# Optimal Trade Execution with Instantaneous Price Impact and Stochastic Resilience

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Vienna, Nov. 18th, 2016

[1]Based on joint work with Paulwin Graewe

# Aims and Goals

- portfolio liquidation model with
  - instantaneous price impact
  - permanent price impact
  - stochastic resilience
  - only absolutely continuous trading strategies
- what's new?
  - value functions can be described by a coupled BSDE system
  - one component has a singular terminal condition
  - solved using asymptotic expansion at the terminal time

# Portfolio liquidation models

- models with instantaneous price impact
  - LQ cost function
  - absolutely continuous strategies
  - value functions described by one-dimensional PDEs, B(S)PDEs
- models with permanent price impact and resilience
  - LQ cost function
  - absolutely continuous and block trades
    - absolutely continuous trades: permanent impact
    - block trades: instantaneous and permanent impact
  - optimal strategy characterised in terms of BSDE (systems)

# A blend ...

- only absolutely continuous strategies (value function)
- persistent price impact with only a.c. strategies:
  - trading rate adds drift to a benchmark price
  - impact of past trades on current prices decreases
- instantaneous impact with only a.c. strategies:
  - no instantaneous impact: trade infinitely fast
  - weak instantaneous impact: trade fast initially
  - large instantaneous impact: “no permanent impact”

# Outline

- the control problem
- the BSDE system for the value function
- solving the BSDE system
- the verification argument

# The control problem

# The control problem

For any initial state $(t, x, y) \in [0, T) \times \mathbb{R} \times \mathbb{R}$ the value function is

$$V_t(x, y) = \operatorname*{ess\,inf}_{\xi \in \mathscr{A}(t,x)} \mathbb{E}\left[ \int_t^T \tfrac{1}{2}\eta\xi_s^2 + \xi_s Y_s + \tfrac{1}{2}\lambda_s X_s^2 \, ds \,\middle|\, \mathscr{F}_t \right]$$

where

$$\begin{cases} dX_s = -\xi_s \, ds, & t \le s \le T; \quad X_t = x; \\ dY_s = \{-\rho_s Y_s + \gamma \xi_s\} \, ds, & t \le s \le T; \quad Y_t = y. \end{cases}$$

The process $\xi \in L^2_{\mathscr{F}}(t, T; \mathbb{R})$ is the *control* and

$$\eta, \gamma \in \mathbb{R}_+; \quad \rho, \lambda \in L^\infty_{\mathscr{F}}(0, T; \mathbb{R}_+).$$

**Remark**

$\rho \equiv 0$ $(y = 0)$*: only temporary impact.*

# The control problem

A control is called *admissible* if the terminal state constraint

$$X_T = 0$$

is satisfied a.s. and

$$\xi \in L^2_{\mathscr{F}}(t, T; \mathbb{R}).$$

The set of all admissible controls is denoted

$$\mathscr{A}(t, x).$$

**Remark**

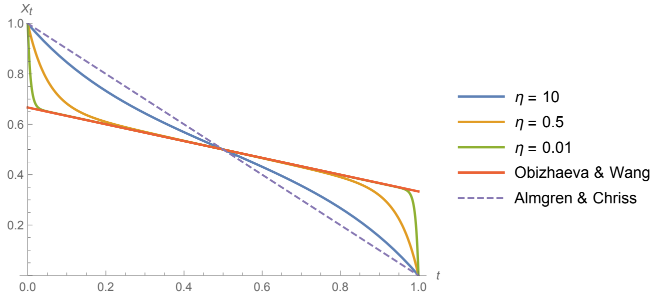
*Since* $\xi \in L^2_{\mathscr{F}}(t, T; \mathbb{R})$ *boundedness of the coefficients guarantees*

$$X, Y \in L^2_{\mathscr{F}}(\Omega; C([t, T]; \mathbb{R})).$$

# The control problem

Recall that

$$V_t(x,y) = \operatorname*{ess\,inf}_{\xi \in \mathscr{A}(t,x)} \mathbb{E}\left[ \int_t^T \tfrac{1}{2}\eta\xi_s^2 + \xi_s Y_s + \tfrac{1}{2}\lambda_s X_s^2 \, ds \,\middle|\, \mathscr{F}_t \right]$$



# The HJB equation

The stochastic HJB equation to our problem is

$$-dV_t(x,y) = \inf_{\xi \in \mathbb{R}} \{ \ldots \partial_x V_t(x,y) \ldots \partial_y V_t(x,y) \ldots \} \, dt - Z_t(x,y) \, dW_t .$$

**Remark**

*The HJB equation is given by the cost function and the state dynamics; the terminal condition by the liquidation constraint.*

# The HJB equation

The LQ structure of the control problem suggest the ansatz

$$V_t(x,y) = \tfrac{1}{2}A_t x^2 + B_t xy + \tfrac{1}{2}C_t y^2$$
$$Z_t(x,y) = \tfrac{1}{2}Z_t^A x^2 + Z_t^B xy + \tfrac{1}{2}Z_t^C y^2$$

for the solution $(V(x,y), Z(x,y))$ to the HJB equation, where

$$\begin{cases} -dA_t = \left\{\lambda_t - \eta^{-1}(A_t - \gamma B_t)^2\right\} dt - Z_t^A \, dW_t \\ -dB_t = \left\{-\rho_t B_t + \eta^{-1}(\gamma C_t - B_t + 1)(A_t - \gamma B_t)\right\} dt - Z_t^B \, dW_t \\ -dC_t = \left\{-2\rho_t C_t - \eta^{-1}(\gamma C_t - B_t + 1)^2\right\} dt - Z_t^C \, dW_t. \end{cases}$$

What is the terminal condition of this BSDE system?

# The terminal condition

- we expect the trading rate $\xi$ to tend to infinity for any non-trivial initial position as $t \to T$.
- we expect the resulting trading cost to dominate any resilience effect.
- we expect that

$$V_t(x, y) \sim V_t^{\rho=0}(x, y) \quad \text{as } t \to T$$

where $V^{\rho=0}$ is the value function corresponding to $\rho \equiv 0$.

## Lemma (The case $\rho \equiv 0$ (Graewe, H and Qiu (2015)))

$$V_t^{\rho=0}(x, y) = \frac{1}{2}(\tilde{A}_t + \gamma)x^2 + xy,$$

*where*

$$\begin{cases} -d\tilde{A}_t = \lambda_t - \eta^{-1}\tilde{A}_t^2\, dt - Z_t\, dW_t \\ \tilde{A}_t \to \infty \text{ in } L^\infty \text{ as } t \to T \end{cases}.$$

## Remark

*From this lemma we see why $\rho \equiv 0$ corresponds to a model with only temporary impact.*

# The terminal condition

Since

$$V_t^{\rho=0}(x,y) = \tfrac{1}{2}(\tilde{A}_t + \gamma)x^2 + xy,$$

and in view of the ansatz

$$V_t(x,y) = \tfrac{1}{2}A_t x^2 + B_t xy + \tfrac{1}{2}C_t y^2$$

we expect the coefficients of the linear-quadratic ansatz to satisfy

$$(A_t, B_t, C_t) \longrightarrow (\infty, 1, 0) \quad \text{in } L^\infty \text{ as } t \to T.$$

**Remark**

*Our approach uses the precise rate of "$A_t \to \infty$ in $L^\infty$".*

## Theorem (Existence and uniqueness of solutions)

- *The above BSDE system imposed with the above singular terminal condition admits at least one solution*

$$
\begin{aligned}
&((A, B, C), (Z^A, Z^B, Z^C)) \\
&\in L^\infty_{\mathscr{F}}(\Omega; C([0, T^-]; \mathbb{R}^3)) \times L^2_{\mathscr{F}}(0, T^-; \mathbb{R}^{3\times m}).
\end{aligned}
$$

- *Suppose a solution exists, then the value function is of the linear quadratic form and the optimal strategy is*

$$
\xi^*_t(x, y) = \eta^{-1}(A_t - \gamma B_t)x - \eta^{-1}(\gamma C_t - B_t + 1)y.
$$

*In particular, the BSDE systems admits at most one solution.*

## Remark

*Recall that*

$$\xi^*_t(x,y) = \eta^{-1}(A_t - \gamma B_t)x - \eta^{-1}(\gamma C_t - B_t + 1)y.$$

*We show that*

$$A_t - \gamma B_t > 0; \quad \gamma C_t - B_t + 1 \geq 0$$

*but we have no result on the sign of $\xi^*$ along $(X^*, Y^*)$. We can not rule our 'price triggered round trips'.*

# Solving the BSDE system

# The BSDE system

We need to solve a fully coupled BSDE system

- with singular terminal condition
- to which multi-dimensional comparison results don't apply

The idea is to find an equivalent BSDE system

- with regular terminal condition
- but singular driver

that can be solved in a suitable space: as $t \to T$:

$$A_t = \frac{\eta}{T-t} + \frac{H_t}{(T-t)^2}, \quad B_t = 1 + \frac{G_t}{T-t}, \quad C_t = P_t$$

where

$$H_t, B_t, C_t = O((T-t)^2).$$

# Asymptotics: some intuition

Recall that

$$\begin{cases} -dA_t = \left\{\lambda_t - \eta^{-1}(A_t - \gamma B_t)^2\right\} dt - Z_t^A \, dW_t \\ -dB_t = \left\{-\rho_t B_t + \eta^{-1}(\gamma C_t - B_t + 1)(A_t - \gamma B_t)\right\} dt - Z_t^B \, dW_t \\ -dC_t = \left\{-2\rho_t C_t - \eta^{-1}(\gamma C_t - B_t + 1)^2\right\} dt - Z_t^C \, dW_t. \end{cases}$$

- $B_T = 1 \Rightarrow A$ behaves as in the one-dimensional case
- rate of convergence of $B$ is slower than that of $C$
- $A_t \sim \frac{1}{T-t} \Rightarrow B_t - 1 = O(T-t)$

# A priori estimates

Let

$$((A, B, C), (Z^A, Z^B, Z^C))$$

denote any solution to our BSDE system that satisfies

$$(A_t, B_t, C_t) \longrightarrow (\infty, 1, 0) \qquad \text{in } L^\infty \text{ as } t \to T.$$

It will be necessary to also consider the processes

$$D := \eta^{-1}(A - \gamma B) \qquad \text{and} \qquad E := \eta^{-1}(\gamma C - B + 1)$$

that appear in the characterization of the candidate strategy.

**Remark**

*We need the full picture: value function and strategy. The BSDEs for $D, E$ and the system for $(B, D)$ allow for comparison.*

# A priori estimates

Using multi-dimensional comparison we can show that a.s.

$$A, D \geq 0, \quad B, -\gamma C, \eta E \in [0,1]$$

Since

$$\xi_t^*(x,y) = D_t x - E_t y$$

this does not guarantee that $\xi^* \geq 0$.

# A priori estimates

- using a comparison principle for quasi-monotone BSDE systems we obtain a priori estimates for $B, D$ and $E$
- from these we conclude that the following asymptotic behaviors hold in $L^\infty$ as $t \to T$:

$$\begin{aligned}(T-t)A_t &= \eta + O(T-t),\\ B_t &= 1 + O(T-t),\\ C_t &= O((T-t)^3).\end{aligned}$$

# Existence

The asymptotic behavior suggests the following ansatz:

$$\begin{aligned} A_t &= \frac{\eta}{T-t} + \frac{H_t}{(T-t)^2}, & H_t &= O((T-t)^2) \\ B_t &= 1 + \frac{G_t}{T-t}, & G_t &= O((T-t)^2) \\ C_t &= P_t, & P_t &= O((T-t)^2) \end{aligned}$$

where $H$, $G$ and $P$ are of the form

$$-dY_t = \left\{ \dots - \left( \frac{Y_t}{T-t} + \text{coupling} \right)^p \right\} dt + Z_t dW_t.$$

# Existence

We have a BSDE system with singular driver,

$$-dY_t = f(\omega, t, Y_t)dt - Z_t dW_t$$

We need to look for a solution in the right space, namely

$$\mathscr{H} = \{Y \in L^\infty_{\mathscr{F}}(\Omega; C([T-\delta, T]; \mathbb{R}^3)) : \|Y\|_{\mathscr{H}} < +\infty\}$$

endowed with the norm

$$\|Y\|_{\mathscr{H}} = \left\|(T-\cdot)^{-2} Y_\cdot\right\|_{L^\infty_{\mathscr{F}}(\Omega; C([T-\delta,T];\mathbb{R}^3))}.$$

This space is complete and the driver $f$ is locally Lipschitz:

$$\|f(\cdot, Y_\cdot) - f(\cdot, X_\cdot)\|_{\mathscr{H}} \leq L\|Y_\cdot - X_\cdot\|_{\mathscr{H}} \qquad \forall Y, X \in \overline{B}_{\mathscr{H}}(R).$$

## Lemma

- *There exists a short-time solution*

$$(Y, Z) \in \mathscr{H}^2 \times L^2_{\mathscr{F}}(T - \delta, T; \mathbb{R}^{3\times m})$$

  *to the above BSDE with singular driver.*
- *In particular, there exists a small time solution to the BSDE system for $(A, B, C)$ with singular terminal value.*
- *The small time solution for $(A, B, C)$ can be extended to a global solution on $[0, T)$.*

## Remark

*The extension uses that $f$ is independent of $Z$, analogously to PDEs, when the inhomogeneity is independent of the gradient.*

# Verification

# Verification

- we can apply Itô on $[0, s]$ for all $s < T$ to get:

$$V_t(x,y) \leq \mathbb{E}_t[V_s(X_s, Y_s)] + \mathbb{E}_t\left[\int_t^s \{\tfrac{1}{2}\eta\xi_r^2 + \xi_r Y_r + \tfrac{1}{2}\lambda_r X_r^2\}\, dr\right].$$

with equality for the candidate strategy $\xi^*$

## Lemma

- *the candidate is admissible and*

$$\xi^*, Y^* \in L^\infty_{\mathscr{F}}(\Omega; C([t,T];\mathbb{R})).$$

- *for every* $\xi \in \mathscr{A}(t,x)$ *it holds that*

$$E_t[A_s X_s^2 + B_s X_s Y_s + C_s Y_s^2] \xrightarrow{s \to T} 0.$$

# Verification

By the Itô-Kunita formula we have (after stopping):

$$V_t(x,y) \le \mathbb{E}_t[V_s(X_s, Y_s)] + \mathbb{E}_t\left[\int_t^s \{\tfrac{1}{2}\eta\xi_r^2 + \xi_r Y_r + \tfrac{1}{2}\lambda_r X_r^2\}\, dr\right].$$

Since $X, Y \in L^2_{\mathscr{F}}(\Omega; C([t,T];\mathbb{R}))$, and $\xi \in L^2_{\mathscr{F}}(t,T;\mathbb{R})$, we get

$$V_t(x,y) \le \mathbb{E}_t\left[\int_t^T \tfrac{1}{2}\eta\xi_r^2 + \xi_r Y_r + \tfrac{1}{2}\lambda_r X_r^2\, dr\right].$$

# Conclusion

- liquidation model with instantaneous and permanent impact
- main results:
  - value function is characterised by a BSDE system
  - BSDE system satisfies a singular terminal condition
  - existence via an asymptotic expansion at the terminal time
- main limiting factors:
  - a priori estimates
  - comparison result
- some open problems:
  - sign of the optimal trading strategy
  - introduce dark pools
  - random order book hight (liquidity)
  - convergence as $\eta \to 0$
  - ...

Many Thanks!