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METRICALLY REGULAR DIFFERENTIAL GENERALIZED EQUATIONS*

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Abstract. In this paper we consider a control system coupled with a generalized equation, which we call Differential Generalized Equation (DGE). This model covers a large territory in control and optimization, such as differential variational inequalities, control systems with constraints, as well as necessary optimality conditions in optimal control. We study metric regularity and strong metric regularity of mappings associated with DGE by focusing in particular on the interplay between the pointwise versions of these properties and their infinite-dimensional counterparts. Metric regularity of a control system subject to inequality state-control constraints is characterized. A sufficient condition for local controllability of a nonlinear system is obtained via metric regularity. Sufficient conditions for strong metric regularity in function spaces are presented in terms of uniform pointwise strong metric regularity. A characterization of the Lipschitz continuity of the control part of the solution mapping as a function of time is established. Finally, a path-following procedure for a discretized DGE is proposed for which an error estimate is derived.

Key Words. variational inequality, control system, optimal control, metric regularity, strong metric regularity, discrete approximation, path-following.

AMS Subject Classification (2010): 49K40, 49J40, 49J53, 49m25, 90C31.

1. Introduction. In the paper we consider the following problem: given a positive real T, find a Lipschitz continuous function x acting from $[0,T]$ to $\mathbb{R}^m$ and a measurable and essentially bounded function u acting from $[0,T]$ to $\mathbb{R}^n$ such that

$$\dot{x}(t) = g(x(t), u(t)), \tag{1}$$

$$f(x(t), x(0), x(T), u(t)) + F(u(t)) \ni 0 \tag{2}$$

for almost every (a.e.) $t \in [0,T]$, where $\dot{x}$ is the derivative of x with respect to t, $g : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^d$ are functions, and $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^d$ is a set-valued mapping. We assume throughout that the functions g and f are twice continuously differentiable everywhere (this assumption could be relaxed in most of the statements in the paper but we keep it as a standing assumption for simplicity). In analogy with the terminology used in control theory, we call the variable $x(t)$ *state* and the variable $u(t)$ *control value*. The independent variable t is thought of as *time* which varies in a finite time interval $[0,T]$ for a fixed $T > 0$. A function $t \mapsto u(t)$ is said to be *control* and a solution $t \mapsto x(t)$ of (1) for some control u is said to be *state trajectory*. At this point we will not make any assumptions for the mapping F. A complete description of the problem should also include the function spaces where the functions x and u reside; we will choose such spaces a bit later.

The model (1)–(2) can be extended to a greater generality by, e.g., adding a set-valued mapping to the right side of (1), making F depend on $x(t)$ etc., but even in the present form it already covers a broad spectrum of problems. When $f = \begin{pmatrix} -x(0) \\ h(x,u) \end{pmatrix}$ and $F \equiv \begin{pmatrix} x_0 \\ -W \end{pmatrix}$, where $x_0 \in \mathbb{R}^m$ is a fixed initial point

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and W is a closed set in $\mathbb{R}^{d-m}$, (1)–(2) describes a control system with pointwise state-control constraints:

$$\left\{ \begin{array}{l} \dot{x}(t) = g(x(t), u(t)), \quad x(0) = x_0, \\ h(x(t), u(t)) \in W \quad \text{ for a.e. } t \in [0, T]. \end{array} \right. \tag{3}$$

Showing the existence of solutions of this problem is known as solving the problem of *feasibility.* There are various extensions of problem (3) involving, e.g., inequality constraints, pure state constrains, mixed constraints, etc. In Section 2 we will have a closer look at this problem for the case when $W = \mathbb{R}^{d-m}_{+} = \{v \in \mathbb{R}^{d-m} \,\big|\, v_i \geq 0,\, i = 1, \ldots, d-m\}$.

When $f(x, x(0), x(T), u) = \begin{pmatrix} -x(0) \\ -x(T) \\ -u \end{pmatrix}$ and $F \equiv \begin{pmatrix} x_0 \\ x_T \\ U \end{pmatrix}$, where U is a closed set in $\mathbb{R}^n$ and $x_T \in \mathbb{R}^m$ with $2m + n = d$, (1)–(2) describes a constrained control system with fixed initial and final states:

$$\left\{ \begin{array}{l} \dot{x}(t) = g(x(t), u(t)), \quad u(t) \in U \quad \text{ for a.e. } t \in [0, T], \\ x(0) = x_0, \quad x(T) = x_T. \end{array} \right. \tag{4}$$

The system (4) is said to be *controllable* at the point x_T for time T when there exists a neighborhood $\mathcal{W}$ of x_T such that for each point $y \in \mathcal{W}$ there exists a feasible control such that the corresponding state trajectory starting from x_0 at time $t = 0$ reaches the target y at time $t = T$. In Section 2 we obtain a necessary and sufficient condition for controllability of system (4).

Recall that, given a closed convex set Ω in a linear normed space X, the normal cone mapping acting from X to its topological dual X^* is

$$N_\Omega(x) = \left\{ \begin{array}{ll} \{y \in X^* \mid \langle y, v - x \rangle \leq 0 \text{ for all } \; v \in \Omega\} & \text{if } \; x \in \Omega, \\ \emptyset & \text{otherwise,} \end{array} \right.$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing. In the particular case when X is the n-dimensional euclidean space $\mathbb{R}^n$, in problem (1)–(2) we have $F = N_\Omega$ (in which case $d = n$) and f is independent of $x(t), x(0)$ and $x(T)$, then the inclusion (2) separates from (1) and the dependence on t becomes superfluous; then (2) reduces to a finite-dimensional variational inequality:

$$f(u) + N_\Omega(u) \ni 0. \tag{5}$$

More generally, for

$$f = \begin{pmatrix} -x(0) \\ h(x, u) \end{pmatrix} \quad \text{ and } \quad F(u) = \begin{pmatrix} x_0 \\ N_\Omega(u) \end{pmatrix},$$

system (1)–(2) takes the form of a Differential Variational Inequality (DVI), a name apparently coined in [2] and used there for a differential inclusion with a special structure. The importance of DVIs as a general model in optimization is broadly discussed in [23].

When F is the zero mapping, system (1)–(2) becomes a Differential Algebraic Equation (DAE). An important class of DAEs are those of index one in which the algebraic equation determines the variable u as a function of x and then, after substitution in the differential equation, the DAE reduces to an initial value problem. In this paper we will not discuss DAEs. We only mention that the property of strong metric regularity which we study in Section 3 of the paper, is closely related to the index one property.

Another particular case of (1)–(2) comes from the first-order optimality conditions in optimal control, e.g., for the following optimal control problem involving an integral functional, a nonlinear state equation, and control constraints:

$$\begin{array}{l} \text{minimize} \left[\varphi(y(T)) + \int_0^T L(y(t), u(t)) dt \right] \\ \text{subject to} \\ \dot{y}(t) = g(y(t), u(t)), \; y(0) = y_0, \quad u(t) \in U \quad \text{for a.e. } \; t \in [0, T]. \end{array} \tag{6}$$

Here, as in the model (1)–(2), the control u is essentially bounded and measurable with values in the closed and convex set U, the state trajectory y is Lipschitz continuous, and the functions φ, L and g are twice continuously differentiable everywhere. Under mild assumptions a first-order necessary condition for a weak minimum for problem (6) (Pontryagin's maximum principle) is described in terms of the Hamiltonian $H(y,p,u) = L(y,u) + p^T g(y,u)$ as a Hamiltonian system coupled with a variational inequality:

$$
(7) \qquad \begin{cases} \dot{y}(t) &= D_p H(y(t),p(t),u(t)), & y(0) = y_0, \\ \dot{p}(t) &= -D_y H(y(t),p(t),u(t)), & p(T) = -D\varphi(y(T)), \\ 0 &\in D_u H(y(t),p(t),u(t)) + N_U(u(t)), \end{cases}
$$

where the function p with values $p(t) \in \mathbb{R}^m$, $t \in [0,T]$, is the so-called *adjoint* variable. To translate (7) into the form (1)–(2), set $x = (y,p)$,

$$
f(x, x(0), x(T), u) = \begin{pmatrix} -y(0) \\ p(T) + D\varphi(y(T)) \\ D_u H(y,p,u) \end{pmatrix} \quad \text{and} \quad F(u) = \begin{pmatrix} y_0 \\ 0 \\ N_U(u) \end{pmatrix}.
$$

We consider in more detail this problem in Section 4.

In the model (1)–(2) we assume that the controls are in $L^\infty([0,T],\mathbb{R}^n)$, the space of essentially bounded and measurable functions on $[0,T]$ with values in $\mathbb{R}^n$. The state trajectories belong to $W^{1,\infty}([0,T],\mathbb{R}^m)$, the space of Lipschitz continuous functions on $[0,T]$ with values in $\mathbb{R}^m$. When the initial state is zero, $x(0) = 0$, then it is convenient to use the space $W_0^{1,\infty}([0,T],\mathbb{R}^m) = \{x \in W^{1,\infty}([0,T],\mathbb{R}^m) \mid x(0) = 0\}$. In this paper we also employ the space $C([0,T],\mathbb{R}^n)$ of continuous functions on $[0,T]$ equipped with the usual supremum (Chebyshev) norm. We use the notation $\|\cdot\|$ for the standard euclidean norm, $\|\cdot\|_\infty$ for the L^∞ norm and $\|\cdot\|_C$ for the supremum norm. Also, $C^1([0,T],\mathbb{R}^n)$ is the space of continuously differentiable functions on $[0,T]$ equipped with the norm $\|x\|_{C^1} = \|\dot{x}\|_C + \|x\|_C$. In the sequel we often use the shorthand notation L^∞ instead of $L^\infty([0,T],\mathbb{R}^n)$, etc.

In a seminal paper [25] S. M. Robinson called the variational inequality (5) a *generalized equation*, but in subsequent publications this name has been attached to the more general inclusion

$$
(8) \qquad f(u) + F(u) \ni 0,
$$

where F is not necessarily a normal cone mapping. The generalized equation (8) turned out to be particularly useful for various models in optimization and control. More importantly, quite a few results originally stated for variational inequalities, including the celebrated Robinson's implicit function theorem [25], a particular case of which we present below as Theorem 3, remain valid in the case when the normal cone mapping N_Ω in (5) is replaced by a general set-valued mapping.

By analogy with the name "differential variational inequality" used in [23] for a system of a differential equation coupled with a variational inequality, we call the model (1)–(2) a *Differential Generalized Equation (DGE)*. Note that the DGE (1)–(2) can be written as a generalized equation in function spaces. Indeed, denoting $z = (x,u) \in W^{1,\infty} \times L^\infty$ and

$$
e(z) = \begin{pmatrix} \dot{x} - g(x,u) \\ f(x, x(0), x(T), u) \end{pmatrix}, \qquad E(z) = \begin{pmatrix} 0 \\ F(u) \end{pmatrix},
$$

we can rewrite (1)–(2) as a generalized equation of the form

$$
(9) \qquad e(z) + E(z) \ni 0.
$$

Suppose that (1)–(2) is a differential variational inequality, i.e., $F = N_U$ for a closed and convex set $U \subset \mathbb{R}^n$. Then, in order to obtain a variational inequality in function spaces, say for $(x,u) \in W^{1,\infty} \times L^\infty$, the function $t \mapsto f(x(t), x(0), x(T), u(t))$ should be an element of the dual to L^∞. The problem can be easily resolved if we introduce the mapping

$$
L^\infty \ni u \mapsto F(u) = \{w \in L^\infty \mid w(t) \in N_U(u(t)) \text{ for a.e. } t \in [0,T]\};
$$

then (9) becomes a generalized equation stated in function spaces which may *not* be a variational inequality.

The name "differential variational inequalities" has been used, along with other names such as evolutionary variational inequalities, projected dynamical systems, sweeping processes, to describe various kinds of differential inclusions, see [4] for a comparison of these models. There is a bulk of literature dealing with DVIs along the lines of the basic theory of differential equations studying existence and uniqueness of a solution, asymptotic behavior, stability properties, etc., see the recent papers [14], [18], [19], [22], the monograph [28], and the references therein. In this paper we introduce the new model (1)–(2) which is more general than DVIs and covers in particular optimal control problems. Our specific goal is to study regularity properties of mappings appearing in its description.

We use standard notations and terminology, mostly from the book [6]. In the paper X and Y are Banach spaces with norms $\|\cdot\|$ unless stated otherwise. The distance from a point x to a set A is $d(x,A) = \inf_{y\in A}\|x-y\|$. The closed ball centered at x with radius r is denoted by $\mathbb{B}_r(x)$, the closed unit ball is $\mathbb{B}$. The interior, the closure, and the convex hull of a set A is denoted by $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{co} A$, respectively. A (generally set-valued) mapping $\mathcal{F}: X \rightrightarrows Y$ is associated with its graph $\operatorname{gph}\mathcal{F} = \{\, (x,y) \in X\times Y \,|\, y \in \mathcal{F}(x)\}$, its domain $\operatorname{dom}\mathcal{F} = \{\, x \in X \,|\, \mathcal{F}(x) \neq \emptyset\}$ and its range $\operatorname{rge}\mathcal{F} = \{\, y \in Y \,|\, \exists\, x \in X \text{ with } y \in \mathcal{F}(x)\}$. The inverse of $\mathcal{F}$ is defined as $y \mapsto \mathcal{F}^{-1}(y) = \{\, x \in X \,|\, y \in \mathcal{F}(x)\}$. The space of all linear bounded (single-valued) mappings acting from X to Y equipped with the standard operator norm is denoted by $\mathcal{L}(X,Y)$. The Fréchet derivative of a function $h: X \to Y$ at $\bar{x} \in X$ is denoted by $Dh(\bar{x})$; the partial Fréchet derivatives with respect to x and u of $h: X\times U \to Y$ at a point $(\bar{x},\bar{u}) \in X\times U$ are denoted by $D_x h(\bar{x},\bar{u})$ and $D_u h(\bar{x},\bar{u})$, respectively.

We consider two regularity properties of mappings appearing in the model (1)–(2): metric regularity and strong metric regularity. In classical analysis, the term *regularity* of a differentiable function at a certain point means that the derivative at that point is onto (surjective). For set-valued and nonsmooth mappings, the meaning of regularity becomes much more intricate. A mapping $\mathcal{F}: X \rightrightarrows Y$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ when $\bar{y} \in \mathcal{F}(\bar{x})$, $\operatorname{gph}\mathcal{F}$ is locally closed at $(\bar{x},\bar{y})$, meaning that there exists a neighborhood W of $(\bar{x},\bar{y})$ such that the set $\operatorname{gph}\mathcal{F}\cap W$ is closed in W, and there is a constant $\tau \geq 0$ together with neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$d\big(x, \mathcal{F}^{-1}(y)\big) \leq \tau d\big(y, \mathcal{F}(x)\big) \quad \text{for every} \quad (x,y) \in U\times V.$$

Note that from this definition it follows that $\mathcal{F}^{-1}(y) \neq \emptyset$ for y close to $\bar{y}$. More precisely, for every neighborhood U of $\bar{x}$ there exists a neighborhood V of $\bar{y}$ such that $\mathcal{F}^{-1}(y)\cap U \neq \emptyset$ for all $y \in V$, see [6, Proposition 3E.1 and Theorem 3E.7].

Metric regularity has emerged in 1980s as a central concept in variational analysis, optimization and control, but is present already in the Banach open mapping principle. It has been first used by Lyusternik [20] as a constraint qualification for abstract minimization problems, and later by Graves [13] to extend the Banach open mapping to nonlinear functions. In nonlinear programming, metric regularity appears as the Mangasarian-Fromovitz constraint qualification, and in control it is linked to controllabilty (see Section 2), but not only. More importantly, metric regularity plays a major role in studying the effects of perturbations and approximations in variational problems with constraints, where the solution is typically not differentiable with respect to parameters. The literature related to metric regularity has grown enormously in the last two decades, including several monographs, e.g. [26], [17], [11], [6], and the recent book [15].

We recall two basic results about metric regularity that will be used further on. The first is the (extended) Lyusternik-Graves theorem, which we present here in a simplified form (for a more general version, see [6, Theorem 5E.6]):

Theorem 1. *Let $h: X \to Y$ with $\bar{x} \in \operatorname{int}\operatorname{dom} h$ be continuously Fréchet differentiable around $\bar{x}$ and let $\mathcal{F}: X \rightrightarrows Y$ be a set-valued mapping with a closed graph and with $\bar{y} \in \mathcal{F}(\bar{x})$. Then the mapping $h+\mathcal{F}$ is metrically regular at $\bar{x}$ for $h(\bar{x})+\bar{y}$ if and only if the linearization $x \mapsto h(\bar{x}) + Dh(\bar{x})(x-\bar{x}) + \mathcal{F}(x)$ is metrically regular at $\bar{x}$ for $h(\bar{x})+\bar{y}$.*

The second result is the Robinson–Ursescu theorem stated, e.g., in [6, Theorem 5B.4].

Theorem 2. *A set-valued mapping $\mathcal{F}: X \rightrightarrows Y$ with a closed convex graph and with $\bar{y} \in \mathcal{F}(\bar{x})$ is metrically regular at $\bar{x}$ for $\bar{y}$ if and only if $\bar{y} \in \operatorname{int}\operatorname{rge}\mathcal{F}$.*

The second property we consider here is the strong metric regularity, a property which basically appears already in the standard inverse function theorem. A mapping $\mathcal{F}: X \rightrightarrows Y$ is said to be *strongly metrically regular* at $\bar{x}$ for $\bar{y}$ if $(\bar{x},\bar{y}) \in \operatorname{gph}\mathcal{F}$ and the inverse $\mathcal{F}^{-1}$ has a Lipschitz continuous single-valued graphical localization around $\bar{y}$ for $\bar{x}$, meaning that there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that the mapping $V \ni y \mapsto \mathcal{F}^{-1}(y)\cap U$ is single-valued and Lipschitz continuous on U. It turns out that a mapping $\mathcal{F}$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ if and only if it is metrically regular at $\bar{x}$ for $\bar{y}$ and the inverse $\mathcal{F}^{-1}$ has a graphical localization around $\bar{y}$ for $\bar{x}$ which is nowhere multivalued, see [6, Proposition 3G.1].

Strong metric regularity has been extensively studied for mappings in nonlinear programming. In his groundbreaking paper [25], Robinson proved that the combination of the strong second-order sufficient optimality condition and the linear independence of the active constraints is a sufficient condition for strong metric regularity of the Karush-Kuhn-Tucker mapping at a critical point paired with an associate Lagrange multiplier. This result was later sharpened to show that if the critical point is a minimizer, then this combination becomes also necessary. In the more general context of variational inequalities over polyhedral convex sets, a necessary and sufficient condition for strong metric regularity has been also found, the so-called critical face condition. The strong metric regularity, together with a broad range of applications is covered in [6, Section 4.8]. It should be noted that strong regularity has an important role in numerical optimization; in particular, it implies superlinear or even quadratic convergence, depending on the smoothness of the data, of the most popular Sequential Quadratic Programming (SQP) method, see [6, Section 6c].

A basic result about the strong metric regularity is Robinson's inverse function theorem which we give here in the form symmetric to the Lyusternik-Graves theorem, with an important exception: the mapping $\mathcal{F}$ is not required to be with closed graph (for a more general statement, see [6, Theorems 5F.5]):

Theorem 3. *Let $h : X \to Y$ with $\bar{x} \in \operatorname{int} \operatorname{dom} h$ be continuously Fréchet differentiable around $\bar{x}$ and let $\mathcal{F} : X \rightrightarrows Y$ be a set-valued mapping with $\bar{y} \in \mathcal{F}(\bar{x})$. Then the mapping $h + \mathcal{F}$ is strongly metrically regular at $\bar{x}$ for $h(\bar{x}) + \bar{y}$ if and only if the linearization $x \mapsto h(\bar{x}) + Dh(\bar{x})(x - \bar{x}) + \mathcal{F}(x)$ is strongly metrically regular at $\bar{x}$ for $h(\bar{x}) + \bar{y}$.*

Going back to the DGE model (1)–(2), observe that it consists of two relations of different nature. The first is a control system (1) described by an ordinary differential equation which is a relation in infinite-dimensional spaces of functions, in our case in L^∞ for the control and $W^{1,\infty}$ for the state. Since we can easily differentiate in these spaces, we can apply both the Lyusternik-Graves and Robinson theorems reducing the analysis to that of a linear system. The generalized equation (2) is defined for each $t \in [0, T]$ — so if we fix t, we could apply the available conditions ensuring (strong) metric regularity in finite dimensions. Metric regularity appears in (2) *pointwisely*, but does it imply metric regularity in the infinite-dimensional spaces where the solutions of DGEs live? It is the primary goal of this paper to study in depth the interplay between metric regularity properties of the mapping associated with the DGE defined pointwisely (in time) in finite dimensions and also in function spaces. To the best of our knowledge, this is a first study of such kind. It also covers DVIs and in particular parameterized variational inequalities as special cases.

A summary of the main results of the paper follows. In Section 2 we present necessary and sufficient conditions for metric regularity of the mapping appearing in (1)–(2). We also consider a mapping associated with a control system subject to inequality state-control constraints for which we present a necessary and sufficient condition for metric regularity. The analysis is then extended to an associated controllability problem for which a sufficient condition for controllability is established.

Strong metric regularity for the mapping defining the DGE (1)–(2) is considered in Section 3 for the case when the initial state $x(0)$ is fixed and the final state $x(T)$ is free. In a central result in this section we establish a sufficient condition for strong metric regularity in function spaces in terms of pointwise in time strong metric regularity of the mapping associated with the generalized equation (2). As a side result, for an optimal control problem with control constraints we obtain a characterization of the property that the optimal control is Lipschitz continuous as a function of time. In the final Section 5 we present an application of the theoretical analysis to numerically solving DGEs. Namely, we propose a path-following procedure for a discretized DGE for which we derive an error estimate. A simple numerical example illustrates the result. In each section we present a discussion of results obtained and relate them to the existing literature.

2. Metric Regularity.

In this section we consider the DGE

$$\dot{x}(t) = g(x(t), u(t)), \quad x(0) = x_0, \tag{10}$$

$$f(x(t), u(t)) + F(u(t)) \ni 0 \quad \text{for a.e. } t \in [0, T], \tag{11}$$

where, as for (1)–(2), $x \in W^{1,\infty}([0, T], \mathbb{R}^m)$ and $u \in L^\infty([0, T], \mathbb{R}^n)$, f and g are twice smooth and F is a set-valued mapping. We study the property of metric regularity of the following mapping associated with (10)–(11) defined as acting from $W^{1,\infty} \times L^\infty$ to the subsets of $L^\infty \times \mathbb{R}^m \times L^\infty$ (we use here the shorthand notation for the spaces remembering that the values of the functions in L^∞ belong to Euclidean spaces with different dimensions):

$$(x,u) \mapsto M(x,u) := \begin{pmatrix} \dot{x} - g(x,u) \\ -x(0) \\ f(x,u) \end{pmatrix} + \begin{pmatrix} 0 \\ x_0 \\ F(u) \end{pmatrix}. \tag{12}$$

Given a reference solution $(\bar{x}, \bar{u})$ of (10)–(11), define $\bar{g}(t) = g(\bar{x}(t), \bar{u}(t))$, $\bar{f}(t) = f(\bar{x}(t), \bar{u}(t))$, $A(t) = D_x g(\bar{x}(t), \bar{u}(t))$, $B(t) = D_u g(\bar{x}(t), \bar{u}(t))$, $H(t) = D_x f(\bar{x}(t), \bar{u}(t))$, $E(t) = D_u f(\bar{x}(t), \bar{u}(t))$. The assumptions on the functions g and f allow us to differentiate in $W^{1,\infty} \times L^\infty$ obtaining the mapping

$$W^{1,\infty} \times L^\infty \ni (x,u) \mapsto \begin{pmatrix} \dot{x} - \bar{g} - A(x-\bar{x}) - B(u-\bar{u}) \\ -x(0) \\ \bar{f} + H(x-\bar{x}) + E(u-\bar{u}) \end{pmatrix} + \begin{pmatrix} 0 \\ x_0 \\ F(u) \end{pmatrix}.$$

Substituting $z = x - \bar{x}$ we obtained the following simplified description of the latter mapping:

$$W_0^{1,\infty} \times L^\infty \ni (z,u) \mapsto \mathcal{M}(z,u) := \begin{pmatrix} \dot{z} - Az - B(u-\bar{u}) \\ \bar{f} + Hz + E(u-\bar{u}) \end{pmatrix} + \begin{pmatrix} 0 \\ F(u) \end{pmatrix}. \tag{13}$$

From the Lyusternik-Graves Theorem 1 we immediately obtain the following result:

Corollary 4. *The mapping M defined in (12) is metrically regular at $(\bar{x}, \bar{u})$ for 0 if and only if the mapping $\mathcal{M}$ defined in (13) is metrically regular at $(0, \bar{u})$ for 0.*

Clearly, it is easier to handle the partially linearized mapping (13) than (12); this becomes more apparent in the specific cases considered further: the case of inequality constraints and the case of controllability. Note that, taking into account the comment right after the definition of metric regularity in Introduction, we obtain that metric regularity of the mapping M implies solvability of a perturbation of (10)–(11). Specifically, we have that for every (y,v) with a sufficiently small L^∞ norm there exists a solution of the DGE

$$\begin{aligned} &\dot{x}(t) = g(x(t), u(t)) + y(t), \quad x(0) = x_0, \\ &f(x(t), u(t)) + F(u(t)) + v(t) \ni 0 \quad \text{for a.e. } t \in [0,T]. \end{aligned}$$

The following theorem specializes Corollary 4 taking into account the linear differential operator appearing in the definition of the mapping $\mathcal{M}$. Let Φ be the fundamental matrix solution of the linear equation $\dot{x} = A(t)x$, that is, $\frac{d}{dt}\Phi(t,\tau) = A(t)\Phi(t,\tau)$, $\Phi(\tau,\tau) = I$.

Theorem 5. *Consider the mapping $\mathcal{K}$ acting from L^∞ to L^∞ and defined for a.e. $t \in [0,T]$ as*

$$(\mathcal{K}u)(t) := \bar{f}(t) + H(t)\int_0^t \Phi(t,\tau)(B(\tau)(u(\tau) - \bar{u}(\tau))d\tau + E(t)(u(t) - \bar{u}(t)) + F(u(t)). \tag{14}$$

Then the mapping M is metrically regular at $(\bar{x}, \bar{u})$ for 0 if and only if $\mathcal{K}$ is metrically regular at $\bar{u}$ for 0.

Proof. By Corollary 4, metric regularity of M at $(\bar{x}, \bar{u})$ for 0 is equivalent to metric regularity of the partial linearization $\mathcal{M}$ given in (13) at $(0, \bar{u})$ for 0. Using the fundamental matrix solution for the linear system, given $r \in L^\infty$ and $a \in \mathbb{R}^m$, one has that $\dot{z}(t) - A(t)z(t) = r(t), z(0) = a$ if and only if $z(t) = \Phi(t,0)a + \int_0^t \Phi(t,\tau)r(\tau)d\tau$. This implies that having $(p,a,q) \in \mathcal{M}(z,u)$ is the same as having $v(t) \in (\mathcal{K}u)(t)$ for

$$v(t) = q(t) + H(t)\left(\Phi(t,0)a - \int_0^t \Phi(t,\tau)p(\tau)d\tau\right),$$

that is, we can replace the differential expression in $\mathcal{M}$ with the integral one and then drop the variable z. Noting that local closedness of gph M is equivalent to that of $\mathcal{K}$ and that $\|v\|_\infty$ is bounded by a quantity proportional to $\|(p,a,q)\|$, we complete the proof. ∎

A further specialization of the result in Corollary 4 is obtained when the mapping F has a closed and convex graph, by applying Robinson-Ursescu Theorem 2. To simplify the presentation, we restrict our attention to the case of inequality state-control constraints and the initial state fixed to zero, $x(0) = 0$.

Then the mapping F is a constant mapping equal to the set of all functions in L^∞ with values in $\mathbb{R}^d_+$, which we denote by L^∞_+. That is, we assume that $(\bar{x}, \bar{u}) \in W^{1,\infty}_0 \times L^\infty$ and study the following mapping associated with the feasibility problem (3) in the notation of (10)-(11):

$$W_0^{1,\infty} \times L^\infty \ni (x,u) \mapsto \begin{pmatrix} \dot{x} - g(x,u) \\ f(x,u) \end{pmatrix} + \begin{pmatrix} 0 \\ L^\infty_+ \end{pmatrix}. \tag{15}$$

Theorem 6. *The mapping in (15) is metrically regular at $(\bar{x}, \bar{u})$ for 0 if and only if there exist a constant $\alpha > 0$, and a function $v \in L^\infty$ such that, for a.e. $t \in [0,T]$ and for all $i = 1, 2, \ldots, d$,*

$$[\bar{f}(t) + H(t)\int_0^t \Phi(t,\tau)B(\tau)v(\tau)d\tau + E(t)v(t)]_i \le -\alpha. \tag{16}$$

Proof. By the Lyusternik-Graves Theorem 1, metric regularity of the mapping in (15) at $(\bar{x}, \bar{u})$ for 0 is equivalent to metric regularity at $(0, \bar{u})$ for 0 of the linearized mapping

$$W_0^{1,\infty} \times L^\infty \ni (z,u) \mapsto \begin{pmatrix} \dot{z} - Az - B(u - \bar{u}) \\ \bar{f} + Hz + E(u - \bar{u}) \end{pmatrix} + \begin{pmatrix} 0 \\ L^\infty_+ \end{pmatrix} \subset L^\infty. \tag{17}$$

The mapping (17) has closed and convex graph, hence we can apply Robinson-Ursescu Theorem 2, which in this particular case says that its metric regularity at $(0, \bar{u})$ for 0 is equivalent to the existence of $\delta > 0$ such that for any $(r,q) \in L^\infty$ with $\|(r,q)\|_\infty \le \delta$ the following problem has a solution: find $(z,u) \in W^{1,\infty}_0 \times L^\infty$ such that

$$\begin{aligned} &\dot{z}(t) = A(t)z(t) + B(t)(u(t) - \bar{u}(t)) + r(t), \\ &\bar{f}(t) + H(t)z(t) + E(t)(u(t) - \bar{u}(t)) + q(t) \le 0, \quad \text{for a.e.} \quad t \in [0,T]. \end{aligned} \tag{18}$$

Taking $r = 0$, $q = (\alpha, \ldots, \alpha)$ with $\alpha > 0$ such that $\|q\|_\infty \le \delta$, and then $v = u - \bar{u}$, this property of (18) implies condition (16) in the statement of the theorem.

Conversely, let v satisfy (16) for some $\alpha > 0$, let $y = (r,q)$ be given and let z be the solution of the differential equation in (18) corresponding to the control $u = v + \bar{u}$ and $z(0) = 0$. Note that $z = Q(Bv + r)$ where Q is a bounded linear mapping from L^∞ to $W^{1,\infty}$ defined as $(Qp)(t) = \int_0^t \Phi(t,\tau)p(\tau)d\tau$ for $t \in [0,T]$. Hence, slightly abusing notation, for $\bar{\alpha} = (\alpha, \ldots, \alpha) \in \mathbb{R}^d$,

$$\bar{f} + HQ(Bv + r) + Ev + q \le \bar{f} + HQ(Bv) + Ev + HQ(r) + q \le -\bar{\alpha} + HQ(r) + q \le 0$$

for (r,q) with a sufficiently small norm. This completes the proof. ∎

An analogous argument can be applied to study the controllability problem (4) where we set $x(0) = 0$ for simplicity. Consider the control system

$$\dot{x}(t) = g(x(t), u(t)), \quad x(0) = 0, \tag{19}$$

supplied with feasible controls u from the set

$$\mathcal{U} = \{u \in L^\infty([0,T], \mathbb{R}^n) \mid u(t) \in U \text{ for a.e. } t \in [0,T]\},$$

where U is a convex and compact set in $\mathbb{R}^n$. Given a target point $x_T \in \mathbb{R}^m$ we add to the constraints the condition to reach the target at time T: $x(T) = x_T$. To that problem we associate the mapping

$$W_0^{1,\infty} \times L^\infty \ni (x,u) \mapsto D(x,u) := \begin{pmatrix} \dot{x} - g(x,u) \\ -x(T) \\ -u \end{pmatrix} + \begin{pmatrix} 0 \\ x_T \\ \mathcal{U} \end{pmatrix} \subset L^\infty \times \mathbb{R}^m \times L^\infty. \tag{20}$$

Theorem 7. *The mapping D defined in (20) is metrically regular at $(\bar{x}, \bar{u})$ for 0 if and only if*

$$0 \in \operatorname{int}\{x \in \mathbb{R}^m \mid x = \int_0^T \Phi(T,t)B(t)(u(t) - \bar{u}(t))dt \text{ for some } u \in L^\infty \text{ with } u(t) \in U \text{ for a.e. } t \in [0,T]\}, \tag{21}$$

where Φ is the fundamental matrix solution of $\dot{x} = A(t)x$.

Proof. The first step is the same as in the proof of Theorem 6: by the Lyusternik-Graves Theorem 1 we obtain that the mapping D is metrically regular at $(\bar{x}, \bar{u})$ for 0 as a mapping acting from $W_0^{1,\infty} \times L^\infty$ to the subsets of $L^\infty \times \mathbb{R}^m \times L^\infty$ if and only if its shifted linearization

$$(z, u) \mapsto \mathcal{D}(z, u) := \begin{pmatrix} \dot{z} - Az - B(u - \bar{u}) \\ -z(T) \\ -u \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathcal{U} \end{pmatrix} \subset L^\infty \times \mathbb{R}^m \times L^\infty \tag{22}$$

is metrically regular at $(0, \bar{u})$ for 0 in the same spaces. As in Theorem 6, we apply Robinson-Ursescu Theorem 2 according to which metric regularity of $\mathcal{D}$ at $(0, \bar{u})$ for 0 is equivalent to the existence of $\delta > 0$ such that for any $(r, q) \in L^\infty$ and $y \in \mathbb{R}^m$ with $\|r\|_\infty + \|q\|_\infty + \|y\| \le \delta$ the following problem has a solution: find $(z, u) \in W_0^{1,\infty} \times L^\infty$ such that

$$\begin{aligned} &\dot{z}(t) = A(t)z(t) + B(t)(u(t) - \bar{u}(t)) + r(t), \\ &z(T) = y, \\ &u(t) + q(t) \in U \quad \text{for a.e.} \quad t \in [0, T]. \end{aligned} \tag{23}$$

If (23) has a solution for all such (r, y, q), then, in particular, taking $r = 0$ and $q = 0$ and using the fundamental matrix solution Φ this leads to the property that for every $y \in \mathbb{R}^m$ with a sufficiently small norm there exists $u \in \mathcal{U}$ such that if $z(t) = \int_0^t \Phi(t, \tau)B(\tau)(u(\tau) - \bar{u}(\tau))d\tau$ then $z(T) = y$. This implies (21).

Conversely, let (21) hold. For any $(r, y, q) \in L^\infty \times \mathbb{R}^m \times L^\infty$ with $\|(r, y, q)\|$ sufficiently small, (21) implies the existence of $w \in \mathcal{U}$ such that

$$\int_0^T \Phi(T, \tau)B(\tau)(w(\tau) - \bar{u}(\tau))d\tau = y + \int_0^T \Phi(T, \tau)[B(\tau)q(\tau) - r(\tau)]d\tau.$$

Then system (23) is satisfied with $u = w - q$ and $z(t) = \int_0^t \Phi(t, \tau)[B(\tau)(u(\tau) - \bar{u}(\tau)) + r(\tau)]d\tau$. This completes the proof. ∎

Recall that the reachable set R_T at time T of system (19) is defined as

$$R_T = \{x(T) \mid \text{ there exists } u \in \mathcal{U} \text{ such that } x \text{ is a solution of (19) for } u\}.$$

Also recall that the control system (19) is said to be *locally controllable* at a point $x_T \in \mathbb{R}^m$ whenever $x_T \in \operatorname{int} R_T$. Thus, condition (21) is the same as requiring local controllability at 0 of the shifted linearized system

$$\dot{z}(t) = A(t)z(t) + B(t)(u(t) - \bar{u}(t)), \quad z(0) = 0, \tag{24}$$

with controls from the set $\mathcal{U}$. We obtain:

COROLLARY 8. *Suppose that the linear system (24) is locally controllable at 0 with controls from the set $\mathcal{U}$. Then the nonlinear system (19) has the same property.*

Proof. Local controllability implies, via the theorems of Lyusternik-Graves and Robinson-Ursescu, metric regularity of the mapping (20). The latter property yields that for each y in a neighborhood of x_T there exists a feasible control u such that the corresponding solution x of (19) satisfies $x(T) = y$, that is, the nonlinear system is locally controllable. ∎

That controllability of a linearization of a nonlinear system implies local controllability of the original system is not new: it has been established for various systems, e.g., in [16] and [29]. What is new is the way we prove this implication, namely, by employing much deeper results regarding metric regularity. The converse implication is false in general: local controllability is not stable under linearization the way metric regularity is.

3. Strong metric regularity. In this section we continue to study problem (10)–(11) with the aim to give conditions under which the associated mapping M defined in (12) is *strongly metrically regular*. Our central result is Theorem 17 where we establish a sufficient condition for strong metric regularity of the mapping M in function spaces in terms of pointwise in time strong metric regularity of the parametrized finite-dimensional generalized equation (11). Inasmuch as a number of sufficient conditions, and even necessary and sufficient conditions, for the strong regularity in finite dimensions are available in the literature, with many of them displayed in the books [17], [11], [6], we can now handle accordingly strong metric regularity in function spaces.

In further lines we use the general observation that if a mapping $\mathcal{F}$ is strongly metrically regular at $\bar{x}$ for $\bar{y}$ with a constant $\tau \geq 0$ and neighborhoods $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$ for some positive a and b then for every positive constants $a' \leq a$ and $b' \leq b$ such that $\tau b' \leq a'$ the mapping $\mathcal{F}$ is strongly metrically regular with the constant τ and neighborhoods $\mathbb{B}_{a'}(\bar{x})$ and $\mathbb{B}_{b'}(\bar{y})$. Indeed, in this case any $y \in \mathbb{B}_{b'}(\bar{y})$ will be in the domain of $\mathcal{F}^{-1}(\cdot) \cap \mathbb{B}_{a'}(\bar{x})$.

In the considerations so far, the reference solution $(\bar{x}, \bar{u})$ of (10)–(11) was regarded as an element of the space $W^{1,\infty} \times L^\infty$, thus it is sufficient to require equations (10)–(11) be satisfied almost everywhere. In the remaining part of the paper we consider $\bar{u}$ as a function from $[0,T]$ to $\mathbb{R}^n$, which will be assumed measurable and bounded. In addition, we assume that the reference pair $(\bar{x}, \bar{u})$ satisfies (10)–(11) for each $t \in [0,T]$. This choice of a particular representative of $\bar{u} \in L^\infty$ is needed because the conditions for strong metric regularity of the mapping M and the additional results obtained in this and the next sections are based on assumptions that are to be satisfied for each $t \in [0,T]$. Clearly, considering a reference pair $(\bar{x}, \bar{u})$ with bounded $\bar{u}$ and for which (10)–(11) hold everywhere is not a restriction by itself. Indeed, every $\bar{u} \in L^\infty$ has a bounded representative. If F has a closed graph, then $\bar{u}$ can always be redefined on a set of measure zero so that (11) holds for each t. Then $\dot{\bar{x}}$ can be redefined on a set of measure zero (this leaves $\bar{x}$ unchanged) to satisfy (10) everywhere. What brings a restriction, is that the main assumption below (condition (25)) is in a pointwise form and has to be satisfied for each t.

To start, we state the following corollary of Robinson Theorem 3 which echoes Corollary 4:

Corollary 9. *The mapping M defined in (12) is strongly metrically regular at $(\bar{x}, \bar{u})$ for 0 if and only if the mapping $\mathcal{M}$ defined in (13) is strongly metrically regular at $(0, \bar{u})$ for 0.*

We utilize in further lines the following result, which is a part of [6, Theorem 5G.3][1]:

Theorem 10. *Let a, b, and κ be positive scalars such that F is strongly metrically regular at $\bar{x}$ for $\bar{y}$ with neighborhoods $\mathbb{B}_a(\bar{x})$ and $\mathbb{B}_b(\bar{y})$ and constant κ. Let $\mu > 0$ be such that $\kappa\mu < 1$ and let $\kappa' > \kappa/(1-\kappa\mu)$. Then for every positive α and β such that*

$$\alpha \leq a/2, \quad 2\mu\alpha + 2\beta \leq b \quad \text{and} \quad 2\kappa'\beta \leq \alpha$$

and for every function $g : X \to Y$ satisfying

$$\|g(\bar{x})\| \leq \beta \quad \text{and} \quad \|g(x) - g(x')\| \leq \mu\|x - x'\| \qquad \text{for every } x, x' \in \mathbb{B}_{2\alpha}(\bar{x}),$$

the mapping $y \mapsto (g+F)^{-1}(y) \cap \mathbb{B}_\alpha(\bar{x})$ is a Lipschitz continuous function on $\mathbb{B}_\beta(\bar{y})$ with Lipschitz constant κ'.

We will use Theorem 10 to show that the strong metric regularity of the linearization of (11) at each point of $\operatorname{cl}\operatorname{gph}\bar{u}$ implies *uniform* strong metric regularity. For this we utilize the following condition, which will play an important role in most of the further results:

$$\begin{array}{c} \text{Let } (\bar{x}, \bar{u}) \text{ be a solution of (10)–(11) and let for every } z := (t,u) \in \operatorname{cl}\operatorname{gph}\bar{u} \text{ the mapping} \\ \mathbb{R}^n \ni v \mapsto \mathcal{W}_z(v) := f(\bar{x}(t), u) + D_u f(\bar{x}(t), u)(v - u) + F(v) \\ \text{be strongly metrically regular at } u \text{ for } 0, \text{ thus in particular } 0 \in f(\bar{x}(t), u) + F(u). \end{array} \tag{25}$$

Theorem 11. *Suppose that condition (25) is satisfied. Then there are positive constants a, b, and κ such that for each $z = (t,u) \in \operatorname{cl}\operatorname{gph}\bar{u}$ the mapping*

$$\mathbb{B}_b(0) \ni y \mapsto \mathcal{W}_z^{-1}(y) \cap \mathbb{B}_a(u)$$

is a Lipschitz continuous function with Lipschitz constant κ.

[1] See Errata and Addenda at https://sites.google.com/site/adontchev/

Proof. Let $\Sigma := \operatorname{cl}\operatorname{gph}\bar{u}$. Since Σ is a compact subset of $\mathbb{R}\times\mathbb{R}^n$ (equipped with the box topology), its canonical projection Σ_u onto $\mathbb{R}^n$ is compact as well. This and the continuity of $\bar{x}$ imply the compactness of the set $\Lambda := \operatorname{co}\bar{x}([0,T])\times\operatorname{co}\Sigma_u$. By the continuous differentiability of f there exists $M>0$ such that $\|D_xf(x,u)\|\le M$ for each $(x,u)\in\Lambda$. By the twice continuous differentiability of the function f, the mapping $(x,u)\mapsto D_uf(x,u)$ is locally Lipschitz continuous, and therefore Lipschitz on compact subsets of $\mathbb{R}^m\times\mathbb{R}^n$; denote by $K>0$ its Lipschitz constant on Λ. Finally, let $L>0$ be the Lipschitz constant of $\bar{x}$ on $[0,T]$.

Fix an arbitrary $\bar{z}=(\bar{t},\bar{u})\in\Sigma$ and let $a_{\bar z}$, $b_{\bar z}$ and $\kappa_{\bar z}$ be positive constants such that the mapping

$$\mathbb{B}_{b_{\bar z}}(0)\ni y\mapsto \mathcal{W}_{\bar z}^{-1}(y)\cap\mathbb{B}_{a_{\bar z}}(\bar u) \tag{26}$$

is a Lipschitz continuous function with Lipschitz constant $\kappa_{\bar z}$. Let $\alpha_{\bar z}:=a_{\bar z}/2$ and pick $\rho_{\bar z}\in(0,\alpha_{\bar z}/2)$ such that

$$4\rho_{\bar z}(K\alpha_{\bar z}+ML)<b_{\bar z},\quad 8ML\kappa_{\bar z}\rho_{\bar z}<\alpha_{\bar z}(1-2K\kappa_{\bar z}\rho_{\bar z}),\quad\text{and}\quad K\rho_{\bar z}<2ML. \tag{27}$$

Finally, let $\beta_{\bar z}:=2ML\rho_{\bar z}$ and $\mu_{\bar z}:=2K\rho_{\bar z}$. The second inequality in (27) implies that $\kappa_{\bar z}\mu_{\bar z}<1$.

Pick any $z=(t,u)\in\big(\operatorname{int}\mathbb{B}_{\rho_{\bar z}}(\bar t)\times\operatorname{int}\mathbb{B}_{\rho_{\bar z}}(\bar u)\big)\cap\Sigma$. Define $g_{z,\bar z}:\mathbb{R}^n\to\mathbb{R}^d$ as

$$\begin{aligned} g_{z,\bar z}(v) := {} & f(\bar x(t),u)-f(\bar x(\bar t),\bar u)-D_uf(\bar x(t),u)u+D_uf(\bar x(\bar t),\bar u)\bar u\\ & +\big(D_uf(\bar x(t),u)-D_uf(\bar x(\bar t),\bar u)\big)v,\quad v\in\mathbb{R}^n. \end{aligned}$$

Then $\mathcal{W}_z=\mathcal{W}_{\bar z}+g_{z,\bar z}$. Moreover, for any v_1, $v_2\in\mathbb{R}^n$ we have

$$\begin{aligned} \|g_{z,\bar z}(v_1)-g_{z,\bar z}(v_2)\| &= \|(D_uf(\bar x(t),u)-D_uf(\bar x(\bar t),\bar u))(v_1-v_2)\|\le K(\rho_{\bar z}+\rho_{\bar z})\|v_1-v_2\|\\ &=\mu_{\bar z}\|v_1-v_2\|. \end{aligned}$$

Basic calculus gives us

$$\begin{aligned} g_{z,\bar z}(\bar u) &= f(\bar x(t),u)-f(\bar x(\bar t),\bar u)+D_uf(\bar x(t),u)(\bar u-u)\\ &= f(\bar x(t),u)-f(\bar x(t),\bar u)+D_uf(\bar x(t),u)(\bar u-u)+f(\bar x(t),\bar u)-f(\bar x(\bar t),\bar u)\\ &= -\int_0^1\frac{d}{ds}f(\bar x(t),u+s(\bar u-u))ds+D_uf(\bar x(t),u)(\bar u-u)\\ &\qquad+\int_0^1\frac{d}{ds}f(\bar x(\bar t)+s(\bar x(t)-\bar x(\bar t)),\bar u)ds\\ &= \int_0^1\left[D_uf(\bar x(t),u)-D_uf(\bar x(t),u+s(\bar u-u))\right](\bar u-u)ds\\ &\quad+\int_0^1 D_xf(\bar x(\bar t)+s(\bar x(t)-\bar x(\bar t)),\bar u)(\bar x(t)-\bar x(\bar t))ds. \end{aligned}$$

Hence, taking into account the last inequality in (27) we obtain

$$\|g_{z,\bar z}(\bar u)\|<\frac12K\rho_{\bar z}^2+ML\rho_{\bar z}<\big(ML+ML\big)\rho_{\bar z}=\beta_{\bar z}.$$

Let $\kappa'_{\bar z}:=2\kappa_{\bar z}/(1-\kappa_{\bar z}\mu_{\bar z})>\kappa_{\bar z}/(1-\kappa_{\bar z}\mu_{\bar z})$. Applying Theorem 10 we conclude that the mapping

$$\mathbb{B}_{\beta_{\bar z}}(0)\ni y\mapsto\mathcal{W}_z^{-1}(y)\cap\mathbb{B}_{\alpha_{\bar z}}(\bar u) \tag{28}$$

is a Lipschitz continuous function with Lipschitz constant $\kappa'_{\bar z}$. The second inequality in (27) and the choice of $\rho_{\bar z}$ imply that $\mathbb{B}_{\kappa'_{\bar z}\beta_{\bar z}}(u)\subset\mathbb{B}_{\alpha_{\bar z}/2}(u)\subset\mathbb{B}_{\alpha_{\bar z}}(\bar u)$. Since for $z\in\Sigma$, we have $0\in\mathcal{W}_z(u)$, and for every $y\in\mathbb{B}_{\beta_{\bar z}}(0)$ it holds that

$$\|\mathcal{W}_z^{-1}(y)\cap\mathbb{B}_{\alpha_{\bar z}}(\bar u)-u\|\le\kappa'_{\bar z}\|y\|\le\kappa'_{\bar z}\beta_{\bar z},$$

we conclude that for $y\in\mathbb{B}_{\beta_{\bar z}}(0)$ the set $\mathcal{W}_z^{-1}(y)\cap\mathbb{B}_{\kappa'_{\bar z}\beta_{\bar z}}(u)$ is nonempty. Then for each $z=(t,u)\in\big(\operatorname{int}\mathbb{B}_{\rho_{\bar z}}(\bar t)\times\operatorname{int}\mathbb{B}_{\rho_{\bar z}}(\bar u)\big)\cap\Sigma$ the mapping

$$\mathbb{B}_{\beta_{\bar z}}(0)\ni y\mapsto\mathcal{W}_z^{-1}(y)\cap\mathbb{B}_{\alpha_{\bar z}/2}(u) \tag{29}$$

is a Lipschitz continuous function with Lipschitz constant $\kappa'_{\bar{z}}$, that is, the size of neighborhoods and the Lipschitz constant are independent of z in a neighborhood of $\bar{z}$.

From the open covering $\cup_{\bar{z}=(\bar{t},\bar{u})\in\Sigma}\big([\mathrm{int}I\!B_{\rho_{\bar{z}}}(\bar{t})\times \mathrm{int}I\!B_{\rho_{\bar{z}}}(\bar{u})]\cap\Sigma\big)$ of Σ choose a finite subcovering $\mathcal{O}_i := [\mathrm{int}I\!B_{\rho_{\bar{z}_i}}(\bar{t}_i)\times\mathrm{int}I\!B_{\rho_{\bar{z}_i}}(\bar{u}_i)]\cap\Sigma$, $i=1,2,\ldots,k$. Let $a=\min\{\alpha_{\bar{z}_i}/2 \mid i=1,\ldots,k\}$, $\kappa=\max\{\kappa'_{\bar{z}_i} \mid i=1,\ldots,k\}$, and $b=\min\{a/\kappa,\min\{\beta_{\bar{z}_i} \mid i=1,\ldots,k\}\}$. For any $\bar{z}=(\bar{t},\bar{u})\in\Sigma$ there is $i\in\{1,\ldots,k\}$ such that $\bar{z}\in\mathcal{O}_i$. Hence the mapping $I\!B_b(0)\ni y\mapsto \mathcal{W}_{\bar{z}}^{-1}(y)\cap I\!B_a(\bar{u})$ is a Lipschitz continuous function with Lipschitz constant κ. The proof is complete. ∎

The following two results concern uniform strong metric regularity of two mappings related to inclusion (11) along a solution trajectory of (10)–(11). For the linearization of (11) along $(\bar{x}(t),\bar{u}(t))$ we immediately obtain:

Corollary 12. *Let condition* (25) *hold. Then the mapping*

$$\mathbb{R}^n\ni v\mapsto \mathcal{G}_t(v) := \bar{f}(t)+E(t)(v-\bar{u}(t))+F(v) \tag{30}$$

is strongly metrically regular at $\bar{u}(t)$ for 0 uniformly in $t\in[0,T]$, that is, there exist positive constants a, b and κ such that for each $t\in[0,T]$ the mapping $I\!B_b(0)\ni y\mapsto \mathcal{G}_t^{-1}(y)\cap I\!B_a(\bar{u}(t))$ is a Lipschitz continuous function with Lipschitz constant κ.

Proof. It is sufficient to observe that condition (25) involves the closure of the graph of $\bar{u}$ while the strong metric regularity of $\mathcal{G}_t$ is defined for the graph of $\bar{u}$. ∎

Theorem 13. *Let condition* (25) *hold. Then the mapping*

$$\mathbb{R}^n\ni v\mapsto G_t(v) := f(\bar{x}(t),v)+F(v) \tag{31}$$

is strongly metrically regular at $\bar{u}(t)$ for 0 uniformly in $t\in[0,T]$.

Proof. Corollary 12 yields that there exist positive constants a, b and κ such that for each $t\in[0,T]$ the mapping $I\!B_b(0)\ni y\mapsto \mathcal{G}_t^{-1}(y)\cap I\!B_a(\bar{u}(t))$ is a Lipschitz continuous function with Lipschitz constant κ. Since $\mathrm{cl}\,\mathrm{gph}\,\bar{u}$ is a compact set, the function $u\mapsto D_uf(\bar{x}(t),u)$ is Lipschitz continuous on $I\!B_a(\bar{u}(t))$ uniformly in $t\in[0,T]$; let $L>0$ be the corresponding Lipschitz constant.

Choose $\alpha>0$ such that
$$\alpha\le\frac{a}{2},\quad 2L\alpha\kappa<1,\quad\text{and}\quad 4L\alpha^2<b.$$
Fix any $\kappa'>\kappa/(1-2L\alpha\kappa)$ and find $\beta>0$ such that
$$4L\alpha^2+2\beta<b\quad\text{and}\quad 2\kappa'\beta<\alpha.$$
Fix any $t\in[0,T]$ and define the function
$$\mathbb{R}^n\ni v\mapsto g_t(v):=f(\bar{x}(t),v)-\bar{f}(t)-E(t)(v-\bar{u}(t)).$$
Then $g_t(\bar{u}(t))=0$ and for any $v,v'\in I\!B_{2\alpha}(\bar{u}(t))$ we have

$$\begin{aligned}\|g_t(v)-g_t(v')\| &= \|f(\bar{x}(t),v)-f(\bar{x}(t),v')-E(t)(v-v')\|\\ &= \|\int_0^1\big(D_uf(\bar{x}(t),v'+s(v-v'))-D_uf(\bar{x}(t),\bar{u}(t))\big)(v-v')ds\|\\ &\le L\sup_{s\in[0,1]}\|v'+s(v-v')-\bar{u}(t)\|\|v-v'\|\le 2L\alpha\|v-v'\|.\end{aligned}$$

We apply then Theorem 10 (with $\mu:=2L\alpha$) obtaining that the mapping
$$I\!B_\beta(0)\ni y\mapsto(g_t+\mathcal{G}_t)^{-1}(y)\cap I\!B_\alpha(\bar{u}(t))=G_t^{-1}(y)\cap I\!B_\alpha(\bar{u}(t))$$
is a Lipschitz continuous function on $I\!B_\beta(0)$ with Lipschitz constant κ'. It remains to note that α, β and κ' do not depend on t. ∎

The uniform in $t\in[0,T]$ strong metric regularity at $\bar{u}(t)$ for 0 of the mapping (31) implies that the inclusion $0\in G_t(u)$ determines a Lipschitz continuous function which is isolated from other solutions. The isolatedness doesn't have to be true, however, for the reference control $\bar{u}$. To make the presentation more precise, we state the following definition.

Definition 14. *Given a mapping $\mathcal{T} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^d$, a function $u : [0,T] \to \mathbb{R}^n$ is said to be an* isolated solution *of the inclusion*

$$0 \in \mathcal{T}(t,v) \quad \textit{for all } t \in [0,T],$$

whenever there is an open set $\mathcal{O} \subset \mathbb{R}^{n+1}$ such that

$$(32) \qquad \{(t,v) \mid t\in[0,T] \textit{ and } 0 \in \mathcal{T}(t,v)\} \cap \mathcal{O} = \operatorname{gph} u.$$

Our next result shows that under pointwise strong metric regularity of the mapping (31) at $\bar{u}(t)$ for 0 the isolatedness of $\bar{u}$ is equivalent to Lipschitz continuity of $\bar{u}$ as a function of t.

Theorem 15. *Suppose that for each $t \in [0,T]$ the mapping G_t in (31) is strongly metrically regular at $\bar{u}(t)$ for 0. Then the following assertions are equivalent:*
(i) *$\bar{u}$ is an isolated solution of $G_t(v) \ni 0$ for all $t\in[0,T]$;*
(ii) *$\bar{u}$ is continuous on $[0,T]$;*
(iii) *$\bar{u}$ is Lipschitz continuous on $[0,T]$.*

Proof. Let us first show that (i) implies (ii). Choose an open set $\mathcal{O} \subset \mathbb{R}^{n+1}$ such that

$$(33) \qquad \{(t,v) \mid t\in[0,T] \text{ and } 0 \in G_t(v)\} \cap \mathcal{O} = \operatorname{gph} \bar{u}.$$

Let $t\in[0,T]$ and let a_t, b_t and λ_t be positive constants such that the mapping $\mathbb{B}_{b_t}(0) \ni y \mapsto G_t^{-1}(y) \cap \mathbb{B}_{a_t}(\bar{u}(t))$ is a Lipschitz continuous function with Lipschitz constant λ_t. Since $\bar{x}$ is Lipschitz continuous, we have that the functions $\tau \mapsto f(\bar{x}(\tau),v)$ and $\tau \mapsto D_u f(\bar{x}(\tau),v)$ are Lipschitz continuous on $[0,T]$ uniformly in v in the compact set $\mathbb{B}_{a_t}(\bar{u}(t))$; let $L_t > 0$ be a Lipschitz constant for both of them. Note that, due to the boundedness of $\bar{u}$ and the fact that a_t can always be assumed uniformly bounded (say ≤ 1), the Lipschitz constant $L_t = L$ can be chosen independent of t. Since this doesn't change the proof, we keep L_t with subscript t.

Pick $\alpha_t \in (0, a_t/2)$ and then $\rho_t \in (0,1)$ such that $(\tau, v) \in \mathcal{O}$ for every $\tau \in [t-\rho_t, t+\rho_t]$ and $v \in \mathbb{B}_{\alpha_t}(\bar{u}(t))$, and also

$$(34) \qquad \lambda_t L_t \rho_t < 1, \quad L_t\rho_t a_t + 2L_t\rho_t \le b_t, \quad \text{and} \quad 4\lambda_t L_t \rho_t \le \alpha_t(1-\lambda_t L_t \rho_t).$$

Let $\tau \in [t-\rho_t, t+\rho_t] \cap [0,T]$ and define the mapping $g_{\tau,t} : \mathbb{R}^n \to \mathbb{R}^d$ as

$$g_{\tau,t}(v) := f(\bar{x}(\tau),v) - f(\bar{x}(t),v), \quad v \in \mathbb{R}^n.$$

The function $s \mapsto f(\bar{x}(s),\bar{u}(t))$ is Lipschitz continuous on $[0,T]$, hence we have

$$(35) \qquad \|g_{\tau,t}(\bar{u}(t))\| \le L_t|\tau - t| \le L_t\rho_t.$$

Since the function $s \mapsto D_u f(\bar{x}(s), w)$ is Lipschitz continuous on $[0,T]$ uniformly in w from $\mathbb{B}_{a_t}(\bar{u}(t))$, for any $v, v' \in \mathbb{B}_{a_t}(\bar{u}(t))$ we have

$$\begin{aligned}
\|g_{\tau,t}(v) - g_{\tau,t}(v')\| &= \|f(\bar{x}(\tau),v) - f(\bar{x}(\tau),v') - f(\bar{x}(t),v) + f(\bar{x}(t),v')\| \\
&\le \int_0^1 \|D_u f(\bar{x}(\tau), v' + s(v-v')) - D_u f(\bar{x}(t), v' + s(v-v'))\| ds\, \|v'-v\| \\
&\le L_t\rho_t\, \|v'-v\|.
\end{aligned}$$

Let

$$\lambda_t' := 2\lambda_t/(1-\lambda_t L_t\rho_t) \qquad \text{and} \qquad \beta_t := L_t\rho_t.$$

Taking into account (34), we use Theorem 10 with $(a,b,\alpha,\beta,\kappa,\kappa',\mu)$ replaced by $(a_t, b_t, \alpha_t, \beta_t, \lambda_t, \lambda_t', \beta_t)$ obtaining that the mapping

$$\mathbb{B}_{\beta_t}(0) \ni y \mapsto (g_{\tau,t} + G_t)^{-1}(y) \cap \mathbb{B}_{\alpha_t}(\bar{u}(t)) = G_\tau^{-1}(y) \cap \mathbb{B}_{\alpha_t}(\bar{u}(t))$$

is a Lipschitz continuous function on $\mathbb{B}_{\beta_t}(0)$ with Lipschitz constant λ_t', where α_t, β_t and λ_t' defined in the preceding lines do not depend on τ. In particular, there exists exactly one point $w \in \mathbb{B}_{\alpha_t}(\bar{u}(t))$ such that

$0 \in g_{\tau,t}(w) + G_t(w) = G_\tau(w)$. But then $(\tau, w) \in \mathcal{O}$ which is possible only if $w = \bar{u}(\tau)$, by (33). From (35) it follows that $g_{\tau,t}(\bar{u}(t)) \in I\!B_{\beta_t}(0)$. Thus

$$\bar{u}(t) = (g_{\tau,t} + G_t)^{-1}(g_{\tau,t}(\bar{u}(t))) \cap I\!B_{\alpha_t}(\bar{u}(t)).$$

Since $\bar{u}(\tau) = (g_{\tau,t} + G_t)^{-1}(0) \cap I\!B_{\alpha_t}(\bar{u}(t))$, using (35), we conclude that

$$\|\bar{u}(t) - \bar{u}(\tau)\| \le \lambda'_t \|g_{\tau,t}(\bar{u}(t))\| \le \lambda'_t L_t |t - \tau|.$$

Summarizing, we proved that, given $t \in [0, T]$, the function $\bar{u}$ is continuous (even calm) at t. As $t \in [0, T]$ was arbitrary, (ii) is proved. Note that $\bar{u}$ is actually uniformly continuous on $[0, T]$.

To prove that (ii) implies (i), note that if $\bar{u}$ is continuous then its graph is a compact set. Given $t \in [0, T]$, according to Robinson's implicit function theorem [6, Theorems 5F.4] the mapping G_t is strongly metrically regular at $\bar{u}(t)$ for 0 if and only if so is $\mathcal{G}_t$. Hence condition (25) holds with $\mathcal{W}_{(t,\bar{u}(t))} = \mathcal{G}_t$, which in turn, by Theorem 13, implies (i).

Clearly, (iii) implies (ii). To show the converse, we use an argument somewhat parallel to the preceding step but with some important differences. Assume that t, a_t, b_t, λ_t, and L_t are as at the beginning of the proof. Pick $\alpha_t \in (0, a_t/2)$ and then $\rho_t \in (0, 1)$ such that

$$2\lambda_t L_t \rho_t < 1, \quad 2L_t\rho_t a_t + 4L_t\rho_t \le b_t, \quad \text{and} \quad 8\lambda_t L_t \rho_t \le \alpha_t(1 - 2\lambda_t L_t \rho_t); \tag{36}$$

and also that

$$\bar{u}(\tau) \in I\!B_{\alpha_t}(\bar{u}(\theta)) \quad \text{for each} \quad \tau, \theta \in [t - \rho_t, t + \rho_t] \cap [0, T],$$

which is possible thanks to the uniform continuity of $\bar{u}$ on $[0, T]$.

Let τ and θ belong to $[t - \rho_t, t + \rho_t] \cap [0, T]$ and define the mapping $g_{\tau,\theta} : \mathbb{R}^n \to \mathbb{R}^d$ as

$$g_{\tau,\theta}(v) := f(\bar{x}(\tau), v) - f(\bar{x}(\theta), v), \quad v \in \mathbb{R}^n.$$

Since $\bar{u}(\theta) \in I\!B_{\alpha_t}(\bar{u}(t)) \subset I\!B_{a_t}(\bar{u}(t))$, the function $s \mapsto f(\bar{x}(s), \bar{u}(\theta))$ is Lipschitz continuous on $[0, T]$ with constant L_t, which implies that

$$\|g_{\tau,\theta}(\bar{u}(\theta))\| \le L_t |\tau - \theta| \le 2L_t\rho_t. \tag{37}$$

Since the function $s \mapsto D_u f(\bar{x}(s), w)$ is Lipschitz continuous on $[0, T]$ uniformly in w from $I\!B_{a_t}(\bar{u}(t))$, for any v, $v' \in I\!B_{a_t}(\bar{u}(t))$ we have

$$\begin{aligned}
\|g_{\tau,\theta}(v) - g_{\tau,\theta}(v')\| &= \|f(\bar{x}(\tau), v) - f(\bar{x}(\tau), v') - f(\bar{x}(\theta), v) + f(\bar{x}(\theta), v')\| \\
&\le \int_0^1 \|D_u f(\bar{x}(\tau), v' + s(v - v')) - D_u f(\bar{x}(\theta), v' + s(v - v'))\| ds \, \|v' - v\| \\
&\le 2L_t\rho_t \, \|v' - v\|.
\end{aligned}$$

Let $\lambda'_t := 2\lambda_t/(1 - 2\lambda_t L_t \rho_t)$ and $\beta_t := 2L_t\rho_t$. Taking into account (36), we apply Theorem 10 with $(a, b, \alpha, \beta, \kappa, \kappa', \mu)$ replaced by $(a_t, b_t, \alpha_t, \beta_t, \lambda_t, \lambda'_t, \beta_t)$ obtaining that the mapping

$$I\!B_{\beta_t}(0) \ni y \mapsto (g_{\tau,\theta} + G_\theta)^{-1}(y) \cap I\!B_{\alpha_t}(\bar{u}(\theta)) = G_\tau^{-1}(y) \cap I\!B_{\alpha_t}(\bar{u}(\theta))$$

is a Lipschitz continuous function on $I\!B_{\beta_t}(0)$ with Lipschitz constant λ'_t, where α_t, β_t and λ'_t defined in the preceding lines do not depend on τ and θ. Since $\bar{u}(\tau) \in I\!B_{\alpha_t}(\bar{u}(\theta))$, we have $\bar{u}(\tau) = G_\tau^{-1}(0) \cap I\!B_{\alpha_t}(\bar{u}(\theta))$. From (37) it follows that $g_{\tau,\theta}(\bar{u}(\theta)) \in I\!B_{\beta_t}(0)$. Thus $\bar{u}(\theta) = G_\tau^{-1}(g_{\tau,\theta}(\bar{u}(\theta))) \cap I\!B_{\alpha_t}(\bar{u}(\theta))$. Using (37), we conclude that

$$\|\bar{u}(\theta) - \bar{u}(\tau)\| \le \lambda'_t \|g_{\tau,\theta}(\bar{u}(\theta))\| \le \lambda'_t L_t |\theta - \tau|. \tag{38}$$

Summarizing, we proved that, given $t \in [0, T]$, the function $\bar{u}$ is locally Lipschitz continuous around t. Since $[0, T]$ is compact, we obtain condition (iii). ∎

REMARK 3.1. Observe that in the last three theorems $\bar{x}$ does not need to be a solution of (10). It may be any Lipschitz continuous function from $[0, T]$ to $\mathbb{R}^m$ for which condition (25) holds.

For a given positive constant c define the set

$$S_c := \{(z,t,q) \in \mathbb{R}^{m+1+n} \mid t \in [0,T], \|z\| \le c, \|q\| \le c\}.$$

LEMMA 16. *Suppose that condition (25) holds and let the constants a, b, and κ be as in Corollary 12. Then for every $c > 0$ such that $c(\|H\|_C + 1) \le b$ the mapping*

$$S_c \ni (z,t,q) \mapsto u(z,t,q) := \{u \in I\!B_a(\bar{u}(t)) \mid q \in \bar{f}(t) + H(t)z + E(t)(u - \bar{u}(t)) + F(u)\}$$

is a function which is bounded and measurable in t for each (z,q) and Lipschitz continuous with respect to (z,q) uniformly in t with Lipschitz constant $\lambda := \kappa(\|H\|_C + 1)$.

Proof. Choose c as required. Clearly, for each $(z,t,q) \in S_c$ we have $q - H(t)z \in I\!B_b(0)$, and hence, by definition,

$$u(z,t,q) = \mathcal{G}_t^{-1}(q - H(t)z) \cap I\!B_a(\bar{u}(t)).$$

By Robinson's implicit function theorem [6, Theorem 2B.5] the function $(y,t) \mapsto \mathcal{G}_t^{-1}(y)$ is Lipschitz continuous on $[0,T] \times I\!B_b(0)$. Therefore the function $[0,T] \ni t \mapsto u(z,t,q)$ is measurable and bounded for each $\{(z,q) \mid (z,t,q) \in S_c\}$ as a composition of a Lipschitz function with a measurable and bounded function; furthermore, for every $(z_1,t,q_1), (z_2,t,q_2) \in S_c$ we get

$$\|u(z_1,t,q_1) - u(z_2,t,q_2)\| \le \kappa(\|q_1 - q_2\| + \|H(t)(z_1 - z_2)\|) \le \lambda(\|z_1 - z_2\| + \|q_1 - q_2\|).$$

Thus, u has the desired property. ∎

THEOREM 17. *Suppose that condition (25) is satisfied. Then the mapping M defined in (12) is strongly metrically regular at $(\bar{x},\bar{u})$ for 0. If, in addition, one of the equivalent statements (i)–(iii) in Theorem 15 holds, then the mapping M, now considered as acting from $C^1 \times C$ to the subsets of $C \times \mathbb{R}^m \times C$, is strongly metrically regular at $(\bar{x},\bar{u})$ for 0.*

Proof. Let the constants a, b and κ be as in Corollary 12, let λ be as in Lemma 16, and let

$$\nu_0 := \max\{\|A\|_C, \|B\|_C, \|H\|_C, \|E\|_C\} \quad \text{and} \quad c \le b/(\nu_0 + 1). \tag{39}$$

From Lemma 16, for any $(z,t,q) \in S_c$ the inclusion

$$q \in \bar{f}(t) + H(t)z + E(t)(u - \bar{u}(t)) + F(u) \tag{40}$$

has a unique solution $u(z,t,q) \in I\!B_a(\bar{u}(t))$; moreover, the function $S_c \ni (z,t,q) \mapsto u(z,t,q)$ is measurable in t for each (z,q) and Lipschitz continuous in (z,q) with Lipschitz constant λ. Observe that $u(0,t,0) = \bar{u}(t)$ for all $t \in [0,T]$.

From Corollary 9 we know that the mapping M defined in (12) is strongly metrically regular at $(\bar{x},\bar{u})$ for 0 if and only if the mapping $\mathcal{M}$ defined in (13) is strongly metrically regular at $(0,\bar{u})$ for 0. Choose $\delta > 0$ such that

$$e^{(1+\lambda)\nu_0 T}((\nu_0\lambda + 1)T + 1)\delta < c \tag{41}$$

and also $q \in L^\infty([0,T],\mathbb{R}^d)$, $y \in \mathbb{R}^m$ and $r \in L^\infty([0,T],\mathbb{R}^m)$ with $\|q\|_\infty \le \delta$, $\|y\| \le \delta$, $\|r\|_\infty \le \delta$. Consider the initial value problem

$$\dot{z}(t) = A(t)z(t) + B(t)(u(z(t),t,q(t)) - \bar{u}(t)) + r(t) \quad \text{for a.e.} \quad t \in [0,T], \quad z(0) = y. \tag{42}$$

Since the right side of this differential equation is a Carathèodory function which is Lipschitz continuous in z, and also the initial condition $z(0) = y \in \operatorname{int} I\!B_c(0)$, by a standard argument there is a maximal interval $[0,\tau] \subset [0,T]$ in which there exists a solution z of (42) on $[0,\tau]$ with values in $I\!B_c(0)$ and if $\tau < T$ then $\|z(\tau)\| = c$. Let $\tau < T$. But then for $t \in [0,\tau]$ we have

$$\|z(t)\| \le \|y\| + \int_0^t \big(\nu_0\|z(s)\| + \nu_0\lambda(\delta + \|z(s)\|) + \delta\big)ds.$$

Hence, by applying the Grönwall lemma and using (41), we get

$$\|z(t)\| \le e^{(1+\lambda)\nu_0 T}((\nu_0\lambda + 1)T + 1)\delta < c,$$

which contradicts the assumption that $\tau < T$. Hence $\tau = T$ and there exists a solution z of problem (42) on the entire interval $[0,T]$ such that $z(t) \in \operatorname{int} \mathbb{B}_c(0)$ for each $t \in [0,T]$. Then for $u(t) := u(z(t), t, q(t))$, $t \in [0,T]$ we obtain that $(u,z) := (u(t), z(t))$ satisfies (40) for almost every $t \in [0,T]$. In conclusion, for each $(r,q) : [0,T] \to \mathbb{R}^{m+d}$ and $y \in \mathbb{R}^m$ with $\|r\|_\infty\| \le \delta$, $\|q\|_\infty \le \delta$ and $\|y\| \le \delta$ there exists a unique solution $(u,z) \in L^\infty \times W^{1,\infty}$ of the perturbed system

$$\begin{aligned} &\dot{z}(t) = A(t)z(t) + B(t)(u(t) - \bar{u}(t)) + r(t), \quad z(0) = y, \\ &0 \in \bar{f}(t) + H(t)z(t) + E(t)(u(t) - \bar{u}(t)) + q(t) + F(u(t)), \end{aligned} \tag{43}$$

for a.e. $t \in [0,T]$, such that $\|u - \bar{u}\|_\infty \le a$ and $\|z\|_C \le c$.

In the last part of the proof we show Lipschitz continuity of the solution $(u,z) \in L^\infty \times W^{1,\infty}$ of the perturbed system (43) with respect to $(r,y,q) \in L^\infty \times \mathbb{R}^m \times L^\infty$, $\|r\|_\infty \le \delta$, $\|y\| \le \delta$, $\|q\|_\infty \le \delta$. From now on through the end of the proof $\gamma > 0$ is a generic constant which may change in different relations. Choose $(r_i, q_i) \in L^\infty([0,T], \mathbb{R}^{m+d})$ and $y_i \in \mathbb{R}^m$ such that $\|r_i\|_\infty \le \delta$, $\|q_i\|_\infty \le \delta$, $\|y_i\| \le \delta$, and let (z_i, u_i), be the solutions of (43) associated with (r_i, y_i, q_i), $i = 1,2$. Due to (39), for $i = 1,2$ we have

$$-q_i(t) - H(t)z_i(t) \in \mathbb{B}_b(0) \quad \text{ for a.e. } t \in [0,T]$$

and hence

$$u_i(t) = \mathcal{G}_t^{-1}(-q_i(t) - H(t)z_i(t)) \cap \mathbb{B}_a(\bar{u}(t)) \text{ for a.e. } t \in [0,T].$$

Therefore

$$\|u_1(t) - u_2(t)\| \le \kappa\nu_0\|z_1(t) - z_2(t)\| + \kappa\|q_1(t) - q_2(t)\| \text{ for a.e. } t \in [0,T]. \tag{44}$$

Plugging (44) into the integral form of the differential equation in (43), we get

$$\begin{aligned} \|z_1(t) - z_2(t)\| &\le \|y_1 - y_2\| + \int_0^t (\nu_0\|z_1(\tau) - z_2(\tau)\| + \nu_0\|u_1(\tau) - u_2(\tau)\| + \|r_1(\tau) - r_2(\tau)\|)d\tau \\ &\le \|y_1 - y_2\| + \int_0^t \nu_0(1 + \kappa\nu_0)\|z_1(\tau) - z_2(\tau)\| + \kappa\nu_0\|q_1(\tau) - q_2(\tau)\| \\ &\qquad + \|r_1(\tau) - r_2(\tau)\|)d\tau \quad \text{for every } t \in [0,T]. \end{aligned}$$

The Grönwall lemma yields that

$$\|z_1(t) - z_2(t)\| \le \gamma(\|y_1 - y_2\| + \|q_1 - q_2\|_\infty + \|r_1 - r_2\|_\infty) \quad \text{for every} \quad t \in [0,T]. \tag{45}$$

Then (45) substituted in (44) results in

$$\|u_1 - u_2\|_\infty \le \gamma(\|y_1 - y_2\| + \|q_1 - q_2\|_\infty + \|r_1 - r_2\|_\infty). \tag{46}$$

Substituting (45) and (46) in the state equation gives us

$$\|\dot{z}_1 - \dot{z}_2\|_\infty \le \gamma(\|y_1 - y_2\| + \|q_1 - q_2\|_\infty + \|r_1 - r_2\|_\infty).$$

This proves the first part of the theorem.

As for the second part, since in this case $\bar{u}$ is Lipschitz continuous on $[0,T]$, it is sufficient to repeat the above argument changing the L^∞ norm to the C norm, obtaining

$$\|z_1 - z_2\|_C \le \gamma(\|y_1 - y_2\| + \|q_1 - q_2\|_C + \|r_1 - r_2\|_C). \tag{47}$$

Then, from (44) which is valid for all $t \in [0,T]$, we have

$$\|u_1 - u_2\|_C \le \gamma(\|y_1 - y_2\| + \|q_1 - q_2\|_C + \|r_1 - r_2\|_C). \tag{48}$$

Finally, utilizing (47) and (48) in the differential equation we obtain

$$\|\dot{z}_1 - \dot{z}_2\|_C \le \gamma(\|y_1 - y_2\| + \|q_1 - q_2\|_C + \|r_1 - r_2\|_C).$$

This ends the proof. ∎

REMARK 3.2. Note that, by Robinson's theorem, strong metric regularity in L^∞ of the mapping M implies Lipschitz dependence in L^∞ of the control u with respect to perturbations, which yields restrictions on the behavior of u as a function of time. Suppose that the problem in hand is perturbed; then as a consequence of the strong metric regularity, the control for the perturbed problem must be close to $\bar{u}$ in L^∞ which means that it has to have jumps at the same instants of time as $\bar{u}$. If we assume a bit more, namely the local isolatedness of $\bar{u}$, then the function $\bar{u}$ becomes Lipschitz continuous. In the paper [9] we considered a variational inequality of the form (2) without the state variable x and used a condition which is stronger than (25), namely that *each point of the graph of the associated solution mapping is a point of strong metric regularity*. In this case it turned out that there are finitely many Lipschitz continuous functions whose graphs do not intersect each other such that for each value of the parameter the set of values of the solution mapping is the union of the values of these functions. Here we assume less, focusing on a particular solution $\bar{u}$ but still the strong metric regularity imposes restrictions on the way the solution depends on perturbations.

4. Regularity in optimal control. Consider the optimal control problem (6) and the associated optimality system (7) with a reference solution $(\bar{y}, \bar{p}, \bar{u})$. We assume for simplicity that $y_0 = 0$ and $\varphi \equiv 0$. In further lines we use the notation $A(t) = D_{py}\bar{H}(t)$, $B(t) = D_{pu}\bar{H}(t)$, $Q(t) = D_{yy}\bar{H}(t)$, $S(t) = D_{uy}\bar{H}(t)$, $R(t) = D_{uu}\bar{H}(t)$ for the corresponding derivatives of the Hamiltonian H, where the bar means that the function is evaluated at $(\bar{y}(t), \bar{p}(t), \bar{u}(t))$.

We start with a result regarding the Lipschitz continuity of the optimal control $\bar{u}$ with respect to time t, which is a consequence of Theorem 15 and also [6, Theorem 2C.2].

THEOREM 18. *Let $\bar{u}$ be an optimal control for problem (6) which is measurable and bounded on $[0,T]$ and also an isolated solution of the variational inequality*

$$0 \in \mathcal{H}_t(v) := D_u H(\bar{y}(t), \bar{p}(t), v) + N_U(v), \tag{49}$$

where $\bar{y}$ and $\bar{p}$ are the associated optimal state and adjoint variables. Assume that for each $t \in [0,T]$ the mapping $\mathcal{H}_t$ is strongly metrically regular at $\bar{u}(t)$ for 0. Then the optimal control $\bar{u}$ is Lipschitz continuous in t on $[0,T]$.

In addition, let $n = 1$ and suppose that

$$S(t)\bar{g}(t) - B^T(t)D_y\bar{H}(t) \neq 0 \quad \text{for every } t \in [0,T]. \tag{50}$$

Then the converse statement holds as well: if $\bar{u}$ is Lipschitz continuous in $[0,T]$ then for each $t \in [0,T]$ the mapping $\mathcal{H}_t$ is strongly metrically regular at $\bar{u}(t)$ for 0.

Proof. The first part of the statement readily follows from Theorem 15 (see also Remark 3.1). As for the second part, let $\bar{u}$ be Lipschitz continuous on $[0,T]$. Then for each $t \in [0,T]$, by using the assumption that $\bar{u}$ is an isolated solution, the mapping $t \mapsto \{v \mid 0 \in \mathcal{H}_t(v)\}$ has a single-valued localization around t for $\bar{u}(t)$. This in turn implies strong metric regularity of the mapping $\mathcal{H}_t$ at $\bar{u}(t)$ for 0 is provided that the so-called *ample parameterization condition* is satisfied, see [6, Theorem 2C.2]. In the specific case of (7) this condition has the form:

$$\operatorname{rank}\left[S(t)\dot{\bar{y}}(t) + B^T(t)\dot{\bar{p}}(t)\right] = n \quad \text{for every } t \in [0,T]. \tag{51}$$

Since $n = 1$ and on the left side we have a single vector, condition (51) is equivalent to condition (50). ∎

Consider next the mapping appearing in the optimality system (7):

$$W_0^{1,\infty} \times W_T^{1,\infty} \times L^\infty \ni (y,p,u) \mapsto P(y,p,u) := \begin{pmatrix} \dot{y} - g(y,u) \\ \dot{p} + D_y H(y,p,u) \\ D_u H(y,p,u) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ N_U(u) \end{pmatrix}. \tag{52}$$

where $W_T^{1,\infty} = \{p \in W^{1,\infty} \mid p(T) = 0\}$. The associated linearized mapping has the form

$$W_0^{1,\infty} \times W_T^{1,\infty} \times L^\infty \ni (z,q,u) \mapsto \mathcal{P}(z,q,u) :=$$

$$\begin{pmatrix} \dot{z} - Az - B(u-\bar{u}) \\ \dot{q} + Qz + A^T q + S^T(u - \bar{u}) \\ Sz + B^T q + R(u - \bar{u}) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ N_U(u) \end{pmatrix}.$$

As a final result of this section we adopt [7, Theorem 5] to present a sufficient condition for strong metric regularity of the mapping P or, equivalently, the mapping $\mathcal{P}$. This results also serves as an example which illustrates that strong metric regularity can be deduced from the well-known *strong second-order sufficient optimality condition*, sometimes also called *coercivity*. This condition basically requires positive definiteness of a quadratic form on a subspace, and in principle can be checked numerically.

In the statement below L^2 is the usual Lebesque space of measurable and square integrable functions while $W^{1,2}$ is the space of functions x with both x and the derivative $\dot{x}$ in L^2.

THEOREM 19. *Suppose that $\bar{y} \in W_0^{1,\infty}$, $\bar{p} \in W_T^{1,\infty}$, $\bar{u} \in L^\infty$ and consider the mapping P defined in (52) acting from $W_0^{1,\infty} \times W_T^{1,\infty} \times L^\infty$ to the subsets of L^∞. Suppose that the following condition is satisfied: there exists $\alpha > 0$ such that*

$$\int_0^T (y(t)^T Q(t) y(t) + u(t)^T R(t) u(t) + 2y(t)^T S(t) u(t)) dt \geq \alpha \int_0^T \|u(t)\|^2 dt \tag{53}$$

whenever $y \in W^{1,2}, y(0) = 0, u \in L^2, \dot{y} = Ay + Bu, u = v - w$ for some v, $w \in L^2$ with values $v(t)$, $w(t) \in U$ for a.e. $t \in [0,T]$. Then the mapping P in (52) is strongly metrically regular at $(\bar{y}, \bar{p}, \bar{u})$ for 0.

Proof. According to [7, Theorem 5], condition (53) implies that the linearized mapping $\mathcal{P}$ is strongly metrically regular at $(0, 0, \bar{u})$ for 0. Then, by applying Robinson's theorem as in Corollary 9 we obtain the conclusion. ∎

Note that the Remark 3.2 applies also here; having strong metric regularity in L^∞ imposes restrictions on the way the optimal control behaves as a function of time. Also note that the coercivity condition (53) implies pointwise coercivity, namely $u^T R(t) u \geq \alpha \|u\|^2$ for all $u \in U - U$ and a.e. $t \in [0,T]$. But then, if we assume that the components of R, B, S are continuous functions, we will end up with the reference control $\bar{u}$ being Lipschitz continuous on $[0,T]$.

There is a wealth of literature on Lipschitz stability in optimal control, where strong metric regularity plays a major role. Alt [1] was the first to employ strong metric regularity in nonlinear optimal control; his results were broadly extended in [7]. In a series of papers, see e.g. [21], Malanowski studied various optimal control problems including problems with inequality state and control constraints. A characterization of strong metric regularity for an optimal control problem with inequality control constraints is obtained in [10]. For recent results in this direction, see [3], [12], [24] and the references therein.

5. Discrete approximations and path-following. As an application of the analysis given in the preceding two sections, in this section we study a time-stepping procedure for solving the DGE considered in Section 3, namely

$$\dot{x}(t) = g(x(t), u(t)), \qquad x(0) = 0, \tag{54}$$

$$f(x(t), u(t)) + F(u(t)) \ni 0 \qquad \text{for all } t \in [0,T]. \tag{55}$$

Let N be a natural number and let the interval $[0,T]$ be divided into N subintervals $[t_k, t_{k+1}]$, with $t_0 = 0, t_N = T$, and with equal step-size $h = T/N$, that is, $t_{k+1} = t_k + h$, $k = 0, 1, \ldots, N-1$. Consider the following iteration: starting from some (x_0, u_0), given (x_k, u_k) at time t_k obtain the next iterate (x_{k+1}, u_{k+1}) associated with time t_{k+1} as a solution of the system

$$x_{k+1} = x_k + hg(x_k, u_k), \tag{56}$$

$$f(x_{k+1}, u_k) + D_u f(x_{k+1}, u_k)(u_{k+1} - u_k) + F(u_{k+1}) \ni 0, \tag{57}$$

for $k = 0, 1, \ldots, N-1$. Note that (56) determines x_{k+1} by an Euler step from (x_k, u_k) for the differential equation (54). Having x_{k+1}, the control iterate u_{k+1} is obtained as a solution of the linear generalized equation (57) which is a Newton-type step for the discretized generalized equation (55). The iteration (56)–(57) resembles an Euler-Newton path-following (time-stepping) procedure aiming at obtaining a sequence $\{(x_k, u_k)\}_{k=0}^N$ which represents a discrete approximation of a solution to the original DGE (54)–(55). The following theorem gives conditions under which the iteration (56)–(57) produces an approximate solution which is at distance $O(h)$ from the reference solution $(\bar{x}, \bar{u})$.

THEOREM 20. *Consider the DGE (54)–(55) with a reference solution $(\bar{x}, \bar{u})$ at which condition (25) holds together with one of the equivalent statements (i)–(iii) in Theorem 15. Then there exist a natural number N_0 and positive reals $\bar{d}$, α and $\bar{c}$ such that for each $N \geq N_0$, if the starting point is chosen to satisfy*

$$x_0 = 0 \quad \textit{and} \quad \|u_0 - \bar{u}(0)\| \leq \bar{d}h, \tag{58}$$

then the iteration (56)–(57) generates a sequence $\{(x_k, u_k)\}_{k=0}^N$ such that

$$(x_k, u_k) \in \mathbb{B}_\alpha((\bar{x}(t_k), \bar{u}(t_k))), \quad k = 1, \ldots, N;$$

in addition, there is no other sequence in $\mathbb{B}_\alpha((\bar{x}(t_k), \bar{u}(t_k)))$ generated by the method. Moreover, the following error estimates hold:

$$\max_{0 \leq k \leq N} \|u_k - \bar{u}(t_k)\| \leq \bar{d}(\bar{c}+1)h \qquad \textit{and} \qquad \max_{0 \leq k \leq N} \|x_k - \bar{x}(t_k)\| \leq \bar{c}h. \tag{59}$$

Proof. According to Theorem 13 the mapping $v \mapsto G_t(v) = f(\bar{x}(t), v) + F(v)$ is strongly metrically regular at $\bar{u}(t)$ for 0 uniformly in $t \in [0, T]$; that is, there exist positive reals a, b and κ such that for each $t \in [0, T]$ the mapping $\mathbb{B}_b(0) \mapsto G_t^{-1}(y) \cap \mathbb{B}_a(\bar{u}(t))$ is a Lipschitz continuous function with Lipschitz constant κ. Furthermore, from the assumed twice continuous differentiability of g and f there exists $\nu_1 > 0$ such that for every $t \in [0, T]$, every $x \in \mathbb{B}_a(\bar{x}(t))$, and every $u \in \mathbb{B}_a(\bar{u}(t))$ we have

$$\|f(x, u) - f(\bar{x}(t), \bar{u}(t))\| \leq \nu_1(\|x - \bar{x}(t)\| + \|u - \bar{u}(t)\|), \tag{60}$$

$$\|g(x, u) - g(\bar{x}(t), \bar{u}(t))\| \leq \nu_1(\|x - \bar{x}(t)\| + \|u - \bar{u}(t)\|); \tag{61}$$

and also that, for every $t \in [0, T]$, every $x, x' \in \mathbb{B}_a(\bar{x}(t))$ and every $u, u' \in \mathbb{B}_a(\bar{u}(t))$,

$$\|D_u f(x, u) - D_u f(x', u')\| \leq \nu_1(\|x - x'\| + \|u - u'\|). \tag{62}$$

By Theorem 15 , the function $t \to (\bar{x}(t), \bar{u}(t))$ is Lipschitz continuous on $[0, T]$, hence there exists $\nu_2 > 0$ such that

$$\|\bar{x}(s) - \bar{x}(t)\| + \|\bar{u}(s) - \bar{u}(t)\| \leq \nu_2 |t - s| \quad \text{ for all } t, s \in [0, T].$$

Let

$$\kappa' := 4\kappa, \quad \mu := 1/(2\kappa), \quad \text{and} \quad \nu := \max\{1, \nu_1, \nu_2, \kappa'\}, \tag{63}$$

and then set

$$\alpha := \min\{1, a/2, 1/(16\kappa\nu), 4b\kappa/5\} \quad \text{and} \quad \beta := 2\alpha^2\nu. \tag{64}$$

In the next step of the proof we prove the following claim:

$$\begin{array}{c} \text{Given } t \in [0, T], x \in \mathbb{B}_{\alpha^2}(\bar{x}(t)), \text{ and } u \in \mathbb{B}_\alpha(\bar{u}(t)) \\ \text{there is a unique } \tilde{u} \in \mathbb{B}_\alpha(\bar{u}(t)) \text{ such that} \\ f(x, u) + D_u f(x, u)(\tilde{u} - u) + F(\tilde{u}) \ni 0 \\ \text{and } \ \|\tilde{u} - \bar{u}(t)\| \leq \nu^2(\|u - \bar{u}(t)\|^2 + \|x - \bar{x}(t)\|). \end{array} \tag{65}$$

Fix t, x and u as required and consider the function

$$\mathbb{R}^n \ni v \mapsto \Psi(v) = \Psi_{t,x,u}(v) := f(x, u) + D_u f(x, u)(v - u) - f(\bar{x}(t), v) \in \mathbb{R}^d.$$

We utilize Theorem 10 with $(\bar{x}, \bar{y}, F, g)$ replaced by $(\bar{u}(t), 0, G_t, \Psi)$. By (63), $\kappa\mu < 1$ and $\kappa' > 2\kappa = \kappa/(1 - \mu\kappa)$. From (63) and (64) we get

$$\alpha \leq a/2, \quad 2\kappa'\beta = (16\kappa\nu\alpha)\alpha \leq \alpha,$$

and

$$2\mu\alpha + 2\beta = \frac{\alpha}{\kappa} + (4\alpha\nu)\alpha \leq \frac{\alpha}{\kappa} + \frac{\alpha}{4\kappa} = \frac{5\alpha}{4\kappa} \leq b.$$

To apply Theorem 10 we need to show that

$$\|\Psi(\bar u(t))\| < \beta \quad \text{and} \quad \|\Psi(v) - \Psi(v')\| \le \mu\|v - v'\| \quad \text{whenever} \quad v, v' \in \mathbb{B}_{2\alpha}(\bar u(t)). \tag{66}$$

Noting that $x \in \mathbb{B}_{\alpha^2}(\bar x(t)) \subset \mathbb{B}_a(\bar x(t))$ and $u + s(\bar u(t) - u) \in \mathbb{B}_\alpha(\bar u(t)) \subset \mathbb{B}_a(\bar u(t))$ for any $s \in [0,1]$, using (60) and (62) we obtain

$$\begin{aligned} \|\Psi(\bar u(t))\| &= \|f(x,u) + D_u f(x,u)(\bar u(t) - u) - f(\bar x(t), \bar u(t))\| \\ &\le \|f(x,u) - f(x, \bar u(t)) + D_u f(x,u)(\bar u(t) - u)\| \\ &\quad + \|f(x, \bar u(t)) - f(\bar x(t), \bar u(t))\| \\ &\le \textstyle\int_0^1 \|[D_u f(x,u) - D_u f(x, u + s(\bar u(t) - u))](\bar u(t) - u)\| ds + \nu\|x - \bar x(t)\| \\ &\le \nu\|\bar u(t) - u\|^2 \textstyle\int_0^1 s ds + \nu\|x - \bar x(t)\|. \end{aligned} \tag{67}$$

Consequently, $\|\Psi(\bar u(t))\| \le \frac{1}{2}\nu\alpha^2 + \nu\alpha^2 < 2\nu\alpha^2 = \beta$, which is the first inequality in (66). Pick any v, $v' \in \mathbb{B}_{2\alpha}(\bar u(t)) \subset \mathbb{B}_a(\bar u(t))$. Then $v' + s(v - v') \in \mathbb{B}_{2\alpha}(\bar u(t))$ for every $s \in [0,1]$ and $\sup_{s\in[0,1]} \|u - [v' + s(v - v')]\| \le 3\alpha$. Therefore, from (62),

$$\begin{aligned} \|\Psi(v) - \Psi(v')\| &= \|D_u f(x,u)(v - v') - [f(\bar x(t), v) - f(\bar x(t), v')]\| \\ &\le \int_0^1 \|[D_u f(x,u) - D_u f(\bar x(t), v' + s(v - v'))](v - v')\| ds \\ &\le \nu(\|x - \bar x(t)\| + \sup_{s\in[0,1]} \|u - v' - s(v - v')\|)\,\|v - v'\| \\ &\le \nu(\alpha^2 + 3\alpha)\,\|v - v'\| \le 4\alpha\nu\|v - v'\|. \end{aligned}$$

Since $4\alpha\nu \le 1/(4\kappa) < \mu$ by (64), the second inequality in (66) follows. Then Theorem 10 implies that the mapping

$$\mathbb{B}_\beta(0) \ni y \mapsto (f(\bar x(t), \cdot) + \Psi + F)^{-1}(y) \cap \mathbb{B}_\alpha(\bar u(t)) \tag{68}$$

is a Lipschitz continuous function with Lipschitz constant κ' on $\mathbb{B}_\beta(0)$. In particular, there is a unique solution $\tilde u$ in $\mathbb{B}_\alpha(\bar u(t))$ of

$$f(\bar x(t), v) + \Psi(v) + F(v) \ni 0.$$

Taking into account that $\bar u(t)$ is the unique solution in $\mathbb{B}_\alpha(\bar u(t))$ of

$$f(\bar x(t), v) + \Psi(v) + F(v) \ni \Psi(\bar u(t)),$$

and the first inequality in (66), we conclude that

$$\|\tilde u - \bar u(t)\| \le \kappa'\|\Psi(\bar u(t))\|.$$

Using (67) and the fact that $\kappa' \le \nu$, we complete the proof of (65).

Set

$$\bar d := \nu^2, \quad \lambda := \max\{\nu(1 + \bar d), \nu(\nu + \bar d)\}, \quad \text{and} \quad \bar c := T\lambda e^{\lambda T}. \tag{69}$$

Next, choose an integer $N_0 > T$ so that

$$T\bar c \le \alpha^2 N_0 \quad \text{and} \quad T\big(\bar d(2 + \bar c)\big)^2 \le \alpha N_0. \tag{70}$$

Let $N \ge N_0$ and let $h := T/N$. Then we have $h < 1$ and from (70),

$$\bar c h \le \alpha^2 \quad \text{and} \quad \big(\bar d(2 + \bar c)\big)^2 h \le \alpha. \tag{71}$$

Let $c_i := \lambda i h e^{\lambda i h}$, $i = 0, 1, \ldots, N$. We will show that the iteration (56)–(57) is sure to generate points $\{(x_k, u_k)\}_{k=0}^N$ that satisfy the following inequalities:

$$\|x_i - \bar x(t_i)\| \le c_i h \quad \text{and} \quad \|u_i - \bar u(t_i)\| \le \bar d(1 + c_i)h \qquad \text{for } i = 0, 1, \ldots, N. \tag{72}$$

Let (x_0,u_0) satisfy (58); since $c_0=0$, (72) hold for $i=0$. Now assume that for some $k<N$ the point (x_k,u_k) satisfies (72) for $i=k$. We will find a point (x_{k+1},u_{k+1}) generated by (56)–(57) such that inequalities (72) hold for $i=k+1$. Define x_{k+1} by (56). Clearly, $\bar c=\max_{0\le i\le N} c_i$. By (71) and (64), we have $x_k\in \mathbb{B}_a(\bar x(t_k))$ and $u_k\in \mathbb{B}_a(\bar u(t_k))$. Since $\nu\ge\ 1$, the second inequality in (71) implies that

$$\nu h\le \nu^4 h=\bar d^2 h<\big(\bar d(2+\bar c)\big)^2 h\le \alpha\le a/2.$$

Therefore $\bar x(s)\in \mathbb{B}_a(\bar x(t_k))$ and $\bar u(s)\in \mathbb{B}_a(\bar u(t_k))$ for all $s\in[t_k,t_{k+1}]$. Then, using (61),

$$\begin{aligned}
\|x_{k+1}-\bar x(t_{k+1})\| &= \left\|x_k+hg(x_k,u_k)-\bar x(t_k)-\int_{t_k}^{t_{k+1}} g(\bar x(s),\bar u(s))ds\right\|\\
&\le \|x_k-\bar x(t_k)\|+\left\|\int_{t_k}^{t_{k+1}}(g(\bar x(s),\bar u(s))-g(x_k,u_k))ds\right\|\\
&\le c_kh+\int_{t_k}^{t_{k+1}}\left(\|g(\bar x(s),\bar u(s))-g(\bar x(t_k),\bar u(t_k))\|+\|g(\bar x(t_k),\bar u(t_k))-g(x_k,u_k)\|\right)ds\\
&\le c_kh+\int_{t_k}^{t_{k+1}}\nu(\|\bar x(s)-\bar x(t_k)\|+\|\bar u(s)-\bar u(t_k)\|+\|\bar x(t_k)-x_k\|+\|\bar u(t_k)-u_k\|)ds\\
&\le c_kh+\nu\int_{t_k}^{t_{k+1}}(2\nu(s-t_k)+c_kh+\bar dh(c_k+1))ds\\
&= c_kh+\nu h^2(c_k+\bar d(c_k+1))+\nu^2h^2=c_kh(1+\nu(1+\bar d)h)+h^2\nu(\bar d+\nu)\\
&\le c_kh(1+\lambda h)+h^2\lambda=h^2\lambda ke^{kh\lambda}(1+\lambda h)+h^2\lambda\\
&\le h^2\lambda ke^{(k+1)h\lambda}+h^2\lambda e^{(k+1)h\lambda}=h^2\lambda(k+1)e^{(k+1)h\lambda}=c_{k+1}h.
\end{aligned}$$

In particular, from the first inequality in (71), we get

$$\|x_{k+1}-\bar x(t_{k+1})\|\le \bar ch\le \alpha^2.$$

Since $\nu\ge 1$, we also have

$$\begin{array}{rcl}\|u_k-\bar u(t_{k+1})\| &\le& \|u_k-\bar u(t_k)\|+\|\bar u(t_k)-\bar u(t_{k+1})\|\le \bar d(1+c_k)h+\nu h\\ &<& \bar d(2+\bar c)h<(\bar d(2+\bar c))^2h\le\alpha.\end{array} \tag{73}$$

Using (65) with $(t,x,u):=(t_{k+1},x_{k+1},u_k)$ we obtain that there is u_{k+1} which is unique in $\mathbb{B}_\alpha(\bar u(t_{k+1}))$ and satisfies (57). Combining the estimate from (65), (73), and the second inequality in (71), we get that

$$\begin{aligned}
\|u_{k+1}-\bar u(t_{k+1})\| &\le \nu^2(\|u_k-\bar u(t_{k+1})\|^2+\|x_{k+1}-\bar x(t_{k+1})\|)\\
&\le \nu^2\big((\bar d(1+c_k)h+\nu h)^2+c_{k+1}h\big)\\
&= \nu^2h\big(c_{k+1}+(\bar d(1+c_k)+\nu)^2h\big)<\nu^2h\big(c_{k+1}+(\bar d(2+\bar c))^2h\big)\\
&\le \nu^2h(c_{k+1}+\alpha)\le \bar dh(c_{k+1}+1).
\end{aligned}$$

The induction step is complete and so is the proof. ∎

The obtained error estimate of order $O(h)$ is sharp in the sense that the optimal control $\bar u$ is at most a Lipschitz continuous function of time in the presence of constraints. If however, $\bar u$ has better smoothness properties, in line with the analysis in [8], by applying a Runge-Kutta scheme to the differential equation (54) and an adjusted Newton iteration to the generalized equation (55) would lead to a higher-order accuracy. This topic is left for future research.

Finally, we note that time-stepping procedures for solving DVIs have been considered already in [23], see also the more recent papers [5] and [27] dealing with various discretization schemes. An extensive overview to time-stepping strategies for time-dependent variational inequalities is presented in [9]. The Euler-Newton path following procedure we deal here is different from the time-stepping schemes considered in those papers and the error estimate obtained is a first result in the direction of rigorous numerical analysis of dynamical systems of the kind of DGE.

A numerical example. As an illustration we consider a slight (nonlinear) modification of the model of a half-ware rectifier considered in [28, Chapter 1.3.1]. It consists of the differential variational system

$$\dot{\bar{x}}(t) = \begin{pmatrix} -0.5 & -1 \\ 2 & 0 \end{pmatrix} \bar{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t),$$
$$x_1(t) + \arctan(u(t)) \in F(u(t)),$$

where $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$, and

$$F(u) = \begin{cases} \emptyset & \text{if } u < 0, \\ [0, +\infty) & \text{if } u = 0, \\ \{0\} & \text{if } u > 0. \end{cases}$$

We mention that the inclusion in the above system is equivalent to the complementarity condition

$$0 \le (x_1(t) + \arctan(u(t))) \perp u(t) \ge 0.$$

The graphs of the exact solution $(\bar{x}(t), u(t))$ and of two approximate solutions are presented in Fig. 6.1.

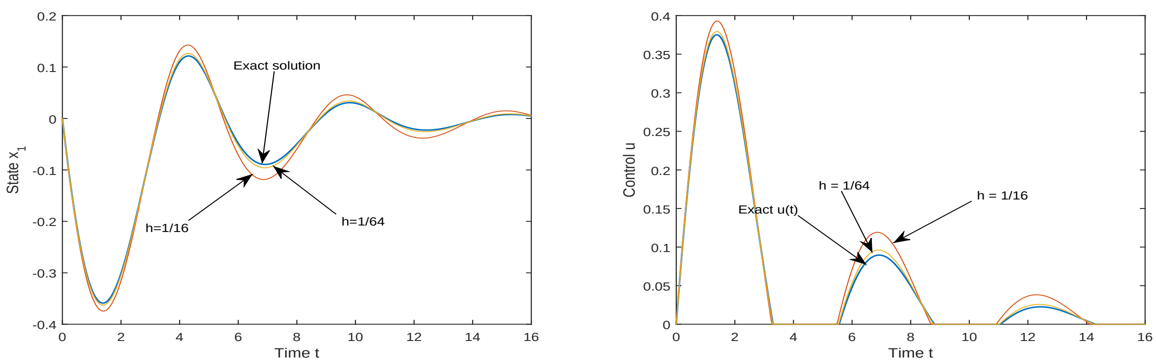

FIG. 6.1. *The exact solution (state x_1 on the left and control u on the right) and the Euler-Newton approximations with step sizes $h = 1/16$ and $h = 1/64$.*

The table below presents the errors $e_u^h = \max_{k=0,N}\{\|u_k - u(t_k)\|\}$ and $e_{\bar{x}}^h = \max_{k=0,N}\{\|x_k - \bar{x}(t_k)\|\}$ for various values of $h = T/N$. On the last line we give the values of the ratios $r_u^h = e_u^h / e_u^{h/4}$, which, due to the estimation in Theorem 20, are expected to be in average not smaller than 4. This is supported by the computation.

TABLE 1
The errors e_u^h and $e_{\bar{x}}^h$ for various values of h and the ratios r_u^h.

h	1/4	1/16	1/64	1/256	1/1024	1/4096
e_u^h	0.1980	0.0302	0.0068	0.0016	0.000384	0.00007
$e_{\bar{x}}^h$	0.1908	0.0299	0.0067	0.0016	0.000382	0.00007
r_u^h	6.55	4.44	4.25	4.19	5.00	

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