PDMP based risk models

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Overview

Risk Models

PDMPs

QMC integration

Outlook

Risk models and ruin concept

Surplus of insurance portfolio given by process $X = (X_t)_{t \geq 0}$

Determine:

time and probability of ruin ...classical risk measure
(indication of problems with liquidity)

$$\tau = \inf\{t > 0 \mid X_t < 0\}$$
$$\psi(x) = P(\tau < \infty \mid X_0 = x), \qquad \psi(x,T) = P(\tau \leq T \mid X_0 = x)$$

or in general *Gerber-Shiu* functions:

$$g(x) := \mathbb{E}_x\left(e^{-\delta\tau} w(X_{\tau-}, |X_\tau|)\, \mathbb{1}_{\{\tau<\infty\}}\right)$$

w ... *function of time of, deficit at and surplus prior to ruin*

$\Rightarrow$ allows for mutual analysis of risk relevant quantities

(Gerber & Shiu 1998-classical, 2005-renewal)

Classical risk or Cramér-Lundberg model

Use $X = (X_t)_{t \geq 0}$ of the form

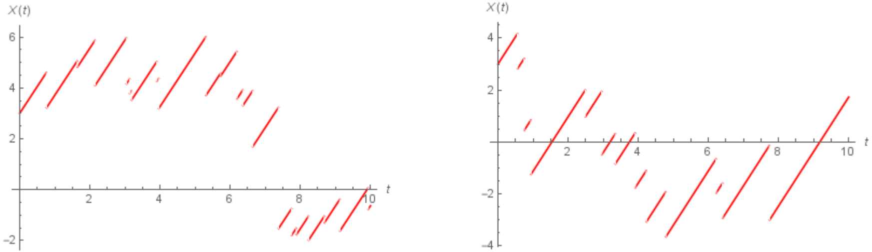
$$X_t = x + c\,t - \sum_{k=1}^{N_t} Y_k, \quad t \geq 0$$

Ingredients:

- deterministic initial capital $x \geq 0$ and premium rate $c \geq 0$
- counting process $N = (N_t)_{t \geq 0}$ homogeneous Poisson process with intensity $\lambda > 0$
- claims $\{Y_k\}_{k \in \mathbb{N}}$, $Y_k \overset{iid}{\sim} F_Y$ with $F_Y(0) = 0$, $\mathbb{E}(Y_1) = \mu$
- crucial assumption: N and $\{Y_k\}$ are independent

(Lundberg 1903, Cramér 1955, net profit condition: $c > \lambda\mu$)

Sample paths



Surplus paths with Exp and Par distributed claims

Asymptotic behaviour of ruin probability

Classical results depend on nature of claims

- light-tailed claims ($\exists\, s > 0$ with $\mathbb{E}[e^{sY_1}] < \infty$)

$$\lim_{x\to\infty} e^{Rx}\psi(x) = C$$

with $R > 0$ s.t. $\lambda(\mathbb{E}[e^{RY_1}] - 1) - c\,R = 0$

- heavy tailed claims (if $F_I(x) = \frac{1}{\mu}\int_0^x (1 - F_Y(y))dy \in \mathcal{S}$)

$$\lim_{x\to\infty} \frac{\psi(x)}{1 - F_I(x)} = \frac{\rho}{1-\rho}$$

$$\psi(x) \sim \frac{\rho}{\alpha(1-\rho)} \left(\frac{x}{c}\right)^{-(\alpha-1)}$$

... if $f_Y(x) = \frac{\alpha}{c}\left(\frac{c}{x}\right)^{\alpha+1}\ (x > c > 0)$

Excursion: reinsurance control

Goal: minimize penalty function

$$\Phi(x) = \inf_{u \in \mathcal{U}} \Phi^u(x) := \inf_{u \in \mathcal{U}} \mathbb{E}_x \left[e^{-\delta \tau_x^u} \, w(X^u_{\tau_x^u -}, |X^u_{\tau_x^u}|) \right]$$

$$X_t^u = x + \int_0^t c(u_s) ds - \sum_{i=1}^{N_t} r(Y_i, u_{T_i})$$

Control by dynamic reinsurance, where

- parametrized retention function

$$r : [0, \infty) \times U \to [0, \infty) \text{ with } 0 \leq r(y, u) \leq y$$

- admissible controls

$$\mathcal{U} = \{ u = (u_t)_{t \geq 0} \mid u_t \in U \text{ and } u \text{ is } \mathcal{F}^X \text{ previsible} \}$$

(Preischl & Th. 2019)

HJB-equation:

$$0 = \inf_{u\in U}\left\{c(u)f'(x) - (\delta+\lambda)f(x) + \lambda\int_0^{\rho(x,u)} f(x - r(y,u))\,dF_Y(y) + \lambda\int_{\rho(x,u)}^{\infty} w(x, r(y,u) - x)\,dF_Y(y)\right\}$$

Operator for uniqueness:

$$\mathcal{G}f(x) := \inf_{u\in\mathcal{U}}\left\{\mathbb{E}_x\left[e^{-\delta T_1} f(X^u_{T_1})\mathbb{1}_{\{T_1<\tau^u_x\}}\right] + \mathbb{E}_x\left[e^{-\delta T_1} w(X^u_{T_1-}, |X^u_{T_1}|)\mathbb{1}_{\{T_1=\tau^u_x\}}\right] + \mathbb{E}_x\left[e^{-\delta\tau^u_x} w(0,0)\mathbb{1}_{\{T_1>\tau^u_x\}}\right]\right\}$$ ... contraction on $\mathcal{C}^{+,b}[0,\infty)$

Theorem

In $\mathcal{C}^{+,b}[0,\infty)$, Φ is unique fixed point of $\mathcal{G}$ and unique positive, (Lipschitz) continuous solution to HJB-equation that **is not greater** *than $w(0,0)$.*

Why do we need more general processes?

- numerical approach via policy iteration:

  fix u_0, compute V^{u_0} $\rightarrow$ improve control, fix u_1, compute V^{u_1} ...

- Markovian controls $u_t = u(X_{t-})$ lead to controlled processes of PDMP type

- on the way we need classical cost functions

$$v^i(x) = \mathbb{E}_x \left[\int_0^\tau e^{-\delta t} \ell(X_t^{u_i}) dt + e^{-\delta \tau} \Psi(X_\tau^{u_i}) \right]$$

- also here $v^i(0)$ is crucial

- use MC simulations for approximation of $v^i(0)$
  ($\rightarrow$ approximate $(\mathcal{G}^{u_i})^n f(0)$ with MC)

Illustration of results

$F_Y(x) = 1 - (1+x)^{-3}$, $\delta = 0.1$ and penalty $w_2(x,y) = \min\{10^{10}, (x+0.5)(y+1)^2\}$

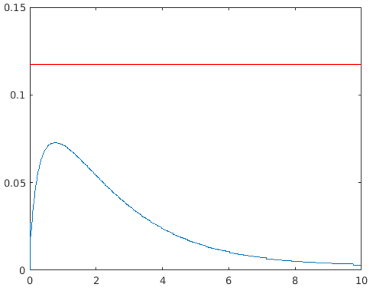


Figure: Optimal strategy for Pareto claims

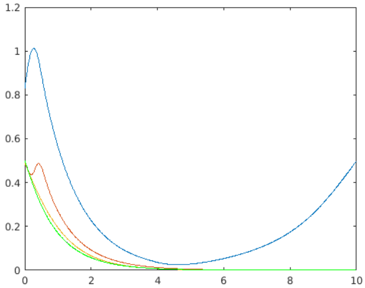


Figure: Functions Φ^{u2} to Φ^{u5}

Need for model extensions

- analyze risk models in unified framwork
- keep Markov property
  (at least by adding not too many components)
- allow for flexible behaviour between jumps
- include more complex jumps
  (intensity and jump size distributions)
- incorporate control opportunities

Piecewise deterministic Markov processes

... introduced as *finite variation sample path* alternative to diffusions

Construction of $X = (X_t)_{t\geq 0}$:

- state space $E = \{(k,y)\,|\,k \in K \text{ and } y \in E_k\}$ (K finite set, $E_k \subset \mathbb{R}^{d_k}$)
- $\phi = \{\phi_k\}$ ... deterministic trajectories (ϕ_k specified by vector field $\mathcal{X}_k$ on E_k)

$$X_t = (k, \phi_k(y,t)), \quad X_0 = (k,y), \quad \frac{\partial}{\partial t}\phi_k(y,t) = g_k(\phi_k(y,t))$$

- $\lambda = \{\lambda_k\}$ ... jump intensities

$$\text{time of 1st jump } T_1 \overset{d}{\sim} P_{k,y}(T_1 > t) = e^{-\int_0^t \lambda_k(\phi_k(y,s))ds}$$

- $Q : (E, \mathcal{E}) \to [0,1]$ ... jump kernel

$$X_{T_1} \overset{d}{\sim} Q(\phi_k(y,T_1), \cdot)$$

- piecewise construction (starting anew in X_{T_1})

(PDMPs introduced by Davis 1984)

Additional features

- *active boundary* Γ: points at boundary of E which can be reached along ODE paths (good for bing-bang controls)

- at time $t^*(x) = \inf\{t \geq 0 \,|\, \phi_k(t, \zeta) \in \Gamma\}$ $(x = (k, \zeta))$ force jump

$$T_1 \overset{d}{\sim} P_x(T_1 > t) = e^{-\int_0^t \lambda_k(\phi_k(\zeta,s))ds} \mathbb{1}_{\{t < t^*(x)\}}$$

- embedded pure jump Markov process η with

$$\eta_t = (X_{T_n}, n) \quad \text{for } T \leq t < T_{n+1}$$

(something to be exploited later)

Sometimes easier to deal with *generator* of X

Theorem (Davis 1984/92)

Let X be a PDMP with $\mathbb{E}_x[N_t] < \infty$ for all $t \geq 0,\ x \in E$. Then $\mathcal{D}(\mathcal{A})$ consists of functions f which fulfill

- $f(x) = \lim_{t\to 0} f(\phi_\nu(-t,\zeta))$ *for* $x = (\nu,\zeta) \in E$,
- $t \mapsto f(\phi_\nu(t,\zeta))$ *is absolutely continuous for* $x = (\nu,\zeta) \in E$,
- $f(x) = \int_E f(y)Q(x,dy)$ *for* $x \in \Gamma$,
- $\mathcal{B}f \in L_1^{loc}(p)$,

and $\mathcal{A}f$ is

$$\mathcal{A}f(x) = \mathcal{X}f(x) + \lambda(x)\int_E (f(y) - f(x))Q(x,dy).$$

$\left(p(t,A) = \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \leq t\}} \mathbb{1}_{\{X_{T_i} \in A\}}\right.$ and $\left.p^*(t) = \sum_{i=1}^{\infty} \mathbb{1}_{\{T_i \leq t\}} \mathbb{1}_{\{X_{T_i-} \in \Gamma\}}\right)$

Cost functions

Consider

- cemetery state $E^c \neq \emptyset$ (process absorbed)
- running reward/cost function $\ell \colon E \to \mathbb{R}$ with $\ell|_{E^c} \equiv 0$
- terminal cost function $\Psi \colon E^c \to \mathbb{R}$ with $\Psi|_{E \setminus E^c} \equiv 0$

Corresponding cost functional:

$$v(x) = \mathbb{E}_x \left[\int_0^\tau e^{-\delta t} \ell(X_t) dt + e^{-\delta \tau} \Psi(X_\tau) \right]$$

$$\tau = \inf\{t \geq 0 \colon X_t \in E^c\}$$

Goal: determine $v(x)$ by means of integration instead of IDE

Iterated integrals

Exploit Markov property of $\{X_{T_i}\} \Rightarrow$

$$
\begin{aligned}
v(x) =& \mathbb{E}_x \Bigg[\left(\int_0^{T_1} e^{-\delta t} \ell(\phi(x,t)) dt + e^{-\delta T_1} v(X_{T_1}) \right) \mathbb{1}_{\{T_1 < \tau\}} \\
& + \left(\int_0^{\tau} e^{-\delta t} \ell(\phi(x,t)) dt + e^{-\delta \tau} \Psi(\phi(x,\tau)) \right) \mathbb{1}_{\{\tau < T_1\}} \\
& + \left(\int_0^{T_1} e^{-\delta t} \ell(\phi(x,t)) dt + e^{-\delta T_1} \Psi(X_{T_1}) \right) \mathbb{1}_{\{T_1 = \tau\}} \Bigg] \\
=:& \mathcal{H}(x) + \mathcal{G}v(x)
\end{aligned}
$$

$\mathcal{H}$ ... collects costs/rewards between jumps

$\mathcal{G}$ ... shifts problem forward by one jump (time)

In total we arrive at:

$$
v(x) = \underbrace{\mathcal{G}^n v(x)}_{\to 0} + \sum_{i=1}^{n} \underbrace{\mathcal{G}^{i-1}\mathcal{H}(x)}_{2i-1\, dim \text{ integral}}
$$

Identify integrand (unfortunately complicated):

$$\mathcal{G}^{i-1}\mathcal{H}(x_0) =$$

$$\int_{t_1=0}^{\infty} f_W(t_1, x_0)e^{-\delta t_1} \int_{x_1 \in E} \int_{t_2=0}^{\infty} f_W(t_2, x_1)e^{-\delta t_2} \int_{x_2 \in E} \cdots \int_{t_{i-1}=0}^{\infty} f_W(t_{i-1}, x_{i-2})e^{-\delta t_{i-1}}$$

$$\int_{x_{i-1} \in E} \mathcal{H}(x_{i-1})Q(\phi(x_{i-2}, t_{i-1}), dx_{i-1})dt_{i-1} \cdots Q(\phi(x_0, t_1), dx_1)dt_1$$

$$= \int_{t_1=0}^{\infty} \int_{x_1 \in E} \cdots \int_{t_{i-1}=0}^{\infty} \int_{x_{i-1} \in E} \left(\prod_{j=1}^{i-1} f_W(t_j, x_{j-1})e^{-\delta t_j} \right)$$

$$\mathcal{H}(x_{i-1})Q(\phi(x_{i-2}, t_{i-1}), dx_{i-1})dt_{i-1} \cdots Q(\phi(x_0, t_1), dx_1)dt_1$$

... but still it can be beneficial to exploit

$$v(x) \approx \sum_{i=1}^{n} \mathcal{G}^{i-1}\mathcal{H}(x)$$

for some x - but certainly not too many

QMC integration

Numerically evaluate

$$\int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x} \quad \text{for} \quad f : [0,1]^s \to \mathbb{R}$$

using point set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\} \subset [0,1]^s$, $N \in \mathbb{N}$

Quality of points measured by D_N^* (distance to *uniformity*):

$$D_N^* = \sup_{J \subset [0,1]^s} \left| \frac{1}{N} \sharp \{ n \leq N : \boldsymbol{x}_n \in J \} - \lambda(J) \right|$$

$\ldots \sup$ taken over axis-aligned boxes J with one vertex in $\mathbf{0}$

Koksma-Hlawka inequality provides error bound:

$$\left| \frac{1}{N} \sum_{n=1}^{N} f(\boldsymbol{x}_n) - \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x} \right| \leq \mathcal{V}(f) D_N^*$$

(low discrepancy sequence achieve $D_N^* \leq C (\ln N)^s N^{-1}$)

Comparison of point sets

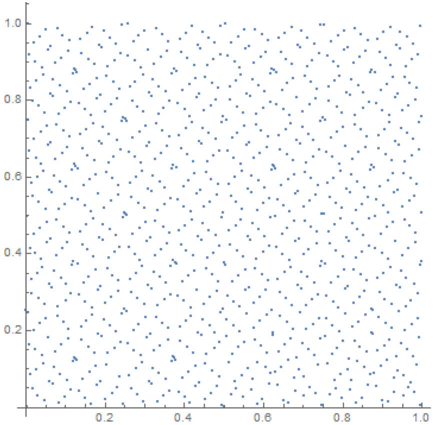


Figure: 1000 Sobol points

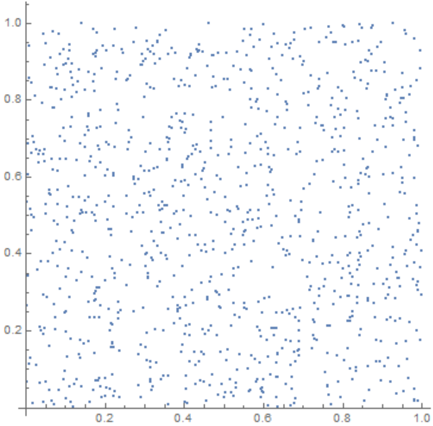


Figure: 1000 $U([0,1]^2)$ points

Complications

Form of error bound appealing:

- contribution of point set via D_N^*
- contribution of integrand via its variation $\mathcal{V}(f)$

Drawback: $\mathcal{V}(f)$ in *Hardy-Krause sense* is hard to deal with

... best case $f : [0,1]^s \to \mathbb{R}$ continuous derivatives up to order s, then

$$\sum_{\emptyset \neq u \subset \{1,\ldots,s\}} \int_{[0,1]^{|u|}} \left| \frac{\partial^{|u|}}{\partial \mathbf{x}_u} f(\mathbf{x}_u, \mathbf{1}) \right| d\mathbf{x}_u$$

- difficult to estimate
- many integrands are known to have unbounded variation

Modified approach

For $f \in \mathcal{C}^2([0,1]^s)$ one gets:

$$V_{\mathcal{K}}(f) \leq \sup f - \inf f + \frac{s}{16} \sup\{\|Hess(f,x)\| \, | x \in [0,1]^s\}$$

such that error bound is

$$\left| \int_{[0,1]^s} f(\boldsymbol{x}) d\boldsymbol{x} - \frac{1}{N} \sum_{i=1}^{N} f(\boldsymbol{x}_i) \right| \leq$$

$$\left(\sup_{\boldsymbol{x} \in [0,1]^s} f(\boldsymbol{x}) - \inf_{\boldsymbol{x} \in [0,1]^s} f(\boldsymbol{x}) + \frac{s}{16} \sup\{\|Hess(f,\boldsymbol{x})\| \, | \boldsymbol{x} \in [0,1]^s\} \right) \tilde{D}_N$$

with *isotropic discrepancy*

$$\tilde{D}_N = \sup_{J \in \mathcal{K}} \left| \frac{1}{N} \sharp\{n \leq N : \boldsymbol{x}_n \in J\} - \lambda(J) \right|$$

(notice $D_N^* \leq \tilde{D}_N \leq (4s\sqrt{s}+1)(D_N^*)^{1/s}$, concept due to Pausinger & Svane 2015)

Observations

Message: integrand part of $\mathcal{G}^i\mathcal{H}(x)$ should be $\mathcal{C}^2$

includes: first i jump times and $i-1$ post-jump locations

$\Rightarrow$ interplay between ODE sensitivities

$$\frac{\partial}{\partial t}\phi(y,t),\ \frac{\partial^2}{\partial t^2}\phi(y,t),\ \frac{\partial}{\partial y}\phi(y,t),\ \frac{\partial^2}{\partial t \partial y}\phi(y,t),\ \frac{\partial^2}{\partial y^2}\phi(y,t)$$

and probabilistic ingredients (λ, Q)

We have 2 choices:

Let $\{X^n\}_{n\in\mathbb{N}}$ be *smooth-coefficient-approximating* PDMPs

- Use weak convergence to show convergence of expected values
- Show directly $\lim_{n\to\infty} v^n(x) \to v(x)$

Theorem (Kritzer et al. 2019)

Let X be a Feller PDMP with local characteristics (ϕ, λ, Q) and let X^n, $n \in \mathbb{N}$, be Feller PDMPs with local characteristics (ϕ^n, λ^n, Q^n). Further, let the following assumptions hold:

(i) *$g^n \to g$ and $\lambda^n \to \lambda$ as $n \to \infty$, uniformly in $x \in E$,*

(ii) *for all $f \in C_b^\infty(E, \mathbb{R})$,*

$$\lim_{n\to\infty} \sup_{x \in E} \left| \int_E f(y) Q^n(dy, x) - \int_E f(y) Q(dy, x) \right| = 0,$$

(iii) *$X_0^n \xrightarrow{d} X_0$ in E.*

Then $X^n \xrightarrow{d} X$ in $D([0,\infty), E)$ and if ℓ, Ψ are bounded and continuous

$$\mathbb{E}_x \left(\int_0^\tau e^{-\delta t} \ell(X_t^n) dt + e^{-\delta\tau} \Psi(X_\tau^n) \right) \to \mathbb{E}_x \left(\int_0^\tau e^{-\delta t} \ell(X_t) dt + e^{-\delta\tau} \Psi(X_\tau) \right)$$

as $n \to \infty$.

Current work and outlook

Use PDMP techniques to analyze risk models with stochastic intensities

Surplus process $(X, \lambda, \cdot) = ((X_t, \lambda_t, t))_{t \geq 0}$ with generators:

$$\begin{aligned} \mathcal{A}^{SN} f(x,\lambda,t) =& c\frac{\partial f(x,\lambda,t)}{\partial x} - \delta\lambda \frac{\partial f(x,\lambda,t)}{\partial \lambda} + \frac{\partial f(x,\lambda,t)}{\partial t} - (\lambda+\rho) f(x,\lambda,t) \\ &+ \lambda \int_0^\infty f(x-u,\lambda,t) dF_U(u) + \rho \int_0^\infty f(x,\lambda+y,t) dF_Y(y) \end{aligned}$$

$$\begin{aligned} \mathcal{A}^{H} f(x,\lambda,t) =& c\frac{\partial f(x,\lambda,t)}{\partial x} + \delta(a-\lambda) \frac{\partial f(x,\lambda,t)}{\partial \lambda} + \frac{\partial f(x,\lambda,t)}{\partial t} - \lambda f(x,\lambda,t) \\ &+ \lambda \int_0^\infty \int_0^\infty f(x-u,\lambda+y,t) dF_U(u) dF_Y(y) \end{aligned}$$

(Shot-noise: Pojer & Th. 2022, Hawkes: Palmowski, Pojer & Th. 2022 working paper)

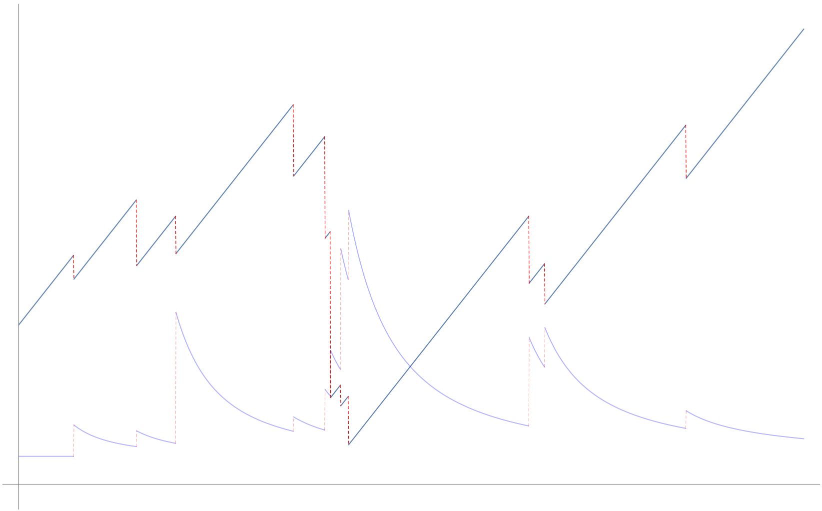


Figure: Surplus with stochastic intensity

(Plot by Simon Pojer: Hawkes or Shot-Noise?)

Under *meaningful* assumptions on parameters we can derive:

$$\lim_{x \to \infty} e^{Rx} \psi(x, \lambda) = C^{\lambda}$$

Proofs use:

- exponential martingales and suitable change of measure
- recurrence of intensities to get rid of λ_t
- renewal theorem of Schmidli (1997) for the equation

$$Z(u) = z(u) + \int_0^u Z(u-y)(1 - p(u, y))B(dy)$$

(results are surprising, since suitable renewal structure is not obvious)

References

Kritzer, Leobacher, Szölgyenyi & Th., Approximation methods for piecewise deterministic Markov processes and their costs, 2019.

Preischl & Th., Optimal reinsurance for Gerber-Shiu functions in the Cramér-Lundberg model, 2019.

Pojer & Th., Ruin probabilities in a Markovian shot-noise environment, 2022.

Thank you for your attention