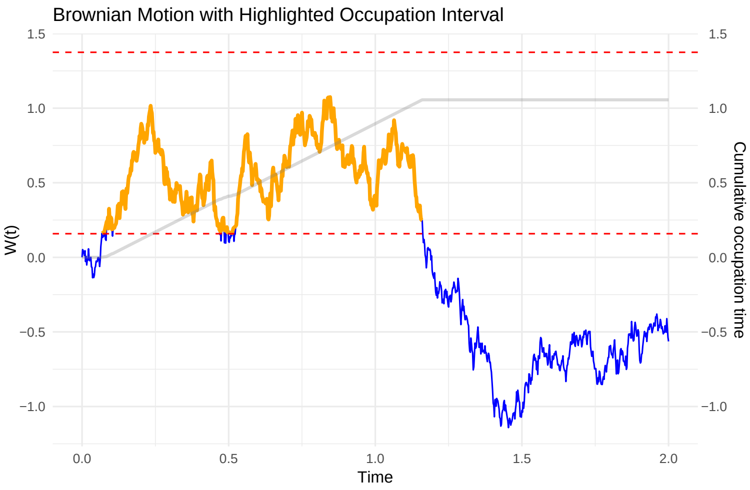
- Measures how much time the process spends in set A up to time t.
- Naturally depends on sample path ω.

BM with highlighted iccupation interval



Occupation Density

Definition ((Expected) Occupation Density)

- If there exists a measurable function $\lambda_t(x)$ such that

$$\Lambda_t(A) = \int_A \lambda_t(x)\,dx, \qquad A \in \mathcal{B}(\mathbb{R}^d),$$

then $\lambda_t(x)$ is called the *occupation density* at time $t \geq 0$, $x \in \mathbb{R}^d$.

- If there exists a measurable function $\ell_t(x)$ such that

$$\int_0^t P(X_s \in A)ds = \mathrm{E}[\Lambda_t(A)] = \int_A \ell_t(x)\,dx, \qquad A \in \mathcal{B}(\mathbb{R}^d),$$

then $\lambda_t(x)$ is called the expected occupation density at time $t \geq 0$, $x \in \mathbb{R}^d$.

Remarks on (expected) occupation measure and density

- The occupation measure Λ_t is a finite random measure with total mass t and its expectation $\mathrm{E}[\Lambda_t(\cdot)]$ is a finite measure with total mass t.
- If X is a 1-dimensional diffusion $dX_t = b_t dt + \sigma_t dW_t$ with $\sigma_t \geq \sigma_{\min} > 0$, then for any $t \geq 0$ the occupation density has a locally bounded version.
- If X is a 1-dimensional process such that X_t has density ρ_t, then

$$\int_0^t \rho_s(x) ds = \ell_t(x).$$

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The main reason to be interested in expected occupation densities is the following formula which holds for any measurable $f : \mathbb{R} \to [0, \infty)$:

$$\mathrm{E}\left[\int_0^t f(X_s) ds\right] = \int_{-\infty}^{\infty} f(x) \ell_t(x) dx.$$

Usage of expected occupation densities

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Let X be a continuous sample path process which has expected occupation density $\ell_t(x)$ and γ measurable with $\ell_t(x) \leq \gamma_t(x)$, that is γ is some upper bound for the expected occupation density.

- For measurable $f : \mathbb{R} \to [0, \infty)$:

$$\mathrm{E}\left[\int_0^t f(X_s)ds\right] \leq \int_{-\infty}^{\infty} f(x)\gamma_t(x)dx.$$

- Upper bounds are usually easier to obtain than the expected occupation density itself.
- If an upper bound is known up to some constants (implicit upper bound), then it can be used to get qualitative upper bounds as appear in stability schemes for SDEs.

Relation with expected values

Lemma

Let X be a Markovian diffusion $dX_t = b(X_t)dt + \sigma(X_t)dW_t$ and $f : \mathbb{R} \to \mathbb{R}$ be a C^2-function with b, σ bounded measurable. Then

$$
\begin{aligned}
\mathrm{E}[f(X_t)] &= \mathrm{E}[f(X_0)] + \int_0^t \mathrm{E}[\mathcal{A}f(X_s)]ds \\
&\leq \mathrm{E}[f(X_0)] + \int_{-\infty}^{\infty} \max\{\mathcal{A}f(x), 0\}\ell_t(x)dx
\end{aligned}
$$

where $\mathcal{A}$ is the generator $\mathcal{A}f(x) = b'(x)f(x) + \frac{1}{2}\sigma^2(x)f''(x)$.

- The formula is a useful tool for bounds, control and sensitivity analysis.

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Kernel based bounds (Strook Varadhan, 1979)

Setting. Let $(X_t)_{t\geq 0}$ solve the SDE

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t,$$

where $b, \sigma \in C_b^\infty(\mathbb{R}^d)$ and the diffusion matrix $a = \sigma\sigma^\top$ is *uniformly elliptic*:

$$m_1|\xi|^2 \leq \xi^\top a(x)\xi \leq m_2|\xi|^2.$$

Gaussian heat kernel bounds. There exist unknown constants $c_i > 0$ such that density ρ_t of X_t admits

$$\rho_t(x) \leq c_1 t^{-d/2} \exp\left(-c_2 \frac{|x - X_0|^2}{t}\right), \qquad t \in (0, T].$$

Expected occupation density bound. Consequently,

$$\ell_t(x) \leq \int_0^t c_1 s^{-d/2} \exp\left(-c_2 \frac{|x - X_0|^2}{s}\right) ds.$$

Failure of densities (McNamura 1983)

Counterexample. There exists a one-dimensional diffusion

$$dX_t = \sigma(t, X_t)\, dW_t, \qquad X_0 = 0,$$

with

$$\sigma(t, x) \in \{\sigma_1, \sigma_2\},$$

such that for some fixed $T > 0$ the random variable X_T *does not admit a density* with respect to Lebesgue measure.

Key properties.

- Uniform ellipticity holds.
- The diffusion coefficient is bounded and measurable.
- The process is well defined for all $t \geq 0$.

Consequences.

- No classical heat kernel estimate exists at time T.
- Pointwise or Gaussian density estimates are impossible.
- Occupation bounds cannot rely on kernel estimates.

Krylov occupation bounds (Krylov Röckner 2008)

Setting. Let $(X_t)_{t\geq 0}$ be a diffusion

$$dX_t = \beta_t dt + \sigma_t dW_t,$$

where

$$a_t = \sigma_t \sigma_t^\top \quad \text{is uniformly elliptic and bounded,}$$

and b is progressively measurable unbounded.

Krylov estimate. Let $f \in L^p([0,T]\times\mathbb{R}^d)$ with $p > \frac{d}{2} + 1$. Then there exists a unknown constant $C = C(d, p, \lambda, \Lambda, T)$ such that

$$\mathbb{E}\left[\int_0^T |f(t, X_t)|\, dt\right] \leq C\|f\|_{L^p([0,T]\times\mathbb{R}^d)}.$$

Key features.

- No pointwise heat kernel bounds required (or possible).
- Coefficients need only be measurable.
- Applies equally to time-inhomogeneous diffusions.

K. Xu upper bound (explicit)

Setting. Let $(X_t)_{t\geq 0}$ be a 1-dimensional diffusion

$$dX_t = \beta_t dt + \sigma_t \, dW_t,$$

where $\sigma_t \in [\sigma_{\min}, \sigma_{\max}]$ with $0 < \sigma_{\min} \leq \sigma_{\max}$ and $|\beta_t|$ is bounded by K.

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Theorem (K., Xu (2025))

Let X be as above. Then its expected local time ℓ_t exists and

$$\alpha_t(x) \leq \ell_t(x) \leq \gamma_t(x)$$

for some continuous functions α, γ
with explicit form given by ...

Explicit upper bound

Theorem

Let X be as above. Then its expected local time ℓ_t exists and a version satisfies

$$\ell_t(x) \leq \gamma_t(x)$$

where

$$\gamma_t(x) = \frac{\sigma_{\max}}{\sigma_{\min}^2} \int_0^t \left(\frac{1}{\sqrt{s}} \varphi\Big(v(|x - X_0|, s)\Big) + K\Phi\Big(v(|x - X_0|, s)\Big) \right) ds,$$

where φ, Φ are the standard normal density and distribution function and

$$v(r,s) := K\sqrt{s} - \frac{r}{\sigma_{\max}\sqrt{s}} \quad s > 0, r \geq 0$$

Explicit upper bound

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Further comments

- Optimal expected and path-wise interval occupation has been found in (Ankirchner, Wendt 25). The bound for

$$\int_0^t P(X_s \in [a,b])ds = \mathrm{E}[\Lambda_t([a,b])] \leq Q_t(a,b,\sigma_{\min},\sigma_{\max},K), \qquad a < b, t \geq 0$$

  looks quite different — they only coincide to our bound in a limiting sense. However, their bound was used to find the bound in (Krühner, Xu 25).

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Application: Pre-estimates for Control Problems

Setting: Let

$$X_t^\sigma = X_0 + \int_0^t \sigma(s, X_s) dW_s$$

where $\sigma : \mathbb{R}^2 \to [a, b]$ measurable where $0 < a \leq b$ is given. Denote the set of measurable functions σ where such a process X^σ exists by $\mathcal{C}$.

Question: How large is

$$\sup_{\sigma \in \mathcal{C}} \mathrm{E}\left[f(X_t^\sigma) + \int_0^t g(X_s^\sigma) ds \right] \leq ?$$

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A common numerical approach: Find C^2-functions $V_N(t, x)$ and hope that

$$V_N(u, x) \approx \mathrm{E}\left[f(X\sigma_t) + \int_u^t g(X_s^\sigma) ds \middle| X_u^\sigma = x \right], \qquad V_N(t, x) = f(x)$$

Application: Pre-estimates for Control Problems, cont.

A common numerical approach: Need to find $h_N : \mathbb{R} \to [0, \infty)$ such that

$$\sup_{m \in [a,b]} \left(\partial_1 V_N(s,x) + \frac{1}{2} \partial_2^2 V_N(s,x) m^2 + g(x) \right) \leq h(x)$$

Application: Pre-estimates for Control Problems, cont.

A common numerical approach: Need to find $h_N : \mathbb{R} \to [0,\infty)$ such that

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Error control: For some $\sigma \in \mathcal{C}$ we find

$$\begin{aligned} \mathrm{E}[f(X_t^\sigma)] &= \mathrm{E}[V_N(t, X_t^\sigma)] \\ &= V_N(0, X_0) + \mathrm{E}\left[\int_0^t \partial_1 V_N(s, X_s^\sigma) + \frac{1}{2}\partial_2^2 V_N(s, X_s)\sigma^2(X_s) ds\right] \\ &\leq V_N(0, X_0) + \mathrm{E}\left[\int_0^t h_N(X_s) ds\right] - \mathrm{E}\left[\int_0^t g(X_s) ds\right] \end{aligned}$$

So we get for any $\sigma \in \mathcal{C}$:

$$\mathrm{E}\left[f(X_t) + \int_0^t g(X_s) ds\right] \leq V_N(0, X_0) + \int_{-\infty}^{\infty} h(x)\gamma_t(x) dx.$$

Summary and Conclusions

- Expected occupation densities bounds γ allow for upper mean path integral bounds:

$$\mathrm{E}\left[\int_0^t f(X_s)ds\right] \leq \int_{-\infty}^{\infty} f(x)\gamma_t(x)dx$$

- When drift/diffusion are known to be smooth, then
- (Krühner Xu 2025) bounds give explicit upper estimates under bounded drift and diffusion in the elliptic regime under no further regularity assumption on the drift/diffusion.

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Thank you for your attention!

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Questions?