Locally risk-minimizing strategies for defaultable claims under incomplete information

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Contents

The default-free financial market model

- $(\Omega, \mathcal{F}, \mathbf{P})$ probability space, $T > 0$ fixed time horizon;
- Let W and B be two one-dimensional, independent Brownian motions;
- Reference filtration: $\mathbb{F} = \mathbb{F}^W \vee \mathbb{F}^B$ where $\mathbb{F}^W$ and $\mathbb{F}^B$ denote the natural filtrations of the processes W and B.
- one riskless asset with price equal to 1;
- one default-free risky asset S satisfying

$$\mathrm{d}S_t = S_t \left(\mu(t, S_t, X_t)\mathrm{d}t + \sigma(t, S_t)\mathrm{d}W_t \right), \quad S_0 = s_0 > 0,$$

- X unobservable exogenous stochastic factor satisfying

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + a(t, X_t) \left[\rho \mathrm{d}W_t + \sqrt{1 - \rho^2}\mathrm{d}B_t \right], \quad X_0 = x_0 \in \mathbb{R},$$

- Accessible information to investors: $\mathbb{F}^S$

The defaultable market model

- Let τ be a nonnegative random variable $\tau : \Omega \to [0, T] \cup \{+\infty\}$ satisfying $\mathbf{P}(\tau = 0) = 0$ and $\mathbf{P}(\tau > t) > 0$, for every $t \in [0, T]$;
- $H_t = \mathbb{I}_{\{\tau \le t\}}$ denotes the default process and $\mathbb{F}^H = \{\mathcal{F}_t^H,\ t \in [0, T]\}$ the natural filtration of H;

  Remark: τ is not necessarily a stopping time with respect to the reference filtration $\mathbb{F}$

- Progressive enlargement of filtration approach
  introduced by Jeulin and Yor (1978-1985) and widely applied to reduced-form models for credit risk, as in [Bielecki, Jeanblanc and Rutkowski (2004-2006)] and in [Elliott, Jeanblanc and Yor (2000)]. Recently applied in Insurance see [Choulli, Daveloose, and Vanmaele (2015)].
- Global market information:
  enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{F}^H = \mathbb{F}^W \vee \mathbb{F}^B \vee \mathbb{F}^H$
  (smallest filtration which contains $\mathbb{F}$, such that τ is a $\mathbb{G}$-stopping time)

- Conditional probability of the event $\{\tau \leq t\}$ given $\mathcal{F}_t$:

$$F_t = \mathbf{P}(\tau \leq t | \mathcal{F}_t) = \mathbb{E}\left[H_t | \mathcal{F}_t\right]$$

  we assume $F_t < 1$ for every $t \in [0, T]$ (this excludes the case where τ is an $\mathbb{F}$-stopping time, see e.g [Bielecki and Rutkowski 2002]);

- $\mathbb{F}$-*hazard process* $\Gamma_t = -\ln(1 - F_t), \quad t \in [0, T]$;
- We assume that Γ has a density, i.e. $\Gamma_t = \int_0^t \gamma_u \mathrm{d}u$ for some nonnegative $\mathbb{F}$- predictable process γ such that $\mathbb{E}\left[\int_0^T \gamma_u \mathrm{d}u\right] < \infty$
  (γ is known as the $\mathbb{F}$-*intensity* or the $\mathbb{F}$-*hazard rate*);
- the $\mathbb{F}$-*survival process* $\mathbf{P}(\tau > t | \mathcal{F}_t) = 1 - F_t = e^{-\int_0^t \gamma_u \mathrm{d}u}$.

Remark

- The process F is a continuous and increasing, then by [Bielecki and Rutkowski 2002], Γ is also an $(\mathbb{F}, \mathbb{G})$-martingale hazard process that is

$$M_t = H_t - \Gamma_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \gamma_u \mathrm{d}u = H_t - \int_0^t \lambda_u \mathrm{d}u, \quad t \in [0,T],$$

  is a $(\mathbb{G}, \mathbf{P})$-martingale, where

$$\lambda_t = \gamma_t \mathbb{I}_{\{\tau \geq t\}} = \gamma_t(1 - H_{t^-})$$

  is the $\mathbb{G}$-intensity.
- τ is a totally inaccessible $\mathbb{G}$-stopping time (the default event comes as a total surprise).

Defaultable claims

Definition

A *defaultable claim* is a triplet (ξ, Z, τ), where

- $\xi \in L^2(\mathcal{F}_T^S, \mathbf{P})$ is the *promised payoff* paid at maturity T, if default has not happened before or at time T;
- the component Z is the *recovery process*, which is paid at the default time if default has happened prior to or at time T (Z is square integrable and $\mathbb{F}^S$-predictable);
- τ is the default time.

$N = \{N_t,\ t \in [0, T]\}$ models the payment stream arising from the defaultable claim, i.e.

$$N_t = Z_\tau \mathbb{I}_{\{\tau \leq t\}} = \int_0^t Z_s \mathrm{d}H_s, \quad 0 \leq t < T, \quad \text{and} \quad N_T = \xi \mathbb{I}_{\{\tau > T\}}, \quad t = T. \quad (1)$$

- Defaultable claims may describe also unit-linked life insurance contracts where the insurance benefits depend on the price of some specific traded stock and so the insurer is exposed to a financial risk.

  Pure endowment contract: the sum insured is paid at T if the insured is still alive, $\xi \mathbb{I}_{\{\tau > T\}}$ and $Z = 0$;

  Term insurance contract: the sum insured is paid at death of the insured if it has happened before or at time T, $Z_\tau \mathbb{I}_{\{\tau \leq T\}}$ and $\xi = 0$.

- We do not assume independence between the financial market and the insurance model (in some recent papers [Biagini, Botero, and Schreiber, 2015], [Li and Szimayer, 2011] this assumption is dropped). As in [Choulli, Daveloose, and Vanmaele 2015] the correlation between the market and time of death is described by the $\mathbb{F}$-hazard rate in a full information setting by an enlargement filtration approach.

- Our model catches real features such as dependence between market model and time of death and partial information.

Partial information setting

Investors do not have a complete information on the market: they cannot observe neither the stochastic factor X nor the Brownian motions W and B which drive the dynamics of the pair (S, X) and as a consequence they cannot observe the $\mathbb{F}$-hazard rate γ.

At any time t, they may observe the risky asset price and know if default has occurred or not. The available information is given by

$$\widetilde{\mathbb{G}} := \mathbb{F}^S \vee \mathbb{F}^H \subseteq \mathbb{G} = \mathbb{F} \vee \mathbb{F}^H := \mathbb{F}^W \vee \mathbb{F}^B \vee \mathbb{F}^H$$

- We do not assume *the martingale invariance property* (Hypothesis H): every $(\mathbb{F}, \mathbf{P})$-martingale is $(\mathbb{G}, \mathbf{P})$-martingale;
- As in [Biagini and Cretarola 2012] we assume that hedging stops after default, hence we work on the stopped interval $[\![0, \tau \wedge T]\!]$;
- Since F is increasing, for any $(\mathbb{F}, \mathbf{P})$-martingale m the stopped process $m^\tau := \{m_{t\wedge\tau},\ t \in [0, T]\}$ is a $(\mathbb{G}, \mathbf{P})$-martingale, see Lemma 5.1.6 in [Bielecki and Rutkowski 2002]. In particular, the stopped processes W^τ and B^τ are $(\mathbb{G}, \mathbf{P})$-Brownian motions on $[\![0, \tau \wedge T]\!]$.

The semimartingale decompositions of the stopped risky asset price process

- The $(\mathbb{G}, \mathbf{P})$-semimartingale decomposition of $S^\tau_t := S_{t\wedge\tau}$

$$S^\tau_t = s_0 + \int_0^{t\wedge\tau} S^\tau_u \mu(u, S^\tau_u, X^\tau_u)\mathrm{d}u + \int_0^{t\wedge\tau} S^\tau_u \sigma(u, S^\tau_u)\mathrm{d}W^\tau_u, \quad t \in [0, T],$$

where

$$X^\tau_t = x_0 + \int_0^{t\wedge\tau} b(u, X^\tau_u)\mathrm{d}u + \int_0^{t\wedge\tau} a(u, X^\tau_u)\left[\rho \mathrm{d}W^\tau_u + \sqrt{1-\rho^2}\mathrm{d}B^\tau_u\right], \quad t \in [0, T].$$

- Since S^τ is $\widetilde{\mathbb{G}} = \mathbb{F}^S \vee \mathbb{F}^H$-adapted it admits also a $(\widetilde{\mathbb{G}}, \mathbf{P})$-semimartingale decomposition.

Definition

Given any subfiltration $\mathbb{B} \subseteq \mathbb{G}$, ${}^{o,\mathbb{B}}Y$ (resp. ${}^{p,\mathbb{B}}Y$) denotes the optional (resp. predictable) projection of a given **P**-integrable, $\mathbb{G}$-adapted process Y with respect to $\mathbb{B}$ and **P**, defined as the unique $\mathbb{B}$-optional (resp. $\mathbb{B}$-predictable) process such that ${}^{o,\mathbb{B}}Y_{\widehat{\tau}} = \mathbb{E}\left[Y_{\widehat{\tau}} | \mathcal{B}_{\widehat{\tau}}\right]$ **P**-a.s. (resp. ${}^{p,\mathbb{B}}Y_{\widehat{\tau}} = \mathbb{E}\left[Y_{\widehat{\tau}} | \mathcal{B}_{\widehat{\tau}-}\right]$ **P**-a.s.) on $\{\widehat{\tau} < \infty\}$ for every $\mathbb{B}$-optional (resp. $\mathbb{B}$-predictable) stopping time $\widehat{\tau}$.

Lemma (Innovation process)

The process $I^\tau = \{I^\tau_t,\ t \in [0,T]\}$ defined by

$$I^\tau_t := W^\tau_t + \int_0^{t\wedge\tau} \frac{\mu(u, S^\tau_u, X^\tau_u) - {}^{p,\widetilde{\mathbb{G}}}\mu_u}{\sigma\left(u, S^\tau_u\right)} \mathrm{d}u, \quad t \in [0,T],$$

is a $(\widetilde{\mathbb{G}}, \mathbf{P})$-Brownian motion on $[\![0, \tau \wedge T]\!]$.

- $(\widetilde{\mathbb{G}}, \mathbf{P})$-semimartingale decomposition of S^τ,

$$S_t^\tau = s_0 + \int_0^{t\wedge\tau} S_u^\tau \, {}^{p,\widetilde{\mathbb{G}}}\mu_u \mathrm{d}u + \int_0^{t\wedge\tau} S_u^\tau \, \sigma(u, S_u^\tau)\mathrm{d}I_u^\tau, \quad t \in [0,T],$$

- S^τ satisfies the structure condition with respect to both filtrations $\mathbb{G}$ and $\widetilde{\mathbb{G}}$

$$S_t^\tau = s_0 + M_t^{\mathcal{G}} + \int_0^{t\wedge\tau} \alpha_u^{\mathcal{G}} \mathrm{d}\langle M^{\mathcal{G}}\rangle_u, \quad t \in [\![0, \tau\wedge T]\!],$$

$$S_t^\tau = s_0 + M_t^{\widetilde{\mathcal{G}}} + \int_0^{t\wedge\tau} \alpha_u^{\widetilde{\mathcal{G}}} \mathrm{d}\langle M^{\widetilde{\mathcal{G}}}\rangle_u, \quad t \in [\![0, \tau\wedge T]\!],$$

$$M_t^{\mathcal{G}} := \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau)\mathrm{d}W_u^\tau, \quad M_t^{\widetilde{\mathcal{G}}} := \int_0^{t\wedge\tau} S_u^\tau \sigma(u, S_u^\tau)\mathrm{d}I_u^\tau \qquad (2)$$

$$\alpha_t^{\mathcal{G}} := \frac{\mu(t, S_t^\tau, X_t^\tau)}{S_t^\tau \sigma^2(t, S_t^\tau)} \quad \text{and} \quad \alpha_t^{\widetilde{\mathcal{G}}} := \frac{{}^{p,\widetilde{\mathbb{G}}}\mu_t}{S_t^\tau \sigma^2(t, S_t^\tau)}$$

Brief overview on (local) risk-minimization

- Contingent claim: $\xi \in L^2(\mathcal{F}_T, \mathbf{P})$
- Risk-minimization [introduced by Föllmer and Sondermann (1986)]: the risky asset S is described by a martingale
- $\psi = (\theta, \eta)$ an admissible strategy, $V(\psi) := \theta S + \eta$ its value process
- Cost process: $C_t(\psi) := V_t(\psi) - \int_0^t \theta_u \mathrm{d}S_u$
- An admissible strategy such that $V_T(\psi) = \xi$ is risk-minimizing if minimizes the risk process: $\mathbb{E}[(C_T(\psi) - C_t(\psi))^2 | \mathcal{F}_t]$ (conditional expected value of the squared future costs)
- θ^* is given by the Galtchouk-Kunita-Watanabe decomposition of ξ:

$$\xi = \mathbb{E}[\xi] + \int_0^T \theta^*_u \mathrm{d}S_u + A_T \quad \mathbf{P} - a.s.$$

  where A is a mg strongly orthogonal to S; ψ^* is mean-self-financing (that is $C_t(\psi^*)$ is a mg) and $C_t(\psi^*) = \mathbb{E}[\xi] + A_t$.

  Recently considered in credit risk and insurance frameworks (see Frey and Schmidt 2012, Biagini and Cretarola 2007, Møller 2001).

- In the semimartingale case such a strategy does not exist, hence Schweizer (1991) introduced the weaker concept of locally risk-minimizing strategy (under suitable assumptions it is equivalent to pseudo optimality).
- An admissible strategy ψ such that $V_T(\psi) = \xi$ is called pseudo optimal if and only if $C(\psi)$ is mean-self-financing and strongly orthogonal to the mg part of S.
- θ^* is given by the Föllmer-Schweizer decomposition of ξ:

$$\xi = \mathbb{E}[\xi] + \int_0^T \theta^*_u \mathrm{d}S_u + A_T, \quad \mathbf{P} - a.s$$

  where A is a mg strongly orthogonal to the mg part of S and the optimal cost $C_t(\psi^*) = \mathbb{E}[\xi] + A_t$.
- When S is continuous the Föllmer-Schweizer decomposition coincides with the Galtchouk-Kunita-Watanabe decomposition under the minimal martingale measure.
- We consider this approach in the case of a defaultable claim and in partial information framework.

Local risk-minimization for payment streams under p.i.

- We assume that hedging stops after default. This allows to work with hedging strategies only up to time $T \wedge \tau$.
- For any h, $\mathbb{G}$ (resp. $\widetilde{\mathbb{G}}$)-predictable process $\exists$ an $\mathbb{F}$ (resp.$\mathbb{F}^S$)-predictable process $\widehat{h}$ such that $\mathbb{I}_{\{\tau \geq t\}} h_t = \mathbb{I}_{\{\tau \geq t\}} \widehat{h}_t$, for each $t \in [0, T]$.

Definition

- Denote by $\Theta^{\mathbb{F},\tau}$ the space of all $\mathbb{F}$-predictable processes θ satisfying

$$\mathbb{E}\left[\int_0^{T\wedge\tau} (\theta_u \sigma(u, S_u^\tau) S_u^\tau)^2 \, \mathrm{d}u + \left(\int_0^{T\wedge\tau} |\theta_u \; \mu(u, S_u^\tau, X_u^\tau) S_u^\tau| \mathrm{d}u\right)^2\right] < \infty.$$

- Denote by $\Theta^{\mathbb{F}^S,\tau}$ of all $\mathbb{F}^S$-predictable processes θ satisfying

$$\mathbb{E}\left[\int_0^{T\wedge\tau} (\theta_u \sigma(u, S_u^\tau) S_u^\tau)^2 \, \mathrm{d}u + \left(\int_0^{T\wedge\tau} |\theta_u \; {}^{p,\widetilde{\mathbb{G}}}\mu_u \; S_u^\tau| \mathrm{d}u\right)^2\right] < \infty.$$

We assume that trader invests in the risky asset according to her/his knowledge on the asset prices before the default and rebalances the portfolio also upon the default information.

Definition

A $(\mathbb{G}, L^2)$-strategy (resp. $(\widetilde{\mathbb{G}}, L^2)$-strategy) is a bidimensional process $\varphi = (\theta, \eta)$ where $\theta \in \Theta^{\mathbb{F},\tau}$ (resp. $\theta \in \Theta^{\mathbb{F}^S,\tau}$) and η is a real-valued $\mathbb{G}$-adapted (resp. $\widetilde{\mathbb{G}}$-adapted) process s.t. the value process $V(\varphi) := \theta S^\tau + \eta$ is right-continuous and square integrable over $[\![0, T \wedge \tau]\!]$.

Definition

The cost process $C(\varphi)$ of a $(\mathbb{G}, L^2)$-strategy (resp. $(\widetilde{\mathbb{G}}, L^2)$-strategy) $\varphi = (\theta, \eta)$ is given by

$$C_t(\varphi) := N_t + V_t(\varphi) - \int_0^t \theta_u \mathrm{d}S_u^\tau, \quad t \in [\![0, T \wedge \tau]\!].$$

where N is defined in (1).

A $(\mathbb{G}, L^2)$-strategy (resp. $(\widetilde{\mathbb{G}}, L^2)$-strategy) φ is called *mean-self-financing* if its cost process $C(\varphi)$ is a $(\mathbb{G}, \mathbf{P})$-martingale (resp. $(\widetilde{\mathbb{G}}, \mathbf{P})$-martingale).

The extension of the local risk-minimization approach to payment streams requires to look for admissible strategies with the 0-achieving property, that is

$$V_{\tau\wedge T}(\varphi) = 0, \quad \mathbf{P} - \text{a.s.}.$$

Then, by Theorem 1.6 in [Schweizer 2008] we give the following equivalent definition of locally risk-minimizing strategy.

Definition

Let N be the payment stream given in (1) associated to the defaultable claim (ξ, Z, τ). We say that a $(\mathbb{G}, L^2)$-strategy (resp. $(\widetilde{\mathbb{G}}, L^2)$-strategy) φ is $(\mathbb{F}, \mathbb{G})$-locally risk-minimizing (resp. $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing) for N if

(i) φ is 0-achieving and mean-self-financing,

(ii) the cost process $C(\varphi)$ is strongly orthogonal to the $\mathbb{G}$-martingale part $M^{\mathcal{G}}$ (respectively $\widetilde{\mathbb{G}}$-martingale part $M^{\widetilde{\mathcal{G}}}$) of S^τ, both given in (2).

The Föllmer-Schweizer decompositions

Definition (Stopped Föllmer-Schweizer decomposition with respect to $\mathbb{G}$)

Given a random variable $\zeta \in L^2(\mathcal{G}_T, \mathbf{P})$, we say that ζ admits a stopped FS-decomposition w.r.t. $\mathbb{G}$, if there exist a process $\theta^{\mathcal{F}} \in \Theta^{\mathbb{F},\tau}$, a square integrable $(\mathbb{G}, \mathbf{P})$-mg $A^{\mathcal{G}} = \{A^{\mathcal{G}}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ null at zero, strongly orthogonal to the martingale part of S^τ, $M^{\mathcal{G}}$ and $\zeta_0 \in \mathbb{R}$ such that

$$\zeta = \zeta_0 + \int_0^T \theta^{\mathcal{F}}_u \mathrm{d}S^\tau_u + A^{\mathcal{G}}_{T\wedge\tau}, \quad \mathbf{P} - a.s.,$$

Definition (Stopped Föllmer-Schweizer decomposition with respect to $\widetilde{\mathbb{G}}$)

Given a random variable $\zeta \in L^2(\widetilde{\mathcal{G}}_T, \mathbf{P})$, we say that ζ admits a stopped FS-decomposition w.r.t. $\widetilde{\mathbb{G}}$, if there exist a process $\theta^{\mathcal{F}^S} \in \Theta^{\mathbb{F}^S,\tau}$, a square integrable $(\widetilde{\mathbb{G}}, \mathbf{P})$-mg $A^{\widetilde{\mathcal{G}}} = \{A^{\widetilde{\mathcal{G}}}_t,\ t \in [\![0, T \wedge \tau]\!]\}$ null at zero, strongly orthogonal to the martingale part of S^τ,$M^{\widetilde{\mathcal{G}}}$, and $\zeta_0 \in \mathbb{R}$ such that

$$\zeta = \zeta_0 + \int_0^T \theta^{\mathcal{F}^S}_u \mathrm{d}S^\tau_u + A^{\widetilde{\mathcal{G}}}_{T\wedge\tau}, \quad \mathbf{P} - a.s..$$

Adapting the results proved in [Biagini and Cretarola 2012] to the partial information setting, we get the following characterization of the optimal hedging strategy.

Proposition

Let N be the payment stream associated to the defaultable claim (ξ, Z, τ). Then, N admits an $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy $\varphi^* = (\theta^*, \eta^*)$ if and only if $N_{T\wedge\tau} = \xi \mathbb{I}_{\{\tau > T\}} + Z_\tau \mathbb{I}_{\{\tau \le T\}}$ admits a stopped Föllmer-Schweizer decomposition with respect to $\widetilde{\mathbb{G}}$, i.e.

$$N_{T\wedge\tau} = N_0 + \int_0^T \theta_u^{\mathcal{F}^S} \mathrm{d}S_u^\tau + A_{t\wedge\tau}^{\widetilde{\mathcal{G}}} \quad \mathbf{P} - a.s..$$

The strategy φ^*, the value process and the minimal cost are given resp.

$$\theta^* = \theta^{\mathcal{F}^S}, \quad \eta^* = V(\varphi^*) - \theta^{\mathcal{F}^S} S^\tau,$$

$$V_t(\varphi^*) = N_0 + \int_0^t \theta_u^{\mathcal{F}^S} \mathrm{d}S_u^\tau + A_t^{\widetilde{\mathcal{G}}} - N_t, \quad C_t(\varphi^*) = N_0 + A_t^{\widetilde{\mathcal{G}}} \quad t \in [\![0, T\wedge\tau]\!].$$

The Galtchouk-Kunita-Watanabe (GKW) decompositions under the MMM

Definition

A martingale measure $\widehat{\mathbf{P}}$ equivalent to $\mathbf{P}$ with s.i.-density is called *minimal* for S^τ if any s.i. $(\mathbb{G}, \mathbf{P})$-martingale which is strongly orthogonal to the mg part of S^τ, $M^{\mathcal{G}}$, under $\mathbf{P}$ is also a $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale.

- Minimal martingale measure: $\widehat{\mathbf{P}}$

$$L_t^\tau = \frac{\mathrm{d}\widehat{\mathbf{P}}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{G}_{\tau\wedge t}} = \mathcal{E}\left(-\int_0^\cdot \frac{\mu(u, S_u^\tau, X_u^\tau)}{\sigma(u, S_u^\tau)}\mathrm{d}W_u^\tau\right)_{t\wedge\tau}, \quad t \in [0, T].$$

- $\widehat{W}_t^\tau = W_t^\tau + \int_0^{t\wedge\tau} \frac{\mu(u, S_u^\tau, X_u^\tau)}{\sigma(u, S_u^\tau)}\mathrm{d}u, \quad t \in [0, T]$, is $(\mathbb{G}, \widehat{\mathbf{P}})$-Brownian motion.

Theorem

Assume $N_{T\wedge\tau}$ and S^τ to be $\widehat{\mathbf{P}}$-square integrable. Consider the GKW-decomposition of $N_{T\wedge\tau}$ with respect to $(\mathbb{G}, \widehat{\mathbf{P}})$, i.e.

$$N_{T\wedge\tau} = \widehat{\mathbb{E}}[N_{T\wedge\tau}] + \int_0^T \widehat{\theta}_u^{\mathcal{F}} \mathrm{d}S_u^\tau + \widehat{A}_{T\wedge\tau}^{\mathcal{G}}, \quad \widehat{\mathbf{P}} - a.s.,$$

where $\widehat{\theta}^{\mathcal{F}}$ is an $\mathbb{F}$-predictable process integrable w.r.t. S^τ and $\widehat{A}^{\mathcal{G}}$ is a $(\mathbb{G}, \widehat{\mathbf{P}})$-mg strongly orthogonal to S^τ. Then, $N_{T\wedge\tau}$ has the following GKW-decomposition with respect to $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$

$$N_{T\wedge\tau} = \widehat{\mathbb{E}}[N_{T\wedge\tau}] + \int_0^T \widehat{\theta}_u^{\mathcal{F}^S} \mathrm{d}S_u^\tau + \widehat{A}_{T\wedge\tau}^{\widetilde{\mathcal{G}}}, \quad \widehat{\mathbf{P}} - a.s.,$$

where

$$\widehat{\theta}_t^{\mathcal{F}^S} = \frac{{}^{\widehat{p},\mathbb{F}^S}\left(\widehat{\theta}_t^{\mathcal{F}} e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}, \quad t \in [\![0, T\wedge\tau]\!],$$

and the $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$-mg $\widehat{A}^{\widetilde{\mathcal{G}}}$ is given by

$$\widehat{A}_t^{\widetilde{\mathcal{G}}} = \widehat{\mathbb{E}}\Big[\widehat{A}_t^{\mathcal{G}} + \int_0^t (\widehat{\theta}_u^{\mathcal{F}} - \widehat{\theta}_u^{\mathcal{F}^S}) \mathrm{d}S_u^\tau \Big| \widetilde{\mathcal{G}}_t\Big], \quad t \in [\![0, T\wedge\tau]\!].$$

The proof consists in two steps:

- we first prove that the GKW-decomposition with respect to $(\widetilde{\mathbb{G}}, \widehat{\mathbf{P}})$ has integrand ${}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_t$;
- next, we prove the following representation

$$\mathbb{I}_{\{\tau \geq t\}} {}^{\widehat{p},\widetilde{\mathbb{G}}}\widehat{\theta}^{\mathcal{F}}_t = \mathbb{I}_{\{\tau \geq t\}} \frac{{}^{\widehat{p},\mathbb{F}^S}\left(\widehat{\theta}^{\mathcal{F}}_t e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}$$

Theorem

The $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy $\varphi^* = (\theta^*, \eta^*)$ s given by

$$\theta^*_t = \theta^{\mathcal{F}^S}_t = \frac{{}^{\widehat{p},\mathbb{F}^S}\left(\theta^{\mathcal{F}}_t e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma_u \mathrm{d}u}\right)}, \quad t \in [\![0, T \wedge \tau]\!],$$

$$\eta^*_t = V_t(\varphi^*) - \theta^*_t S^\tau_t, \quad t \in [\![0, T \wedge \tau]\!],$$

and the optimal value process $V(\varphi^*)$ is given by

$$V_t(\varphi^*) = \widehat{\mathbb{E}}[N_{T\wedge\tau}] + \int_0^t \theta^*_u \mathrm{d}S^\tau_u + \widehat{A}^{\widetilde{\mathcal{G}}}_t - N_t, \quad t \in [\![0, T \wedge \tau]\!],$$

Application to Markovian Models

We assume $\gamma_t = \gamma(t, X_{t-})$, $\xi = G(T, S_T)$, $Z_t = \Phi(t, S_{t-})$.

Proposition (full information case)

Let $g(t,s,x) \in \mathcal{C}_b^{1,2,2}([0,T] \times \mathbb{R}^+ \times \mathbb{R})$ *be a solution of the problem*

$$\begin{cases} \widehat{\mathcal{L}}^{S,X} g(t,s,x) - \gamma(t,x)\ g(t,s,x) + \Phi(t,s)\gamma(t,s) = 0, & (t,s,x) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R} \\ g(T,s,x) = G(T,s), \end{cases} \tag{3}$$

then the first component of the $(\mathbb{F},\mathbb{G})$*-locally risk-minimizing strategy is given by*

$$\theta_t^{\mathcal{F}} = \frac{\partial g}{\partial s}(t, S_t, X_t) + \frac{\rho a(t, X_t)}{S_t \sigma(t, S_t)} \frac{\partial g}{\partial x}(t, S_t, X_t), \quad t \in [\![0, T \wedge \tau]\!] \tag{4}$$

Here $\widehat{\mathcal{L}}^{S,X}$ *denotes the* $(\mathbb{F}, \widehat{\mathbf{P}})$*-Markov generator of* (S, X)

- Existence and uniqueness of classical solutions to (3) can be obtained under suitable assumptions by applying results in [Heath-Schweizer 2000].

Sketch of the proof

- By the GKW-decomposition under $\widehat{P}$ in the full infomation case we get that the process $\widehat{V}_t := \widehat{\mathbb{E}}[N_{T\wedge\tau}|\mathcal{G}_t]$ satisfies $\forall t \in [\![0, T\wedge\tau]\!]$,

$$\widehat{V}_t = \widehat{\mathbb{E}}[N_{T\wedge\tau}] + \int_0^t \theta_u^{\mathcal{F}} \mathrm{d}S_u^\tau + \widehat{A}_t^{\mathcal{G}}, \quad \widehat{\mathbf{P}} - a.s. \tag{5}$$

where $\widehat{A}^{\mathcal{G}}$ is strongly orthogonal to S^τ under $\widehat{P}$.

- The optimal strategy under full information: $\theta_t^{\mathcal{F}} = \frac{\mathrm{d}\langle \widehat{V}, S^\tau \rangle_t^{\widehat{P}}}{\mathrm{d}\langle S^\tau \rangle_t^{\widehat{P}},}$
- How to compute the martingale decomposition (5) under $\widehat{P}$?

For instance in the case where: $\tau = inf\{t \geq 0 : \int_0^t \gamma(r, X_r)\mathrm{d}r \geq \Theta\}$, where Θ unit-mean exponential random variable independent of $\mathcal{F}_T$ (*Cox model*).

In this framework the *Hypothesis H* is fulfilled and

$$\widehat{\mathbf{P}}(\tau \leq t|\mathcal{F}_t) = \mathbf{P}(\tau \leq t|\mathcal{F}_t) = 1 - e^{-\int_0^t \gamma(r,X_r)\mathrm{d}r}$$

and we can apply the results in [Biagini-Cretarola 2012] under full information and Feymann-Kac formula for compute conditional expectation under $\widehat{P}$ w.r.t. the filtration $\mathbb{F}$.

We propose an alternative method:

- Recall $H_t := \mathbb{I}_{\{\tau \le t\}}$ and $M_t = H_t - \int_0^t (1-H_r)\gamma(r, X_r)\mathrm{d}r$ is $(\mathbb{G}, \widehat{\mathbf{P}})$-mg
- Hence

$$\widehat{V}_t = \widehat{\mathbb{E}}[N_{T\wedge\tau}|\mathcal{G}_t] = \int_0^t \Phi(r, S_r)(1-H_r)\gamma(r, X_r)\mathrm{d}r +$$

$$\mathbb{E}[G(T, S_T)(1-H_T) + \int_t^T \Phi(r, S_r)(1-H_r)\gamma(r, X_r)\mathrm{d}r|\mathcal{G}_t]$$

- (S^τ, X^τ, H) is an $(\mathbb{G}, \widehat{\mathbf{P}})$-Markov process with generator $\widehat{\mathcal{L}}^{S,X,H}$ given by

$$\widehat{\mathcal{L}}^{S,X,H} f(t,s,x,z) = \widehat{\mathcal{L}}^{S,X} f(t,s,x,z)(1-z) + \{f(t,s,x,z+1) - f(t,s,x,z)\}\gamma(t,x)(1-z)$$

for any fixed $z \in \{0,1\}$, $f(t,s,x,z) \in \mathcal{C}_b^{1,2,2}([0,T]\times\mathbb{R}^+\times\mathbb{R})$.
$\widehat{\mathcal{L}}^{S,X}$ denotes the $(\mathbb{F}, \widehat{\mathbf{P}})$-Markov generator of (S, X).

- We apply Feymann-Kac formula. Let $g(t,s,x,z)$ be a solution of the problem

$$\begin{cases} \widehat{\mathcal{L}}^{S,X,H} g(t,s,x,z) + \Phi(t,s)(1-z)\gamma(t,x) = 0, (t,s,x,z) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R} \times \{0,1\} \\ g(T,s,x,z) = G(T,s)(1-z), \end{cases}$$

Then $\widehat{\mathbb{E}}[G(T,S_T)(1-H_T) + \int_t^T \Phi(r,S_r)(1-H_r)\gamma(r,X_r)\mathrm{d}r|\mathcal{G}_t] = g(t, S_t^\tau, X_t^\tau, H_t)$ and by Ito formula we write down the $(\mathbb{G}, \widehat{\mathbf{P}})$-martingale decomposition of $\widehat{V}$.

- The optimal strategy under full information is given by

$$\theta_t^{\mathcal{F}} \mathbb{I}_{\{\tau \geq t\}} = \{\frac{\partial g}{\partial s}(t, S_t, X_t, 0) + \frac{\rho a(t,X_t)}{S_t \sigma(t,S_t)} \frac{\partial g}{\partial x}(t, S_t, X_t, 0)\} \mathbb{I}_{\{\tau \geq t\}}$$

- since $g(t,s,x,1) = 0$, $g(t,s,x,0)$ solves the PDE given by (3).

Partial information case: filtering approach

How to compute $\theta^*_t = \theta^{\mathcal{F}^S}_t = \frac{{}^{\widehat{p},\mathbb{F}^S}\left(\theta^{\mathcal{F}}_t e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\right)}{{}^{\widehat{p},\mathbb{F}^S}\left(e^{-\int_0^t \gamma(u,X_u)\mathrm{d}u}\right)}, \quad t \in [\![0, T\wedge\tau]\!]$?

(Here $\theta^{\mathcal{F}}_t = \frac{\partial g}{\partial s}(t, S_t, X_t) + \frac{\rho a(t,X_t)}{S_t \sigma(t,S_t)}\frac{\partial g}{\partial x}(t, S_t, X_t)$).

Remark: the filter provides the conditional law of the state process X_t given the σ-algebra $\mathcal{F}^S_t$. Here we have a functional of the trajectories of X.

- Let $Y_t := e^{-\int_0^t \gamma(r,X_r)\mathrm{d}r}$, we consider as state process (X, Y).
- the triple $(S, X, Y)$is an $(\mathbb{F}, \widehat{\mathbf{P}})$-Markov process with generator $\widehat{\mathcal{L}}^{S,X,Y}$ given by

$$\widehat{\mathcal{L}}^{S,X,Y} f(t, s, x, y) = \frac{\partial f}{\partial t} + \left[b(t, x) - \rho\,\frac{\mu(t, s, x)a(t, x)}{\sigma(t, s)}\right]\frac{\partial f}{\partial x} - y\gamma(t, x)\frac{\partial f}{\partial y} +$$

$$\frac{1}{2}a^2(t, x)\frac{\partial^2 f}{\partial x^2} + \rho a(t, x)\sigma(t, s)s\frac{\partial^2 f}{\partial x \partial s} + \frac{1}{2}\sigma^2(t, s)\, s^2\frac{\partial^2 f}{\partial s^2}$$

- For any $f(t, s, x, y)$ let the filter be defined as

$$\pi_t(f) := \widehat{\mathbb{E}}[f(t, S_t, X_t, Y_t)|\mathcal{F}^S_t] = \int_{\mathbb{R}\times\mathbb{R}^+} f(t, S_t, x, y)\pi_t(\mathrm{d}x, \mathrm{d}y), \quad t \in [0, T]$$

(probability measure valued stochastic process)

Theorem (Kushner-Stratonovich equation)

Under suitable assumptions for every function $f(t,s,x,y) \in \mathcal{C}_b^{1,2,2,1}([0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+)$ *and* $t \in [0,T]$, *the filter* π *is the unique strong solution of the following equation*

$$\pi_t(f) = f(0, s_0, x_0, 1) + \int_0^t \pi_u(\widehat{\mathcal{L}}^{S,X,Y} f)\mathrm{d}u +$$

$$\int_0^t \left[\rho \pi_u \left(a \, \frac{\partial f}{\partial x} \right) + S_u \sigma(t, S_u) \pi_u \left(\frac{\partial f}{\partial s} \right) \right] \mathrm{d}\widehat{W}_u .$$

proof We extend results contained in [Ceci and Colaneri 2012].

Theorem (incomplete information case)

The first component θ^* of the $(\mathbb{F}^S, \widetilde{\mathbb{G}})$-locally risk-minimizing strategy is given by

$$\theta^*_t = \frac{\pi_t\left(id_y \frac{\partial g}{\partial s}\right) + \frac{\rho}{\sigma(t,S_t)S_t}\pi_t\left(a\ id_y \frac{\partial g}{\partial x}\right)}{\pi_t(id_y)},$$

for every $t \in [\![0, T \wedge \tau]\!]$, where $id_y(t,s,x,y) := y$ and g is the solution to PDE (3) .

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