Towards responsible game theory –
from Kant to a parametric QP (copositive view)

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Overview

1. Games and the categorical imperative
2. Equilibrium à la Nash ...
3. ... need not exist !
4. Characterization via copositivity
5. Equilibrium refinements
6. Partnership games: sth. between local and global

A simple two-actor game

Finitely many elementary actions $i \in N = \{1, \ldots, n\}$;

if i played against j, payoff is a_{ij}; payoff matrix $\mathbf{A} = [a_{ij}]_{(i,j) \in N \times N}$.

"Rational" behavior: given j, select i with maximal a_{ij};

randomizing strategies: given distribution

$$\mathbf{x} \in \Delta = \left\{ \mathbf{x} \in \mathbb{R}^n : x_i \geq 0 \,,\ \mathbf{e}^\top \mathbf{x} = \sum_i x_i = 1 \right\} .$$

Then select $\mathbf{y} \in \Delta$ with maximal expected payoff $\mathbf{y}^\top \mathbf{A} \mathbf{x}$ ... is LP, so w.l.o.g. $\mathbf{y} = \mathbf{e}_i$ vertex of Δ. Note $\mathbf{e} = \sum_i \mathbf{e}_i = [1, \ldots, 1]^\top \in \mathbb{R}^n$.

Nash and HM-equilibrium

Nash equilibrium [Nash 1951]: $\mathbf{x}$ best response to itself,
$\mathbf{x}$ maximizes $\mathbf{y}^\top \mathbf{A}\mathbf{x}$ over $\mathbf{y} \in \boldsymbol{\Delta}$ (and some $\mathbf{e}_i$ too). Exists always.

But why should opponents' behaviour completely discouple ?

Categorical imperative [Kant 1785]: act such that your behaviour can be a model for the general society.

Grundlegung
zur
Metaphysik
der Sitten

von

Immanuel Kant.

Riga,
bey Johann Friedrich Hartknoch
1785.

Nash and HM-equilibrium

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Bit of Kant: given *morality parameter* $\theta \in [0, 1]$ and $\mathbf{x}$, maximize

$$u_\theta(\mathbf{y}|\mathbf{x}) := (1-\theta)\,\mathbf{y}^\top \mathbf{A}\mathbf{x} + \theta\,\mathbf{y}^\top \mathbf{A}\mathbf{y} \quad \text{over } \mathbf{y} \in \Delta\,.$$

Homo Moralis (HM)-equilibrium [Alger/Weibull 2013]:
$\mathbf{x}$ itself maximizes $u_\theta(\mathbf{y}|\mathbf{x})$ over $\mathbf{y} \in \Delta$.

Existence of HM-equilibrium ...

... asks whether there is $\mathbf{x} \in \Delta$ maximizing $u_\theta(\cdot|\mathbf{x})$.

For any $\theta \in [0,1]$ and $\mathbf{x} \in \Delta$, the (nonconvex) QP solution set

$$\beta_\theta(\mathbf{x}) = \text{Argmax}\,\{u_\theta(\mathbf{y}|\mathbf{x}) : \mathbf{y} \in \Delta\}$$

is (nonempty and) compact but need not contain $\mathbf{x}$ itself.

Indeed, there are nasty examples even for $|N| = 3$:

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 2 \end{pmatrix}$$

has strictly convex $u_\theta(\cdot|\mathbf{x})$ for any $\theta \in (0,1)$, so $\beta_\theta(\mathbf{x}) \subseteq \{\mathbf{e}_i : i \in N\}$ for all $\mathbf{x} \in \Delta$ but $\beta_\theta(\mathbf{e}_1) = \{\mathbf{e}_3\}$, $\beta_\theta(\mathbf{e}_2) = \{\mathbf{e}_1\}$, $\beta_\theta(\mathbf{e}_3) = \{\mathbf{e}_2\}$.

Optimality conditions

... for QP $\max\{u_\theta(\mathbf{y}|\mathbf{x}) : \mathbf{y} \in \Delta\}$ defining $\beta_\theta(\mathbf{x})$:

First-order necessary/KKT condition: If $\mathbf{y} \in \beta_\theta(\mathbf{x})$, then

$$\frac{\partial}{\partial y_j} u_\theta(\mathbf{y}|\mathbf{x}) \leq \mathbf{y}^\top \nabla_{\mathbf{y}} u_\theta(\mathbf{y}|\mathbf{x}) \quad \text{for all } j$$

with equality if $y_j > 0$.

Have $\mathbf{g}_\theta(\mathbf{y}|\mathbf{x}) := \nabla_{\mathbf{y}} u_\theta(\mathbf{y}|\mathbf{x}) = (1-\theta)\mathbf{A}\mathbf{x} + \theta(\mathbf{A} + \mathbf{A}^\top)\mathbf{y}$.

HM-equilibrium at $\mathbf{x}$ implies that $\mathbf{y} = \mathbf{x}$ is KKT point,
where $\mathbf{g}_\theta(\mathbf{x}|\mathbf{x}) = \mathbf{C}_\theta \mathbf{x}$ with $\mathbf{C}_\theta = \mathbf{A} + \theta\mathbf{A}^\top$.

These are local optimality conditions and **only necessary.**
Need curvature control for QPs over Δ, *aka StQPs* [B. 1997]:

Second-order optimality characterization

A KKT point $\mathbf{y} \in \Delta$ is a **global** maximizer of $u_\theta(\cdot|\mathbf{x})$

if and only if for all i with $y_i > 0$ the symmetric $n \times n$ matrix

$$\mathbf{H}_i(\theta) := \mathbf{e}_i \mathbf{g}_\theta^\top(\mathbf{y}|\mathbf{x}) + \mathbf{g}_\theta(\mathbf{y}|\mathbf{x})\mathbf{e}_i^\top - \theta y_i(\mathbf{A} + \mathbf{A}^\top)$$

satisfies

$$\mathbf{v}^\top \mathbf{H}_i(\theta)\mathbf{v} \geq 0 \quad \text{whenever } \mathbf{v} \in \Gamma_i \,,$$

i.e., if $\mathbf{H}_i(\theta)$ is Γ_i-copositive where

$$\Gamma_i := \left\{ \mathbf{v} \in \mathbb{R}^n : \mathbf{v} \perp \mathbf{e} \text{ and } v_j y_i - v_i y_j \geq 0 \text{ for all } j \in N \right\}$$

is a polyhedral cone.

Now for HM-equilibrium again specialize above for $\mathbf{y} = \mathbf{x}$;

need support $I := \{i \in N : x_i > 0\}$ of $\mathbf{x}$.

Characterization of HM-equilibria

Fix $\theta \in [0,1]$; then the point $\mathbf{x} \in \Delta$ with support I gives rise to HM-equilibrium **if and only if** for some $\gamma \in \mathbb{R}$, both (a) and (b):

(a) the point $(\mathbf{x}, \gamma) \in \mathbb{R}^{n+1}$ solves the linear system of (in)equalities

$$[(\mathbf{A} + \theta \mathbf{A}^\top)\mathbf{x}]_i \quad \begin{cases} = & \gamma\,, \qquad i \in I\,, \\ \leq & \gamma\,, \qquad i \in N \setminus I\,, \end{cases}$$

and (b)

$$\mathbf{H}_i(\theta) \text{ is } \Gamma_i\text{-copositive for all } i \in I\,,$$

where $\mathbf{H}_i(\theta) = \mathbf{e}_i \mathbf{x}^\top (\mathbf{A}^\top + \theta \mathbf{A}) + (\mathbf{A} + \theta \mathbf{A}^\top)\mathbf{x}\mathbf{e}_i^\top - \theta x_i (\mathbf{A} + \mathbf{A}^\top)$ and $\Gamma_i = \left\{ \mathbf{v} \perp \mathbf{e} : v_j x_i \geq v_i x_j\,, \text{ all } j \in N \right\}$.

Difficult to check in general, simpler in special cases.

For $\theta = 0$ reduces to Nash condition as property (b) is automatic.

Antagonism marginalizes morality

Constant-sum games: $a_{ji} = c - a_{ij}$ model antagonistic agents.

Then $\mathbf{C}_\theta = (1-\theta)\mathbf{A} + \theta c\, \mathbf{e}\mathbf{e}^\top$ and $\mathbf{H}_i(\theta) = \mathbf{H}_i(0) - c\theta x_i\, \mathbf{e}\mathbf{e}^\top$, so

$$\mathbf{x} \in \beta_0(\mathbf{x}) \qquad \Leftrightarrow \qquad \mathbf{x} \in \beta_\theta(\mathbf{x}) \quad \text{for all } \theta \in [0,1]\,.$$

All HM-equilibria coincide with classical Nash equilibria for the base game, morality plays no role.

These games are special cases of concave welfare games where existence of HM-equilibria is ensured:

Concave/strictly convex welfare and existence

Let $\mathbf{D} = [\mathbf{I}_{n-1} | -\mathbf{e}]$ and suppose $\lambda_{\text{max}}[\mathbf{D}(\mathbf{A}+\mathbf{A}^\top)\mathbf{D}^\top] \le 0$. Then welfare $\mathbf{y}^\top \mathbf{A}\mathbf{y}$ is concave in $\mathbf{y}$ over Δ and so is $u_\theta(\cdot|\mathbf{x})$ for all $\mathbf{x} \in \Delta$. Thus $\beta_\theta(\mathbf{x})$ is (compact and) convex, so standard fixed point theory implies existence of a $\mathbf{x} \in \beta_\theta(\mathbf{x})$ for any $\theta \in [0, 1]$. So concave welfare ensures HM-equilibrium.

On the other hand, if $\lambda_{\text{min}}[\mathbf{D}(\mathbf{A}+\mathbf{A}^\top)\mathbf{D}^\top] > 0$, then welfare is strictly convex and $\beta_\theta(\mathbf{x}) \subseteq \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Thus HM-equilibrium must yield a vertex $\mathbf{e}_i$, and this holds for $\theta \in (0, 1)$ if and only if

$$a_{ii} \ge \theta a_{kk} + (1-\theta)\, a_{ki} \quad \text{for all } k \in N\,.$$

This fails to hold in counterexample.

HM-equilibrium for small θ yields Nash refinement

Suppose that for all $\theta \searrow 0$ the points $\mathbf{x}(\theta)$ give HM-equilibrium at morality level θ for $\mathbf{A}$.

Then by continuity all accumulation points $\mathbf{x}(0) = \lim_{\theta \searrow 0} \mathbf{x}(\theta)$ yield classical Nash equilibrium for $\mathbf{A}$.

But for general $\mathbf{A}$, not all Nash equil. can be obtained this way (above counterexample).

Open issues:

For which $\mathbf{A}$ with $\lambda_{\max}[\mathbf{D}(\mathbf{A} + \mathbf{A}^\top)\mathbf{D}^\top] > 0$ do $\mathbf{x}(\theta)$ exist ?

Ensured for partnership games where $\mathbf{A} = \mathbf{A}^\top$, see later.

Further properties of $\mathbf{x}(0) = \lim_{\theta \searrow 0} \mathbf{x}(\theta)$: EGT/game dynamics ?

Partnership games and StQPs

If i plays against j, both share payoff: $a_{ij} = a_{ji}$, $\mathbf{A}^\top = \mathbf{A}$.

Observe for $\theta = 1$: $u_1(\mathbf{y}|\mathbf{x}) = \mathbf{y}^\top \mathbf{A}\mathbf{y}$ and β_1 independent of $\mathbf{x}$.

Local version for a neighbourhood $U \subseteq \Delta$ of $\mathbf{x}$:

$$\beta_1^U = \text{Argmax}\left\{\mathbf{y}^\top \mathbf{A}\mathbf{y} : \mathbf{y} \in U\right\} .$$

Have in symmetric case $\mathbf{A}^\top = \mathbf{A}$ for all $\theta \in [0,1]$:

$$\mathbf{x} \in \beta_1^\Delta \quad \Rightarrow \quad \mathbf{x} \in \beta_\theta(\mathbf{x}) \quad \Rightarrow \quad \mathbf{x} \in \beta_1^U .$$

Hence any **global** maximizer of $\mathbf{y}^\top \mathbf{A}\mathbf{y}$ over Δ gives HM-equil.,

and any HM-equilibrium gives a **local** maximizer of $\mathbf{y}^\top \mathbf{A}\mathbf{y}$.

A compromise between local and global optimality in StQPs !

A recent reference

[B./Schachinger/Weibull] Does moral play equilibrate ?
Economic Theory, DOI 10.1007/s00199-020-01246-4 (2020).