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Fitting Regularized Partially Ordinal Regression Models with "Don't Know" Option within the GAMLSS Framework

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- estimation will be based on a mixture model, using (approximate) restricted maximum likelihood
- feature and model selection based on penalization

Short recap on GAMLSS

Generalized Additive Models for Location, Scale and Shape

$$g_1(\mu) \quad = \quad \eta_\mu = \beta_{0\mu} + \sum_{j=1}^{p_1} f_{j\mu}(x_j) \qquad \text{“location”}$$

$$g_2(\sigma) \quad = \quad \eta_\sigma = \beta_{0\sigma} + \sum_{j=1}^{p_2} f_{j\sigma}(x_j) \qquad \text{“scale”}$$

$$\vdots \qquad\qquad\qquad \vdots$$

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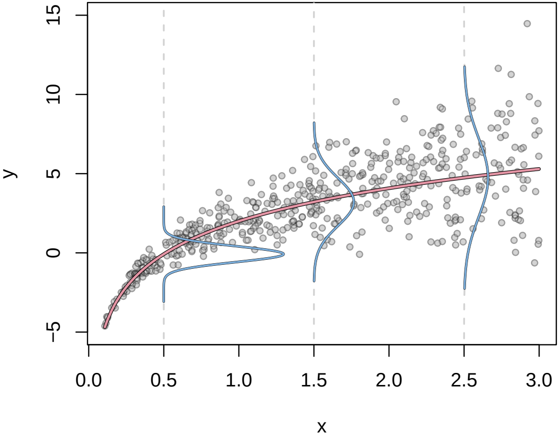
- proposed by Rigby and Stasinopoulos (2005)
- extension of generalized additive models (GAMs)
- distribution parameters are modeled by specific predictors and associated link functions $g_k(\cdot)$.

Example for GAMLSS

$$Y \sim N(\,\mu = \beta_{0\mu} + f_\mu(x),\;\; \sigma = \exp\left(\beta_{0\sigma} + f_\sigma(x)\right))$$



General formulation of a GAMLSS:

- response vector $\mathbf{y} = (y_1, \dots, y_n)^\top$
- corresponding conditional density $f(y_i|\boldsymbol{\theta}_i)$, depending on several distribution parameters $\boldsymbol{\theta}_i = (\theta_{i1}, \dots, \theta_{id})^\top$
- with known monotonic link functions $g_k(\cdot)$:

$$g_k(\theta_{ik}) = \beta_{0k} + \sum_{j=1}^{p_k} \mathbf{x}_{ijk}^\top \boldsymbol{\beta}_{jk} \; = \; \eta_{\theta_{ik}} \, , \qquad k = 1, \dots, d \, .$$

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- estimation of regression parameters: maximize log-likelihood

$$\ell(\boldsymbol{\beta}) = \sum_{i=1}^{n} \log\left(f(y_i|\boldsymbol{\theta}_i)\right) \, , \tag{1}$$

  with $\boldsymbol{\beta}$ collecting all effects of all linear predictors.
- fitting: R-package `gamlss` (Stasinopoulos and Rigby, 2007)

Regularization in the GAMLSS framework

- a gradient boosting approach is provided by Mayr et al. (2012)
- allows for variable selection within GAMLSS framework
- corresponding R-package `gamboostLSS` (Hofner et al., 2015)
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- **new:** an alternative *gradient boosting* approach is implemented in the R-package `bamlss` (Umlauf et al., 2024)
  - embeds many different approaches suggested in literature and software
  - serves as unified conceptional "Lego toolbox" for complex regression models

L1-type penalization in GAMLSS

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Group LASSO (Meier et al., 2008): For a (dummy-encoded) categorical covariate $\mathbf{x}_{jk}$ use

$$J_g(\boldsymbol{\beta}_{jk}) = ||\boldsymbol{\beta}_{jk}||_2\,,$$

with vector $\boldsymbol{\beta}_{jk}$ collecting all corresponding coefficients.

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Fused LASSO (Gertheiss and Tutz, 2010): depending on *nominal* (left) or *ordinal* scale level (right) of the covariate, we use

$$J_f(\boldsymbol{\beta}_{jk}) = \sum_{l>m} w_{lm}^{(jk)} \left|\beta_{jkl} - \beta_{jkm}\right|, \text{ or } J_f(\boldsymbol{\beta}_{jk}) = \sum_{l=1}^{c_{jk}} w_{l}^{(jk)} \left|\beta_{jkl} - \beta_{jk,l-1}\right|,$$

where c_{jk} is the number of levels of categorical predictor $\mathbf{x}_{jk}$ and $w_{lm}^{(jk)}, w_{l}^{(jk)}$ denote suitable weights.

(we choose $l = 0$ as reference $\Longrightarrow \beta_{jk0} = 0$ is fixed)

Fitting algorithm

- implemented in R-package `bamlss` (Umlauf et al., 2024)
- based on Newton-Raphson-type updating
- for very large data sets: IWLS-backfitting scheme
- quadratic penalty approximation (see Oelker & Tutz, 2017)

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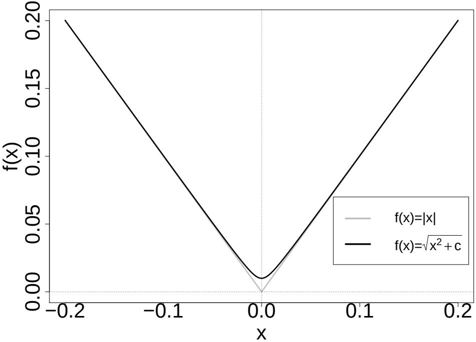

Partially ordinal model

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Ordinal outcomes and simultaneously account for "dk" answers

$\Rightarrow$ represent outcome of individual $i = 1, \ldots, n$ in terms of:

- $Y_{i1} \in \{0, 1\}$: decision of individual to opt for "dk"

  $\Rightarrow Y_{i1} = 1$ if "dk" option is chosen, $Y_{i1} = 0$ otherwise

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- $Y_{i2} \in \{1, \dots, C\}$: ordinal Likert scale outcome if an item is answered

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Responses are dependent / only partially observable:

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Now, link bivariate response vector $\boldsymbol{Y}_i = (Y_{i1}, Y_{i2})^\top$ to (continuous) latent variables $\boldsymbol{Y}_i^* = (Y_{i1}^*, Y_{i2}^*)^\top$, where

$$Y_{i1} = 0 \;\Leftrightarrow\; Y_{i1}^* \leq \alpha_{1,1}$$

with $\alpha_{1,1} \in \mathbb{R}$, and

$$Y_{i2} = c \;\Leftrightarrow\; \alpha_{2,c-1} < Y_{i2}^* \leq \alpha_{2,c},\; c = 1, \ldots, C\,,$$

with ordered thresholds $-\infty = \alpha_{2,0} < \ldots < \alpha_{2,C} = +\infty$.

Partially ordinal model (III)

For this project, we mostly focus on simple linear effects:

$$Y_{id}^{*} = \eta_{id} + \varepsilon_{id},\ d = 1, 2\,,$$

with i.i.d. errors ε_{id} with CDF $G_d(\cdot)$, and predictors $\eta_{id} = \boldsymbol{x}_{id}^{\top}\boldsymbol{\beta}_d$.

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We plan to extend the model by

- nonlinear effects,
- random effects to account for intra-individual heterogeneity and dependence arising from the longitudinal data structure,
- spatial effects.

Partially ordinal model (IV)

We assume:

- $\boldsymbol{\varepsilon}_i = (\varepsilon_{i1}, \varepsilon_{i2})^\top \sim \mathcal{G}_\varepsilon$
- and margins $\varepsilon_{id} \sim G_{\varepsilon_d}, d = 1, 2$.

Then, for given features $\boldsymbol{x}_{id}$, the conditional likelihood for the respondent choosing the “dk” option is

$$
\begin{aligned}
P(Y_{i1} = 1 \,|\, \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}) &= P(Y_{i1}^* > \alpha_{1,1} \,|\, \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}) \\
&= 1 - P(\varepsilon_{i1} \leq \alpha_{1,1} - \eta_{i1}) \\
&= 1 - G_{\varepsilon_1}(\alpha_{1,1} - \eta_{i1})
\end{aligned}
$$

Partially ordinal model (V)

If the respondent decides to answer $(Y_{i1} = 0)$, we get

$$
\begin{aligned}
& P(Y_{i1} = 0, Y_{i2} = c \,|\, \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}) \\
= \quad & P(Y_{i1}^* \leq \alpha_{1,1}, \alpha_{2,c-1} < Y_{i2}^* \leq \alpha_{2,c} \,|\, \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}) \\
= \quad & \mathcal{G}_\varepsilon(\alpha_{1,1} - \eta_{i1}, \alpha_{2,c} - \eta_{i2}) \\
& -\mathcal{G}_\varepsilon(\alpha_{1,1} - \eta_{i1}, \alpha_{2,c-1} - \eta_{i2})
\end{aligned}
$$

Dependence structures

- Correlate error terms via $\boldsymbol{\varepsilon}_i \sim N(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon),\ \boldsymbol{\Sigma}_\varepsilon = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$,
- In this particular case, we obtain the conditional distribution

$$\boldsymbol{Y}_i^* \,|\, \boldsymbol{x}_{i1}, \boldsymbol{x}_{i2} \sim N(\boldsymbol{\eta}_i, \boldsymbol{\Sigma}_\varepsilon).$$

Penalization

- classical shrinkage and variable selection,
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Note that for **estimation** and **inference**

- we use locally quadratic approximations (see Oelker and Tutz, 2017; Groll et al., 2019)
- implemented in `bamlss` (Umlauf et al., 2024)
- now taken over to `gamlss2` (Rigby and Stasinopoulos, 2005)

$\Rightarrow$ employ iterative estimation procedures (NR, FS, IWLS).

Simulation study

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- draw 6 categorical predictors, $x_1, \ldots, x_6$, with 4 to 6 levels each

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  $\boldsymbol{\beta}_1 = (0, 0.5, -0.4, 0.3, -0.4)^\top, \boldsymbol{\beta}_4 = (0, -0.4, 0.4, 0.5, -0.5)^\top$
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- draw 2 latent responses with $\boldsymbol{\Sigma}_\varepsilon = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, with
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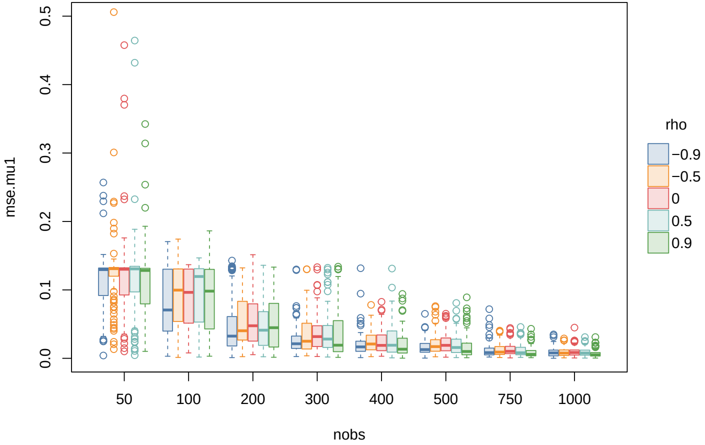
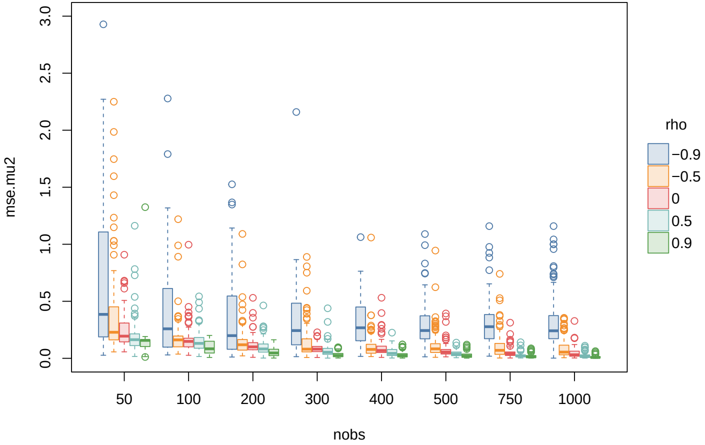
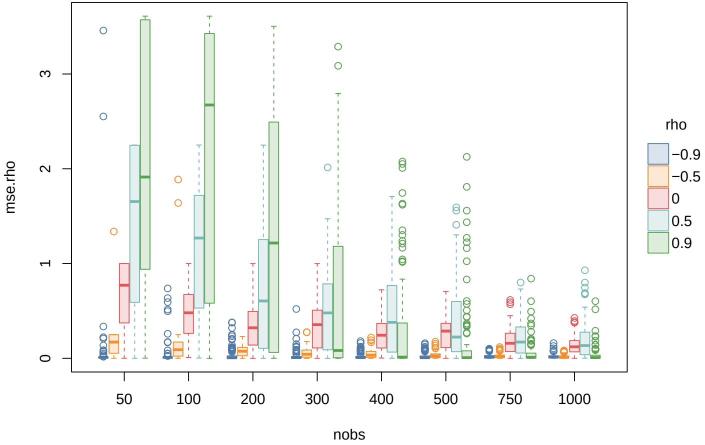
- we fit GAMLSS with group LASSO penalties



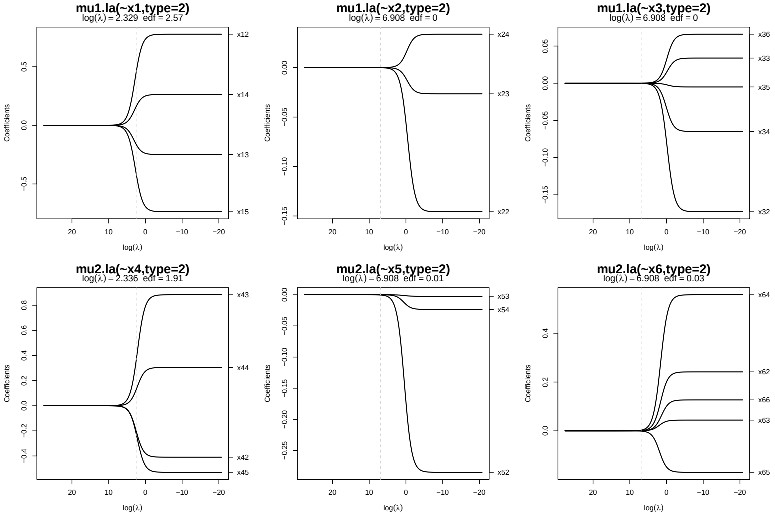
MSEs for η_1-coefficients and varying n, each based on 100 simulation runs



MSEs for η_2-coefficients and varying n, each based on 100 simulation runs



MSEs for ρ and varying n, each based on 100 simulation runs



Coefficient paths of the 6 factors and $\rho = 0.5$; vertical dashed lines: optimal penalty strength

Case Study: `Fintech`

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- x_1: Age; x_2: Gender; x_3: Education; x_4: Family composition; x_5: Income; x_6: Employment

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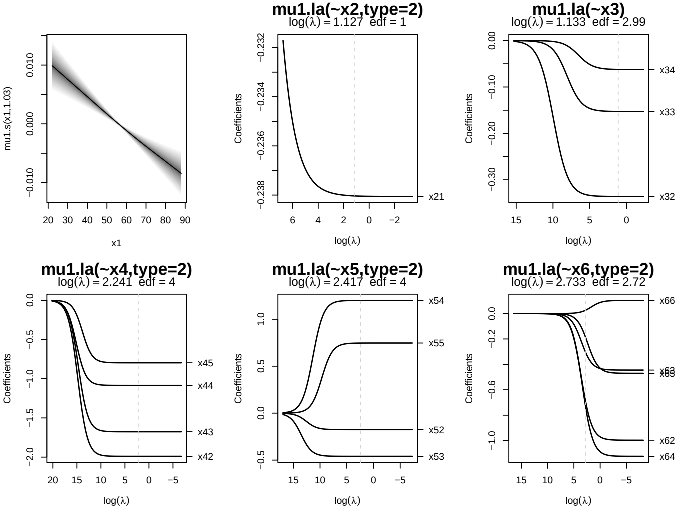
The predictors are specified as follows for $d \in \{1, 2\}$ (in `gamlss2` syntax):

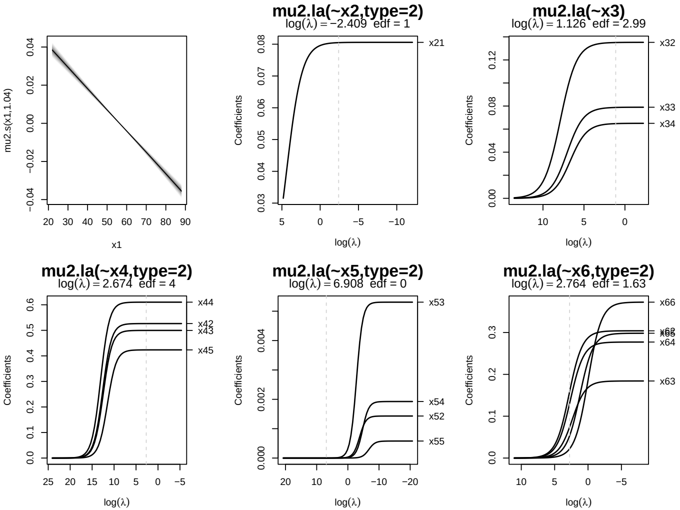
$$\eta_d = s(x_1) + la(x_2) + la(x_3, type = 2) + \ldots + la(x_6, type = 2)$$

Results I



Estimated effects & coefficient paths corresponding to η_1; vertical dashed lines: λ_{opt}

Results II



Estimated effects & coefficient paths corresponding to η_2; vertical dashed lines: λ_{opt}

Summary

- We introduced a regularized bivariate GAMLSS for partially ordered categorical data with a “dk” option.
- The model separates the latent process of “dk” selection from the ordinal response, allowing correlation.
- We investigated model behavior in short simulation study.
- Application to `Fintech` data:
  - We found different covariate effects for η_1 and η_2,
  - Moderate amount of regularization,
  - Latent correlation $\approx 0 \Rightarrow$ supports cond. independence.

Future directions & outlook

- Incorporate additional response options ("No answer")
- Extend to space-time components, random effects, etc.
- Apply in broader survey and behavioral settings.

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