TU WIEN SWM ORCOS Operations Research and Control Systems

Regularity and approximations of generalized equations; applications in optimal control

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Wirtschafts Universität, Wien

Vienna, Nov. 25, 2017

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Plan of the talk

1. “Coercive” problems.
2. “Affine” problems.

Generalized equations

$$0 \in G(x),$$

where $G : X \rightrightarrows Y$, X, Y – metric (Banach) spaces.

Examples:

1. For $X = I\!R^n$, $K \subset X$ – closed, $f : X \to I\!R$ – Fréchet-differentiable

$$\min_{x \in K} f(x) \quad \longrightarrow \quad 0 \in \nabla f(x) + N_K(x).$$

2. Robinson (1980): $0 \in f(x) + F(x)$, with $F(x)$ – set-valued mapping.

3. Differential variational inequalities (e.g. Pang and Steward, 2008):

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t)), \\ 0 &\in h(x(t), u(t)) + N_K(u(t)), \\ 0 &= \Gamma(x(0), x(T)). \end{aligned}$$

$x : [0, T] \to I\!R^n$, $u[0, T] \to I\!R^m$.

$$\text{minimize} \int_0^T l(y(t), u(t))\,\mathrm{d}t$$

$$\dot{y}(t) = g(y(t), u(t)), \quad y(0) = y_0, \quad u(t) \in U \quad t \in [0, T].$$

Hamiltonian: $H(y, p, u) = l(y, u) + p^T g(y, u)$

Optimality conditions:

$$\begin{cases} \dot{y}(t) & = \partial_p H(y(t), p(t), u(t)), & y(0) = y_0, \\ \dot{p}(t) & = -\partial_y H(y(t), p(t), u(t)), & p(T) = 0, \\ 0 & \in \partial_u H(y(t), p(t), u(t)) + N_U(u(t)), & \end{cases}$$

Usual spaces: $u \in L^\infty([0, T]; \mathbb{R}^m)$, $x = (y, p) \in {W_0}^{1,\infty}([0, T]; \mathbb{R}^{2n})$.

Reformulation: Differential Generalized Equation (DGE):

$$\begin{aligned} \dot{x} &= g(x, u), \\ 0 &\in f(x, u) + F(u), \end{aligned}$$

Differential Generalized Equation (DGE):

$$u \in L^\infty([0,T];\mathbb{R}^m), \qquad x = (y,p) \in {W_0}^{1,\infty}([0,T];\mathbb{R}^{2n}).$$

$$\begin{aligned} \dot{x} &= g(x,u), \\ 0 &\in f(x,u) + F(u), \end{aligned}$$

where

$$f(x,u) = \partial_u H(y,p,u), \qquad F(u) = N_{\mathcal{U}}(u),$$

with $\mathcal{U} = \{u \in L^\infty : \ u(t) \in U\}$, and for $u \in L^\infty$

$$N_{\mathcal{U}}(u) = \{w \in L^\infty \mid w(t) \in N_U(u(t)) \text{ for a.e. } t \in [0,T]\}.$$

$N_{\mathcal{U}}(u)$ is not the normal cone to $\mathcal{U}$!

$$f(x,u)(t) = f(x(t),u(t)), \quad F(u)(t) = F(u(t)).$$

A concept of (Lipschitz) regularity

$G : X \rightrightarrows Y$, X, Y – metric spaces.

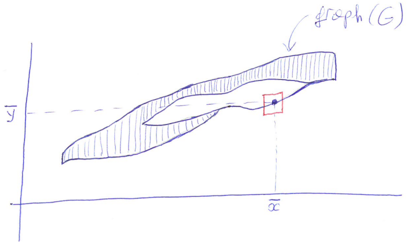
Definition. G is *strongly metrically regular* (SMR) at $\bar{x}$ for $\bar{y} \in G(\bar{x})$ if there are balls $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$, $a, b > 0$ such that the mapping

$$\mathbb{B}_b(\bar{y}) \ni y \to G^{-1}(y) \cap \mathbb{B}_a(\bar{x})$$

is single-valued and Lipschitz continuous (with Lipschitz constant κ).

Here $G^{-1}(y) := \{x : \ y \in G(x)\}$.

SMR means that G^{-1} has a *Lipschitz localization:*



The weaker property of "metric regularity" will not be discussed here.

A Ljusternik-Graves-type theorem (e.g. Dontchev and Rockafellar - 2013)

Theorem

Let a, b, and κ be positive scalars such that G is strongly metrically regular at $\bar{x}$ for $\bar{y}$ with neighborhoods $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$ and constant κ. Let $\mu > 0$ be such that $\kappa\mu < 1$ and let $\kappa' > \kappa/(1-\kappa\mu)$. Then for every positive α and β such that

$$\alpha \leq a/2, \quad 2\mu\alpha + 2\beta \leq b \quad \text{and} \quad 2\kappa'\beta \leq \alpha$$

and for every function $\gamma : X \to Y$ satisfying

$$\|\gamma(\bar{x})\| \leq \beta \;\; \text{and} \;\; \|\gamma(x) - \gamma(x')\| \leq \mu\|x - x'\| \quad \forall\, x, x' \in \mathbb{B}_{2\alpha}(\bar{x}),$$

the mapping $y \mapsto (\gamma + G)^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is a Lipschitz continuous function on $\mathbb{B}_\beta(\bar{y})$ with Lipschitz constant κ'. (Hence $\gamma + G$ is SMR at $\bar{x}$ for $\bar{y}$.)

Qualitative consequences in the case of DGE

R. Cibulka, A. Dontchev, M. Krastanov, V.V., SIAM J. Contr. Opt., (2017(8))

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a solution of the DGE

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t)), \\ 0 &\in f(x(t), u) + F(u(t)). \end{aligned}$$

Assumption (*): $\forall\, (t, u) \in \operatorname{cl} \operatorname{gr} \bar{u}$ the mapping

$$I\!R^m \ni v \mapsto \mathcal{W}_{t,u}(v) := f(\bar{x}(t), u) + \partial_u f(\bar{x}(t), u)(v - u) + F(v)$$

is SMR at u for 0.

Theorem

$\exists a, b, \kappa > 0$: $\forall (t, u) \in \operatorname{cl} \operatorname{gr} \bar{u}$ the mapping $\mathcal{W}_{t,u}(\cdot)$ is SMR at u for 0 with parameters a, b, κ. That is, the mapping $I\!B_b(0) \ni z \mapsto \mathcal{W}^{-1}_{t,u}(z) \cap I\!B_a(u)$ is single-valued and Lipschitz with constant κ.

Theorem

If Assumption (*) is fulfilled then the mapping

$$(x, u) \mapsto \begin{pmatrix} \dot{x} - g(x, u) \\ f(x, u) \end{pmatrix} + \begin{pmatrix} 0 \\ F(u) \end{pmatrix}$$

is SMR at $(\hat{x}, \hat{u})$ *for* 0.

Recall: $u \in L^\infty([0, T]; \mathbb{R}^m), \quad x = (y, p) \in {W_0}^{1,\infty}([0, T]; \mathbb{R}^{2n})$

Other consequences:

Conditions for Lipschitz continuity of $\bar{u}$ ...

Convergence of discrete approximations and “path-following” methods ...
(more detailed analysis in

A. Dontchev, M. Krastanov, R.T. Rockafellar, V.V., SIAM J. Contr. Optim., 2013.)

Extensions for non-differentiable Lipschitz functions f (in terms of the strict prederivative of f): R. Cibulka, A. Dontchev, V.V., SIAM J. Contr. Optim., 2016.

Newton-type methods

R. Cibulka, A. Dontchev, J. Preininger, T. Roubdal, V.V., Journal of Convex Analysis (2018)
X and Y – Banach spaces. Consider the equation $f(x) = 0$, $f : X \to Y$ with a Fréchet-differentiable f.

Newton method: Generate $\{x_k\}$ such that

$$f(x_k) + \partial f(x_k)(x_{k+1} - x_k) = 0, \; x_0 \text{ – given.}$$

Assumption for (quadratic) convergence: a solution $\bar{x}$ exists, $\partial f(\bar{x})$ is invertible, and $\|x_0 - \hat{x}\|$ is small enough.

Kantorovich version: two differences:
(i) the invertibility assumption is posed for $\partial f(x_0)$, some "checkable" assumptions are posed. Then: a solution $\bar{x}$ exists and the convergence is quadratic.

(ii) One can modify the iterations as

$$f(x_k) + \partial f(x_0)(x_{k+1} - x_k) = 0, \; x_0 \text{ – given.}$$

Then the convergence is linear: $\|x_k - \hat{x}\| \leq \alpha^k \|x_0 - \hat{x}\|$, $\alpha \in (0, 1)$.

Further extensions:

– Bartle (1955): $f(x_k) + \partial f(z_k)(x_{k+1} - x_k) = 0$, x_0 – given. Any z_k – ...

– Qi and Sun (1993): f can be only Lipschitz; take $A_k \in \hat{\partial} f(x_k)$ – the Clarke generalized Jacobian ...

Our problem: $0 \in f(x) + F(x)$, where $f : X \to Y$, $F : X \rightrightarrows Y$, X, Y – Banach spaces.

Newton-Kantorovich iterations:

$$f(x_k) + A_k(x_{k+1} - x_k) + F(x_{k+1}) \ni 0,$$

where $A_k = A_k(x_0, \ldots, x_k) \in \mathcal{L}(X, Y)$, together with some $y_0 \in f(x_0) + F(x_0)$ have the following properties:

(i) for very k the mapping

$$x \mapsto f(x_0) + A_k(x - x_0) + F(x)$$

is SMR at x_0 for y_0 with a constant κ and neighborhoods $\mathbb{B}_a(x_0)$, $\mathbb{B}_b(y_0)$;

(ii) $\|f(x) - f(x_k) - A_k(x - x_k)\| \leq \omega(\|x - x_k\|) \, \|x - x_k\| \quad \forall \, x \in \mathbb{B}_a(x_0)$, where $\omega : [0, a] \to [0, \delta]$, $\delta > 0$.

Theorem

Assume that $\kappa\delta < 1$ *and* $\|y_0\| < (1-\kappa\delta)\min\{\frac{a}{\kappa}, b\}$. *Then the Newton-Kantorovich method generates a unique sequence in* $\mathbb{B}_a(x_0)$, *and it* *linearly* *converges to a solution* $\bar{x}$:

$$\|x_k - \bar{x}\| < (\kappa\delta)^k a. \tag{1}$$

If $\lim_{\xi\to 0}\omega(\xi) = 0$, *then the sequence* $\{x_k\}$ *is* *superlinearly* *convergent: there exist sequences of positive numbers* $\{\varepsilon_k\}$ *and* $\{\eta_k\}$ *such that* $\|x_k - \bar{x}\| \le \varepsilon_k$ *and* $\varepsilon_{k+1} \le \eta_k\varepsilon_k$ *for all sufficiently large* k, *and* $\eta_k \to 0$.

If there exists a constant $L > 0$ *such that* $\omega(\xi) \le \min\{\delta, L\xi\}$ *for each* $\xi \in [0, a]$, *then the convergence of* $\{x_k\}$ *is* *quadratic*: *there exists a sequence of positive numbers* $\{\varepsilon_k\}$ *such that* $\|x_k - \bar{x}\| \le \varepsilon_k$ *and* $\varepsilon_{k+1} \le \frac{\alpha L}{\delta}\varepsilon_k^2$ *for all sufficiently large* k.

Theorem

Assume that $\kappa\delta < 1$ *and* $\|y_0\| < (1-\kappa\delta)\min\{\frac{a}{\kappa}, b\}$*.*
Then the Newton-Kantorovich method generates a unique sequence in $\mathbb{B}_a(x_0)$*, and it* *linearly* *converges to a solution* $\bar{x}$*:*

$$\|x_k - \bar{x}\| < (\kappa\delta)^k a. \tag{1}$$

If $\lim_{\xi\to 0}\omega(\xi) = 0$*, then the sequence* $\{x_k\}$ *is* *superlinearly* *convergent: there exist sequences of positive numbers* $\{\varepsilon_k\}$ *and* $\{\eta_k\}$ *such that* $\|x_k - \bar{x}\| \le \varepsilon_k$ *and* $\varepsilon_{k+1} \le \eta_k\varepsilon_k$ *for all sufficiently large* k*, and* $\eta_k \to 0$*.*

If there exists a constant $L > 0$ *such that* $\omega(\xi) \le \min\{\delta, L\xi\}$ *for each* $\xi \in [0, a]$*, then the convergence of* $\{x_k\}$ *is* *quadratic**: there exists a sequence of positive numbers* $\{\varepsilon_k\}$ *such that* $\|x_k - \bar{x}\| \le \varepsilon_k$ *and* $\varepsilon_{k+1} \le \frac{\alpha L}{\delta}\varepsilon_k^2$ *for all sufficiently large* k*.*

Special cases: $A_k = \partial f(x_0)$ – Kantorovich
$A_k = \partial f(x_k)$ – Newton
Other choices of A_k – extended Bartle.

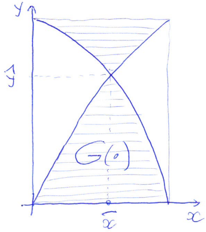
Strong Metric Sub-Regularity (SMs-R)

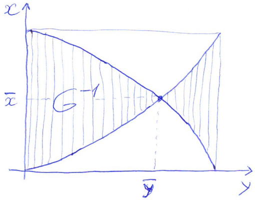
(Cibulka, Dontchev, Kruger (2017(8)))

$G : X \rightrightarrows Y$, X, Y – metric spaces.

Definition. G is *strongly metrically sub-regular* (SMs-R) at $\bar{x}$ for $\bar{y} \in G(\bar{x})$ if there are $\kappa > 0$ and balls $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$, $a, b > 0$, such that

$$G^{-1}(y) \cap \mathbb{B}_a(\bar{x}) \subset \mathbb{B}_{\kappa\, \mathrm{dist}(y,\bar{y})}(\bar{x}) \quad \forall\, y \in \mathbb{B}_b(\bar{y}).$$





This property is enough for many contexts: error analysis of approximations; Newton method.

Newton method for $0 \in f(x) + F(x)$, where $f : X \to Y$, $F : X \rightrightarrows Y$, X, Y – Banach spaces, f has Lipschith Fréchet derivative.

Newton iterations:

$$f(x_k) + \partial f(x_k)(x_{k+1} - x_k) + F(x_{k+1}) \ni 0.$$

Theorem

Assume that linearized mapping $x \to f(\bar{x}) + \partial f(\bar{x})(x - \bar{x}) + F(x)$ is SMs-R at $\bar{x}$ for 0. Then there exists a neighborhood O of $\bar{x}$ such that if a sequence $\{x_k\}$ generated by the Newton method has a tail in O, then x_k is quadratically convergent to $\bar{x}$.

Existence of such a Newton sequence is not granted!

IMPORTANT: When the general results involving SMR or SMs-R are used for

$$\text{minimize} \int_0^T l(y(t), u(t))\, \mathrm{d}$$

$$\dot{y}(t) = g(y(t), u(t)), \quad y(0) = y_0, \quad u(t) \in U \quad t \in [0, T],$$

hence for the optimality conditions

$$\begin{cases} \dot{y}(t) & = \partial_p H(y(t), p(t), u(t)), & y(0) = y_0, \\ \dot{p}(t) & = -\partial_y H(y(t), p(t), u(t)), & p(T) = 0, \\ 0 & \in \partial_u H(y(t), p(t), u(t)) + N_U(u(t)), & \end{cases}$$

the space specifications are always $u \in L^\infty([0, T]; \mathbb{R}^m)$,

$x = (y, p) \in {W_0}^{1,\infty}([0, T]; \mathbb{R}^{2n})$.

The conditions for SMR and SMs-R involve coercivity!

This spaces are not appropriate for problems with discontinuous optimal controls.

Affine problems

$$\min \left\{ \int_0^T [g_0(x(t)) + g(x(t))u(t))] \, \mathrm{d}t \, + \, \Phi(x(T)) \right\}.$$

$$\dot{x} = f_0(x) + u\, f(x), \quad x(0) \; \text{– given}, \qquad u(t) \in U = [0,1].$$

Optimality system:

$$\begin{aligned} 0 &= \dot{x} - f_0(x) + u\, f(x), \\ 0 &= \dot{p} + p^T \partial_x (f_0(x) + u\, f(x)) + \partial_x (g_0(x) + u\, g(x)), \\ 0 &\in g(x(t)) + p(t)^\top f(x(t)) + N_U(u(t)), \\ 0 &= p(T) - \partial \Phi(x(T)). \end{aligned}$$

What are the appropriate spaces?
Under what conditions we have SMR or SMs-R?
Can we apply the Newton method?

Consider the linearized problem:

$$\begin{array}{ll} \text{minimize} & J(x,u) \\ \text{subject to} & \dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t), \quad x(0) = x_0, \\ & u(t) \in U := [-1,1], \end{array}$$

where

$$J(x,u) := \Phi(x(T)) + \int_0^T \left(\frac{1}{2} x(t)^\top W(t)x(t) + x(t)^\top S(t)u(t) \right) \mathrm{d}t.$$

Optimality system:

$$0 \in G(x,p,u) := \begin{pmatrix} \dot{x} - Ax - Bu - d \\ \dot{p} + A^\top p + Wx + Su \\ B^\top p + S^\top x + N_{\mathcal{U}}(u) \\ p(T) - \partial\Phi(x(T)) \end{pmatrix},$$

$$N_{\mathcal{U}}(u) = \{ w \in L^\infty \mid w(t) \in N_U(u(t)), \;\; t \in [0,T] \}.$$

Spaces:

$$\mathcal{X} := W^{1,1}_{x_0} \times W^{1,1} \times L^1, \qquad \mathcal{Y} := L^1 \times L^1 \times L^\infty \times \mathbf{R}^n$$

Sufficient conditions for SMs-R (J. Preininger, T. Scarinci, V.V., 2017(?))

(A1) Continuous differentiability of the data; $W(t)$ symmetric; Φ – differentiable with Lipschitz derivative.

(A2) The functional $J(x, u)$ is convex on the set of admissible control-trajectory pairs.

(A3) For a given reference solution $(\hat{x}, \hat{p}, \hat{u})$ there are numbers $\alpha, \tau > 0$ such that at every zero s of the function

$$H_u(\hat{x}(t), \hat{p}(t), \hat{u}(t)) = \hat{\sigma}(t) = B(t)^\top \hat{p}(t) + S(t)^\top \hat{x}(t)$$

it holds that

$$|\hat{\sigma}(t)| \geq \alpha |t - s| \quad \forall\, t \in [s - \tau, s + \tau] \cap [0, T].$$

Theorem

$\exists c > 0$ such that $\forall y \in \mathcal{Y}$ there exists a solution $(x, p, u) \in \mathcal{X}$ of $y \in G(x, p, u)$ and for every such (x, p, u)

$$\|x - \hat{x}\|_{1,1} + \|p - \hat{p}\|_{1,1} + \|u - \hat{u}\|_1 \leq c\|y\|.$$

(W. Alt, U.Felgenhauer, M. Seidenschwanz, 2016-17)

A (surprising) consequence: (J. Preininger, T. Scarinci, V.V., 2017)

Theorem

Under conditions a bit stronger than (A1)–(A3) for the linearized problem at the solution point $(\hat{x}, \hat{p}, \hat{u})$*, the sequence of any Newton iterates starting from any initial point* (x_0, p_0, u_0) *sufficiently close to* $(\hat{x}, \hat{p}, \hat{u})$ *converges quadratically to* $(\hat{x}, \hat{p}, \hat{u})$*.*

A similar theorem under a number of more restrictive conditions - in [U.Felgenhauer (2017)].

A (surprising) consequence: (J. Preininger, T. Scarinci, V.V., 2017)

Theorem

Under conditions a bit stronger than (A1)–(A3) for the linearized problem at the solution point $(\hat{x}, \hat{p}, \hat{u})$, the sequence of any Newton iterates starting from any initial point (x_0, p_0, u_0) sufficiently close to $(\hat{x}, \hat{p}, \hat{u})$ converges quadratically to $(\hat{x}, \hat{p}, \hat{u})$.

A similar theorem under a number of more restrictive conditions - in [U.Felgenhauer (2017)].

A numerical problem: how to solve the linear-quadratic problem

$$\text{minimize } \Phi(x(T)) + \int_0^T \left(\frac{1}{2} x(t)^\top W(t) x(t) + x(t)^\top S(t) u(t) \right) \mathrm{d}t,$$

$$\text{subject to} \quad \begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)u(t) + d(t), \quad x(0) = x_0, \\ u(t) \in U := [-1, 1], \quad \text{or } U := \{-1, 1\} \end{array}$$

A new discretization scheme

V.V., 1989

A. Pietrus, T. Scarinci, V.V. (SIAM J. CO, 2017(8))

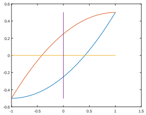
T. Scarinci and V.V. (Comput. Optim. and Appl., 2017)

Basic idea: $\{t_i\}_{i=0}^N$ a mesh with step h on $[\tau, T]$. Consider $w_i = (u_i, v_i)$,

$$u_i = \frac{1}{h}\int_{t_i}^{t_{i+1}} u(t)\,\mathrm{d}t, \qquad v_i = \frac{1}{h^2}\int_{t_i}^{t_{i+1}} (t - t_i)u(t)\,\mathrm{d}t$$

as discrete controls associated with $u(t) \in \{0, 1\}$. When $u(t) \in \{0, 1\}$ or $u(t) \in [0, 1]$, it holds that for $w_i = (u_i, v_i)$

$$w_i \in Z := \text{Aumann-}\int_0^1 \begin{pmatrix} 1 \\ s \end{pmatrix} [-1, 1] ds.$$



Explicit representation:

$$Z = \{(\alpha, \beta) : \alpha \in [-1, 1],\ \beta \in [\varphi_1(\alpha), \varphi_2(\alpha)]\},$$

where $\varphi_1(\alpha) := \frac{1}{4}\left(-1 + 2\alpha + \alpha^2\right)$ and $\varphi_2(\alpha) := \frac{1}{4}\left(1 + 2\alpha - \alpha^2\right)$.

Conversely, there is a mapping $\Phi^h : Z^N \to \{0,1\}$ such that $\forall w := (w_0, \dots, w_{N-1}) = ((u_0, v_0), \dots, (u_{N-1}, v_{N-1})) \in Z^N$

$$u_i = \frac{1}{h} \int_{t_i}^{t_{i+1}} \Phi^h(w)(t)\,\mathrm{d}t, \qquad v_i = \frac{1}{h^2} \int_{t_i}^{t_{i+1}} (t - t_i)\Phi^h(w)(t)\,\mathrm{d}t.$$

$\Phi^h(w)(t) \in \{0,1\}$ has 0, 1 or at most 2 jumps in every interval $[t_i, t_{i+1}]$.

Then we use the 2nd order Volterra-Fliess series to approximate the dynamics and the objective functional.

Under (A1)–(A3), for any solution w^h of the discrete problem it holds that $\|\Phi^h(w^h) - \hat{u}\|_1 \leq ch^2$.

Second order accuracy cannot be provided by any Runge-Kutta scheme! Schemes with second order accuracy (and still “nice” discretized problem) were not known so far.

Next numerical problem: How to solve the resulting mathematical programming problem?

The discretized problem has the general form

$$\min_{w \in K} f(w),$$

where f is a linear-quadratic function (not necessarily convex) and K is strongly convex.

The paper [V.V., P. Vuong, 2018(?)] presents linear convergence results for the GPM and the CGM for such problems in Hilbert spaces.

More specialized methods taking into account the structure of the constraints:

$$K = Z \times Z \ldots \times Z$$

and of the objective function – future work.

Strong Metric Regularity of affine problems

$$\min \left\{ \int_0^T [g_0(x(t)) + g(x(t))u(t)] \,\mathrm{d}t \;+\; \Phi(x(T)) \right\}.$$

$$\dot{x} = f_0(x) + u\, f(x), \quad x(0) \text{ – given}, \quad u(t) \in U = [0,1].$$

Linearized optimality system:

$$0 \in G(x,p,u) := \begin{pmatrix} \dot{x} - Ax - Bu - d \\ \dot{p} + A^\top p + Wx + Su \\ B^\top p + S^\top x + N_{\mathcal{U}}(u) \\ p(T) - \partial\Phi(x(T)) \end{pmatrix},$$

$$N_{\mathcal{U}}(u) = \{ w \in L^\infty \mid w(t) \in N_U(u(t)), \;\; t \in [0,T] \}.$$

SMR in the spaces

$$\mathcal{X} := W^{1,1}_{x_0} \times W^{1,1} \times L^1, \qquad \mathcal{Y} := L^1 \times L^1 \times L^\infty \times \mathbf{R}^n$$

"never" holds!!!

Strong bi-Metric Regularity of affine problems (Sbi-MR

General: $G : X \rightrightarrows Y$, $\quad X, Y$ – metric spaces with metric d_X and d_Y.

Definition. G is *strongly metrically regular* (SMR) at $\bar{x}$ for $\bar{y} \in G(\bar{x})$ if there are balls $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$, $a, b > 0$ such that the mapping

$$\mathbb{B}_b(\bar{y}) \ni y \to G^{-1}(y) \cap \mathbb{B}_a(\bar{x})$$

is single-valued and Lipschitz continuous (with Lipschitz constant κ):

$$d_X(G^{-1}(y) \cap \mathbb{B}_a(\bar{x}), G^{-1}(y') \cap \mathbb{B}_a(\bar{x})) \leq \kappa d_Y(y, y') \quad \forall y, y' \in \mathbb{B}_b(\bar{y}).$$

The bi-metric modification:

M. Quincampoix and V.V., SIAM J. CO (2013)

J. Preininger, T. Scarinci, and V.V., (2018)(??)

Explanation for the two metrics

Consider $\dim(u) = 1$, $U = [-1, 1]$, $\hat{\sigma}(t) = -\frac{1}{2} + t$, $t \in [0, 1]$.

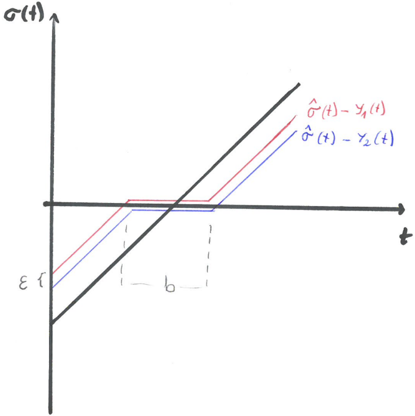
The solution of $y(t) \in \hat{\sigma}(t) + N_U(u(t))$ is

$u(t) = u[y](t) :=$ sgn$(\hat{\sigma}(t) - y(t))$ whenever $\hat{\sigma}(t) - y(t) \neq 0$,
$\hat{u}(t) = u[0](t)$.

When do we have (for some κ and $b > 0$)

$$\|u[y_1] - u[y_2]\|_1 \leq \kappa d_Y(y_1, y_2) \qquad \forall y_1,\ y_2 :\ d_Y(y_i, 0) \leq b.$$

What is the metric space $Y \subset L^\infty$?



Here

$$\|u[y_1] - u[y_2]\|_1 \approx 2b \gg \kappa\varepsilon = \kappa\|y_1 - y_2\|_\infty,$$

thus with $Y = L^\infty$ the mapping $u \to \hat\sigma + N_U(u)$ is not SMR at $\hat u$ for 0!

However, for $y_1,\, y_2 \in Y = W^{1,\infty}$ we have

$$\|u[y_1] - u[y_2]\|_1 \leq \frac{8}{3}\|y_1 - y_2\|_{1,\infty}$$

whenever $\|y_i\|_{1,\infty} \leq b := \frac{1}{4}$.

However, for $y_1, y_2 \in Y = W^{1,\infty}$ we have

$$\|u[y_1] - u[y_2]\|_1 \le \frac{8}{3}\|y_1 - y_2\|_{1,\infty}$$

whenever $\|y_i\|_{1,\infty} \le b := \frac{1}{4}$.

Even more, for $y_1, y_2 \in W^{1,\infty}$

$$\|u[y_1] - u[y_2]\|_1 \le \frac{8}{3}\|y_1 - y_2\|_{\infty}.$$

Thus the Lipschitz property is with respect to the L^∞-norm for y, but the disturbances y should be close to the reference point $\hat{y} = 0$ in the larger norm of $W^{1,\infty}$.

This explains the necessity of using two norms for the disturbances.

(X, d_X), (Y, d_Y) and $(\widetilde{Y}, \widetilde{d}_Y)$ – metric spaces, with $\widetilde{Y} \subset Y$ and $d_Y \leq \widetilde{d}_Y$ on $\widetilde{Y}$.

Definition

The map $G : X \rightrightarrows Y$ is strongly bi-metrically regular (relative to $\widetilde{Y} \subset Y$) at $\bar{x} \in X$ for $\bar{y} \in \widetilde{Y}$ with constants $\varsigma \geq 0$, $a > 0$ and $b > 0$ if $(\bar{x}, \bar{y}) \in \mathrm{graph}(\Phi)$ and the following properties are fulfilled:

1. the mapping $B_{\widetilde{Y}}(\bar{y}; b) \ni y \mapsto G^{-1}(y) \cap B_X(\bar{x}; a)$ is single-valued
2. for all $y, y' \in B_{\widetilde{Y}}(\bar{y}; b)$,

$$d_X(G^{-1}(y) \cap B_X(\bar{x}; a), G^{-1}(y') \cap B_X(\bar{x}; a)) \leq \varsigma d_Y(y, y').$$

Lyusternik-Graves-type theorem (J. Preininger, T. Scarinci, V.V., 2017(?))

Theorem

Let the metric space X be complete, let Y be a subset of a linear space and let both metrices d_Y and $\widetilde{d}_Y$ in Y and $\widetilde{Y} \subset Y$, respectively, be shift-invariant. Let $G : X \rightrightarrows Y$ be strongly bi-metrically regular at $\bar{x}$ for $\bar{y}$ with constants κ, a, b. Let $\mu > 0$ and κ' be such that $\kappa\mu < 1$ and $\kappa' \geq \kappa/(1 - \kappa\mu)$. Then for every positive a', b', and γ such that

$$a' \leq a, \quad b' + \gamma \leq b, \quad \kappa b' \leq (1 - \kappa\mu)a',$$

and for every function $\varphi : X \to \widetilde{Y}$ such that

$$d_Y(g(\bar{x}), 0) \leq b', \qquad \widetilde{d}_Y(g(x), 0) \leq \gamma \qquad \forall x \in B_X(\bar{x}, a'),$$

and

$$d_Y(g(x), g(x')) \leq \mu\, d_X(x, x') \quad \forall x, x' \in B_X(\bar{x}, a'),$$

the mapping $B_{\widetilde{Y}}(\bar{y} + g(\bar{x}); b') \ni y \mapsto (g + G)^{-1}(y) \cap B_X(\bar{x}, a')$ is single-valued and Lipschitz continuous with constant κ' with respect to the metric d_Y. This implies strong bi-metric regularity of $g + G$ ...

X, Y, $\tilde{Y}$ – convex subsets of linear normed spaces, X – complete.

(to be) Theorem. (M. Quincampoix, T. Scarinci, V.V., 2018(?))
Let $f : X \to \tilde{Y}$ be Fréchet differentiable at $\bar{x}$ in the norm of $\tilde{Y}$, and be differentiable in a neighborhood of $\bar{x}$ in the norm of Y, with uniformly continuous (in Y) derivative. Then the mapping $G = f + F$ is strongly bi-metrically regular at $\bar{x}$ for $\bar{y}$ if and only if the mapping $x \mapsto f(\bar{x}) + \partial f(\bar{x})(x - \bar{x}) + F(x)$ is such.

Consequence: Sbi-MR of the affine differential variational inequality is equivalent to that of the linearized one. (A1)–(A3) are sufficient for that.

Conclusions

1. SMR and SMs-R are key concepts of Lipschitz stability: they are themselves stable, enable Newton-Kantorovich methods, analysis of approximations, etc.
2. for DGEs the concepts have been developed and applied in the "coercive" case
3. for "affine" DGEs – recent developments: Newton, new discretization, new problems in mathematical programming
4. A lot more work needed: presence of singular arcs, extensions of the "new discretization", ...

Thank You!