The Elicitation Problem

The Quest of Meaningful Forecast Comparison

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Evaluating and Comparing Forecasts

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  - business
  - government
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  - meteorology

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- Having m different sources of forecasts, one has the prediction-observation-sequences

$$\big(X_t^{(i)}, Y_t\big)_{t=1,\dots,N} \qquad i = 1, \dots, m.$$

  - $X_t^{(i)} \in \mathsf{A}$ (Action domain). For point forecasts, $\mathsf{A} = \mathbb{R}$ or $\mathsf{A} = \mathbb{R}^k$. For probabilisitic forecasts, $\mathsf{A} = \mathcal{F}$ a space of probability distributions.
  - $Y_t \in \mathsf{O}$ (Observation domain). Usually $\mathsf{O} = \mathbb{R}^d$.

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- Ranking of the forecasters in terms of realised scores:

$$\mathbf{S}_N^{(1)} = \frac{1}{N}\sum_{t=1}^{N} S\big(X_t^{(1)}, Y_t\big) \overset{?}{\lessgtr} \mathbf{S}_N^{(2)} = \frac{1}{N}\sum_{t=1}^{N} S\big(X_t^{(2)}, Y_t\big)$$

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- Ranking depends on the choice of the scoring function!
- One should disclose the specific choice of the scoring function to the forecasters *ex ante*.
- $\rightsquigarrow$ We need guidance in the choice of the scoring function.

Consistency and Elicitability

- Specification in terms of
  (i) an intrinsically meaningful scoring function (reflecting the actual economic costs); or
  (ii) a property (mean, median, variance, a risk measure) of the underlying distributions of the observation Y_t.
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- The scoring function should be "unbiased", incentivising truthful forecasts.

Elicitability

Definition 1 (Consistency)

A scoring function $S\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}$ is strictly $\mathcal{F}$-consistent for some functional $T\colon \mathcal{F} \to \mathsf{A}$ if

$$\mathbf{E}_F[S(T(F), Y)] < \mathbf{E}_F[S(x, Y)]$$

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Definition 2 (Elicitability)

A functional $T\colon \mathcal{F} \to \mathsf{A}$ is elicitable if there is a strictly $\mathcal{F}$-consistent scoring function $S\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}$ for T. Then

$$T(F) = \arg\min_{x \in \mathsf{A}} \mathbf{E}_F[S(x, Y)].$$

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- Economics; econometrics; business
- Meteorology
- Machine Learning
- Politics
- Sociology ($\rightsquigarrow$ 'Wisdom of the Crowds')

Regression

Classic situation: There is some parametric model $m \colon \Theta \times \mathbb{R} \to \mathbb{R}$ and we assume that there is some true parameter $\theta^* \in \Theta$ such that

$$Y = m_{\theta*}(X) + \varepsilon, \qquad \text{where} \quad \mathbf{E}[\varepsilon|X] = 0. \tag{1}$$

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Relying in the fact that

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However, instead of squared loss, we could use any strictly consistent scoring function for the mean functional.

Regression II

General situation: There is some parametric model $m\colon \Theta \times \mathbb{R}^\ell \to \mathbb{R}^k$ and we assume that there is some true parameter $\theta^* \in \Theta$ such that

$$T(\mathcal{L}(Y|X)) = m_{\theta*}(X).$$

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The Elicitation Problem

Fix some functional $T \colon \mathcal{F} \to \mathsf{A}$.

(i) Is T elicitable?

(ii) What is the class of (strictly) consistent scoring functions for T?

(iii) What is a particularly good choice of a scoring function?

(iv) What to do if T is not elicitable?

[1] As shown in Fissler and Ziegel (2016, Annals of Statistics)

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identity (probabilistic forecast)	$S(F, y) = -\log(f(y))$

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Brief flavour of (i), (ii), (iii), and (vi).

One-dimensional functionals

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Let $T\colon \mathcal{F} \to \mathsf{A}$ be an elicitable functional and $\mathcal{F}$ be convex. Then, for any $a \in \mathsf{A}$, the level sets

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Remarks:

- This shows that the variance or ES are generally not elicitable.

$$\mathrm{Var}(\delta_x) = \mathrm{Var}(\delta_y) = 0, \quad \mathrm{Var}\big(\lambda\delta_x + (1-\lambda)\delta_y\big) = \lambda(1-\lambda)(x-y)^2\,.$$

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- This argument is independent of the dimension of T.

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- Steinwart et al. (2014) showed that for $k = 1$ and under some regularity assumptions on T, cls are also sufficient for elicitability.
- This argument is independent of the dimension of T.
- For $k > 1$, it is an open question if cls are sufficient.

Vector-valued functionals

Lemma 4

If $T_1, \dots, T_k$ are elicitable, then the vector $(T_1, \dots, T_k)$ is elicitable.

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If $T\colon \mathcal{F} \to \mathsf{A}$ is elicitable then any bijection $g \circ T\colon \mathcal{F} \to \mathsf{A}'$ is elicitable.

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Question

Are there elicitable functionals that are not a bijection of functionals with elicitable components only?

Value at Risk vs. Expected Shortfall

Value at Risk (VaR) and Expected Shortfall (ES) are the most commonly used risk measures in practice.

Definition 6

Let Y be an asset, $Y \sim F$, $\alpha \in (0,1)$. Then

$$\mathrm{VaR}_\alpha(F) := \inf\{x \in \mathbb{R} : F(x) \geqslant \alpha\},$$

$$\mathrm{ES}_\alpha(F) := \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(F)\,\mathrm{d}\beta = \mathbf{E}_F\,[Y|\, Y \leqslant \mathrm{VaR}_\alpha(Y)].$$

- Profits amount to positive values of Y.
- We consider α close to zero (e.g. $\alpha = 0.01$, or $\alpha = 0.025$).
- Risky positions yield large negative values of VaR_α and ES_α.
  $\rightsquigarrow$ We work with utility functions instead of risk measures.

Value at Risk vs. Expected Shortfall (II)

Ongoing debate about the choice of a risk measure for regulatory purposes.

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Properties of VaR_α as a risk measure:

$(+)$ It is elicitable, if the distributions in $\mathcal{F}$ have unique α-quantiles.

$(-)$ It is generally not superadditive (hence, not coherent).

$(-)$ It fails to take the size of losses beyond the level α into account.

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Properties of ES_α as a risk measure:

(+) By definition, it considers the losses beyond the level α.

(+) It is superadditive (it is a coherent and comonotonically additive risk measure).

(−) It fails to have convex level sets and is consequently not elicitable; see Gneiting (2011).

Theorem 7 ((VaR, ES) – Fissler and Ziegel, AoS, 2016)

Let $\alpha \in (0,1)$*. Let* $\mathcal{F}$ *be a class of distribution functions on* $\mathbb{R}$ *with finite first moments. Let* $\mathsf{A}_0 = \{(x_1, x_2) \in \mathbb{R}^2 \colon x_1 \geqslant x_2\}$*, then any scoring function* $S \colon \mathsf{A}_0 \times \mathbb{R} \to \mathbb{R}$ *of the form*

$$\begin{aligned} S(x_1, x_2, y) = &\big(\mathbb{1}\{y \leqslant x_1\} - \alpha\big) g(x_1) - \mathbb{1}\{y \leqslant x_1\} g(y) + a(y) \\ &+ \phi'(x_2)\Big(x_2 + \big(\mathbb{1}\{y \leqslant x_1\} - \alpha\big)\frac{x_1}{\alpha} - \mathbb{1}\{y \leqslant x_1\}\frac{y}{\alpha}\Big) - \phi(x_2), \end{aligned} \tag{2}$$

is strictly $\mathcal{F}$*-consistent for* $T = (\mathrm{VaR}_\alpha, \mathrm{ES}_\alpha)$ *if*

- g *is increasing;*
- ϕ *is strictly increasing and strictly convex.*

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$\rightsquigarrow$ Comparative Backtests of Diebold-Mariano type are possible; see Fissler, Ziegel and Gneiting (2016; Risk).

Identification function (moment function) $V\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}^k$

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Theorem 8 (Osband's Principle; Fissler and Ziegel, AoS; 2016))

Let $T\colon \mathcal{F} \to \mathsf{A} \subseteq \mathbb{R}^k$ be a surjective, elicitable and identifiable functional with a strict $\mathcal{F}$-identification function $V\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}^k$.
Under some regularity assumptions, for any strictly $\mathcal{F}$-consistent scoring function $S\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}$ there exists a matrix-valued function $h\colon \mathrm{int}(\mathsf{A}) \to \mathbb{R}^{k\times k}$ such that

$$\nabla_x \mathbf{E}_F[S(x, Y)] = h(x)\, \mathbf{E}_F[V(x, Y)]$$

for all $x \in \mathrm{int}(\mathsf{A})$ and $F \in \mathcal{F}$.

Second-order Osband's Principle

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Second-order

Under some smoothness conditions, we can even exploit second order conditions: the Hessian

$$\nabla_x^2 \, \mathbf{E}_F\,[S(x, Y)] \in \mathbb{R}^{k\times k}$$

must be symmetric for all $x \in \mathsf{A}$ and for all $F \in \mathcal{F}$. Moreover, it must be positive semi-definite at $x = T(F)$.

$\rightsquigarrow$ This gives a lot of information about the matrix $h(x)$.

Osband's Principle: Examples for $k = 1$

Proposition 10 (Gneiting, 2011)

(a) *Under some regularity conditions, $S\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a strictly consistent scoring function for the mean if and only if*

$$S(x, y) = \phi(y) - \phi(x) + \phi'(x)(x - y) + a(y)$$

$$\partial_x \mathbf{E}_F[S(x, Y)] = \phi''(x)(x - \mathbf{E}_F[Y]),$$

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(b) *Under some regularity conditions, $S\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a strictly consistent scoring function for the α-quantile, $\alpha \in (0, 1)$ if and only if*

$$S(x, y) = (\mathbb{1}\{y \leqslant x\} - \alpha)\big(g(x) - g(y)\big) + a(y)$$

$$\partial_x \mathbf{E}_F [S(x, Y)] = g'(x)(F(x) - \alpha),$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing.

Relevance of Elicitability to Backtesting

Prediction-observation triples

$$(v_t, e_t, Y_t)_{t=1,\dots,N}$$

v_t: VaR_α prediction for time point t
e_t: ES_α prediction for time point t
Y_t: Realization at time point t

Traditional backtesting...

...aims at testing of the null hypothesis

H_0^C: "The risk measure estimates at hand are correct."

- Calculate some test statistic T_1 based on observations $(v_t, e_t, Y_t)_{t=1,\dots,N}$ such that we know the distribution of T_1 (approximately) under H_0^C.
- Backtesting decision: If we do not reject H_0^C, the risk measure estimates at hand are adequate.

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- Elicitability is not relevant.
- Does not respect increasing information sets.
- Does not give guidance for decision between methods.

Comparative backtesting

H_0^- : "The risk measure estimates at hand are *at least as good* as the ones from the standard procedure."

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- Internal model: $(v_t, e_t, Y_t)_{t=1,\dots,N} \rightsquigarrow \mathbf{S}_N = \frac{1}{N}\sum_{t=1}^N S(v_t, e_t, Y_t)$
- Standard model: $(v_t^*, e_t^*, Y_t)_{t=1,\dots,N} \rightsquigarrow \mathbf{S}_N^* = \frac{1}{N}\sum_{t=1}^N S(v_t^*, e_t^*, Y_t)$

(Asymptotically normal) test statistic:

$$T_2 = \frac{\mathbf{S}_N - \mathbf{S}_N^*}{\sigma_N},$$

where σ_N is a suitable estimate of the standard deviation.

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- Under H_0^-: Expectation of T_2 is $\leqslant 0$.
- Backtesting decision: If we do not reject H_0^-, the risk measure estimates at hand are acceptable (compared to the standard).

(Diebold and Mariano, 1995, Giacomini and White, 2006)

Some comments

- Elicitability is crucial.
- Allows for sensible comparison between methods.
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  "[...] the null hypothesis is never proved or established, but it is possibly disproved, in the course of experimentation." (Fisher, 1949)

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  "[...] the null hypothesis is never proved or established, but it is possibly disproved, in the course of experimentation." (Fisher, 1949)
- We suggest a reversed onus of proof:
  Banks are obliged to demonstrate the superiority of the internal model.
  (Similar to regulatory practice in the health sector)

Conservative comparative backtesting

H_0^+ : "The risk measure estimates at hand are *at most as good* as the ones from the standard procedure."

- Internal model: $\mathbf{S}_N = \frac{1}{N}\sum_{t=1}^{N} S(v_t, e_t, Y_t)$
- Standard model: $\mathbf{S}_N^* = \frac{1}{N}\sum_{t=1}^{N} S(v_t^*, e_t^*, Y_t)$

(Asymptotically normal) test statistic:

$$T_2 = \frac{\mathbf{S}_N - \mathbf{S}_N^*}{\sigma_N},$$

where σ_N is a suitable estimate of the standard deviation.

- Under H_0^+: Expectation of T_2 is $\geqslant 0$.
- Backtesting decision: If we reject H_0^+, the risk measure estimates at hand are acceptable (compared to the standard).

(Diebold and Mariano, 1995, Giacomini and White, 2006)

Three zone approaches

BIS three zone approach for VaR_α

- Traditional backtest: One-sided binomial test.
- Backtesting decision:

	Red	Yellow	Green
p-value	very small	moderately small	sufficiently big

- Generalisation of three zone approach for ES_α by Costanzino and Curran (2015).

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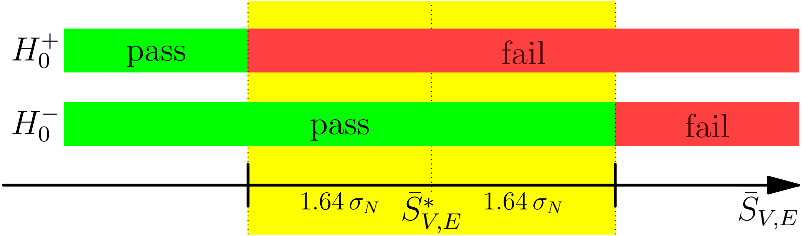
Three zone approach for comparative backtesting



A numerical illustration

$(\mu_t)_{t=1,\dots,N}$ iid standard normal,

$$Y_t \sim \mathcal{N}(\mu_t, 1), \quad \text{conditional on } \mu_t.$$

Scenario A	
(v_t, e_t)	$= (\mathrm{VaR}_\alpha(\mathcal{N}(\mu_t, 1)), \mathrm{ES}_\alpha(\mathcal{N}(\mu_t, 1)))$
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Scenario B	
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A numerical illustration – cont'd

$N = 250$; 10'000 simulations

Scenario A		**Green**	Yellow	Red
Traditional	$\mathrm{VaR}_{0.01}$	89.35	10.65	0.00
Traditional	$\mathrm{ES}_{0.025}$	93.62	6.36	0.02
Comparative	$\mathrm{VaR}_{0.01}$	88.23	11.77	0.00
Comparative	$(\mathrm{VaR}_{0.025}, \mathrm{ES}_{0.025})$	87.22	12.78	0.00

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Scenario B		Green	Yellow	**Red**
Traditional	$\mathrm{VaR}_{0.01}$	89.33	10.67	0.00
Traditional	$\mathrm{ES}_{0.025}$	93.80	6.18	0.02
Comparative	$\mathrm{VaR}_{0.01}$	0.00	11.77	88.23
Comparative	$(\mathrm{VaR}_{0.025}, \mathrm{ES}_{0.025})$	0.00	12.78	87.22

Summary

- Elicitability is not relevant for traditional backtesting.
- Elicitability is useful for model selection, estimation, forecast comparison and ranking.
- Comparative backtesting relies on elicitability, using H_0^+ it is conservative in nature and gives (more) incentive to improve predictions.

Set-valued functionals

- **Quantiles**

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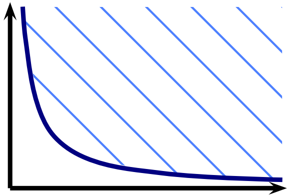

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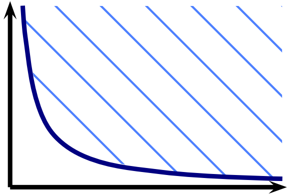

$$R(Y) = \{k \in \mathbb{R}^n \mid \rho(\Lambda(Y + k)) \leqslant 0\}.$$

- **Further spatial examples:** Area of flood, disease, landfall of a hurricane etc.

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- Example of the α-quantile

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$\mathsf{A} \subseteq 2^{\mathbb{R}}$: The forecasts are subsets of $\mathbb{R}$. These are points in the power set $\mathsf{A} \subseteq 2^{\mathbb{R}}$. There is a unique best action namely $x = q_\alpha(F)$.
$\rightsquigarrow$ The functional T is point-valued in some space $\mathsf{A} \subseteq 2^{\mathbb{R}}$, that is,

$$T\colon \mathcal{F} \to \mathsf{A}.$$

Two modes of elicitability

Definition 11

(a) A functional $T\colon \mathcal{F} \to 2^{\mathsf{A}}$ is selectively elicitable if there is a scoring function $S\colon \mathsf{A} \times \mathsf{O} \to \mathbb{R}$ such that

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- For single-valued functionals such as the mean, the notions of selective and exhaustive elicitability are equivalent.
- Forecasting / regression in the exhaustive sense is more ambitious than in the selective sense!

Mutual exclusivity

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- Confidence intervals.

Confidence Intervals

Reminder:

$$\mathcal{I}_\alpha(F) = \{(a, b) \in \mathbb{R}^2 \mid F(b) - F(a) = \alpha\}.$$

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- It is presumably exhaustively elicitable.
- It is not selectively elicitable!
- One needs to have additional properties for selective elicitability:
  - Specify the endpoints as quantiles.
  - Take a 'symmetric' interval.
  - Shortest confidence interval does not work.
  - Centring around the median or mean also fails.

Further Reading

- Good introduction to elicitability:

  T. Gneiting. Making and evaluating point forecasts.
  Journal of the American Statistical Association, 106:746–762, 2011

- Elicitability of vector-valued functionals and elicitability of (VaR, ES):

  T. Fissler and J. F. Ziegel. Higher order elicitability and Osband's principle.
  Annals of Statistics, 44:1680–1707, 2016

- Backtesting and elicitability: T. Fissler, J. F. Ziegel, and T. Gneiting. Expected shortfall is jointly elicitable with value-at-risk: implications for backtesting.
  Risk Magazine, pages 58–61, January 2016

  N. Nolde and J. F. Ziegel. Elicitability and backtesting: Perspectives for banking regulation.
  Annals of Applied Statistics, 11(4):1833–1874, 12 2017

Further Reading II

- Secondary quality criteria:

T. Fissler and J. F. Ziegel. Order-sensitivity and equivariance of scoring functions. *Preprint*, 2017

T. Fissler and J. F. Ziegel. Convex and quasi-convex scoring functions. *In preparation*, 2018

- Measures of Systemic Risk:

Z. Feinstein, B. Rudloff, and S. Weber. Measures of Systemic Risk. *SIAMJ. Financial Math.*, 8:672–708, 2017

T. Fissler, J. Hlavinová, and B. Rudloff. Elicitability and identifiability of systemic risk measures. *In preparation*, 2018

Thank you for your attention!

Looking forward to our discussion!