The Robust Merton Problem of an Ambiguity Averse Investor

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January 27, 2017

Outline

- Overview on portfolio selection under uncertainty aversion.
- Goal: maximize uncertainty averse utility from consumption and terminal wealth. Avoid technicalities as much as possible, and provide explicit solutions.
- Our setup
  - diffusion context, uncertainty about μ and σ estimates.
  - (Ellipsoidal) uncertainty sets for the drift, along a volatility realization.
- A max-min HJB-Isaacs PDE arises. Quite tractable.
- For CRRA utilities: explicit solution, shaped by a rescaled market Sharpe Ratio.
- Examples.

Model uncertainty

- Traditionally, financial modelling heavily relies on the choice of an underlying P, expressing the stochastic nature of future market price evolutions.
- Complexity of the global economic and financial dynamics renders precise identification of P impossible.
- Financial Modelling is inherently subject to model uncertainty, aka Knightian uncertainty or ambiguity (or model risk).
- Incorporating model uncertainty: the single P is replaced by a set of probabilities $\mathcal{P}$, consisting of plausible models.

Philosophy of robust portfolio selection

- The investor has a pessimistic view of the odds, and takes a max-min (or worst-case, or robust) approach to the problem.
- First she minimizes an expected utility functional from wealth X over the plausible models $\mathcal{P}$:

$$\inf_{P \in \mathcal{P}} E_P[U(X)]$$

- Then maximizes wealth $X(\theta)$ over portfolio strategies θ:

$$\sup_{\theta} \left(\inf_{P} E_P[U(X(\theta))] \right)$$

- The problem can be seen as a zero-sum, two players game: the representative agent vs the malevolent nature (market).
- Find a saddle point, and thus the value of the game.

Worst-case preferences axiomatized: uncertainty aversion

- The decision rule of an economic agent is specified via a preference order $\succeq$ on the space of payoffs.
- If the preference order $\succeq$ satisfies:

$$X \sim Y \Rightarrow tX + (1-t)Y \succeq Y \text{ for all } t \in [0,1]$$

  it is called uncertainty averse.
- Gilboa and Schmeidler J. Math. Econ. 1989; Maccheroni, Marinacci and Rustichini, Econ. 2006 show that an ambiguity averse preference order, verifying the Archimedean axiom and a weak certainty independence axiom, admits a numerical representation of worst-case type. To wit,

$$X \succeq Y \text{ iff } \inf_P \{E_P[U(X)] + \gamma(P)\} \geq \inf_P \{E_P[U(Y)] + \gamma(P)\}$$

- When the penalty function γ is either 0 or ∞ (coherent case): the above boils down to an infimization over a subset $\mathcal{P}$ of probabilities, exactly the inner problem we face.

Technical difficulties...

- A main distinction to be drawn in the literature on robust portfolio selection.
- Case $\mathcal{P}$ is a dominated family. All $P \in \mathcal{P}$ are absolutely continuous wrt a reference probability P^0,

$$P \ll P^0$$

  This happens e.g. when:
  0) Ω is discrete, $P^0 = \sum_{n \geq 1} \frac{1}{2^n} \delta_{\{\omega_n\}}$
  1) $\mathcal{P}$ is finite or countable; dominating $P^0 = \sum_n c_n P_n$
  2) in a diffusion context, when there is uncertainty only in the drift.
- Case $\mathcal{P}$ is nondominated. This happens whenever one has to estimate the volatility coefficient! Estimation comes with error intervals.

Literature overview in the dominated case

- Dominated case is easier to treat and has been widely investigated over the last fifteen years.
- *In a diffusion context, stochastic control methods*
  Chen and Epstein (Econ., 2002);
  Maenhout (Rev. Fin. St. 2004);
- *Combination of stochastic control and duality techniques:*
  Quenez (Ascona 2004)
- *Duality methods:*
  Föllmer et alii, Schied various papers 2005-2009.
  Owari (Adv. Math. Econ. 2011) for a thorough analysis of the problem and the 'good' class of strategies.

Nondominated case, I

- Talay and Zheng, FinaSto 2002: diffusion model, uncertainty on drift and sigma. Application to risk management problem, viscosity solution.
- Hernandez-Hernandez and Schied (Stat&Dec. 2006): diffusion model with a non-tradable factor and miss-specified drift and volatility coefficients for the traded asset. DPM applied to the dual problem.
- Matoussi, Possamai and Zhou (MathFin, 2015) - U CRRA, volatility in an interval and known drift – via 2BSDE
- Denis and Kervarec (SIAM J. Con.Opt. 2013) - Ut-max from terminal wealth. Stocks: continuous semimartingale. Uncertainty on the semimartingale characteristics, diagonal quadratic variation. By duality methods (U bounded), there exists a worst case $\overline{P}$.

Nondominated case, II

- Nutz (MathFin, 2016)- general model in discrete time.
- Lin and Riedel, (MathFin 2016) – diffusion with several uncorrelated stocks; box-type uncertainty on the parameters. Treated via robust control, namely with a G-Brownian motion technique.
- Neufeld and Nutz: CRRA utility maximization from terminal wealth for general Levy models (compact-valued strategies) – forthcoming on MathFin.

Market & uncertainty modelling, I

- Canonical space of continuous paths and natural filtration.
- Market model: riskless, constant r; n traded risky assets evolving as

$$dS_t = Diag(S_t)(\mu dt + \sigma dW_t)$$

  with $dimW = n$.
- **Estimate** quadratic variation $\langle W \rangle_t = \hat{\Sigma} t$, and drift $\hat{\mu}$, constants.
- The agent is diffident about these estimates.
- The instantaneous VarCov Σ is no more constant, but

$$\Sigma_t(\omega) \in K$$

  where K is some compact s.t.

$$\hat{\Sigma} \in K \subset Sym_{++}(n)$$

  satisfying a uniform ellipticity condition

$$\min_{\Sigma \in K} y' \Sigma y \geq h^2 \|y\|^2, \; h > 0$$

Market & uncertainty modelling, II

- What about the drift?
- Fix $\Sigma \in K$, and consider the ellipsoid centered at $\hat{\mu}$, shaped by Σ, with radius ϵ:

$$U_\epsilon(\Sigma) = \{u \in \mathbb{R}^n \mid (u - \hat{\mu})' \Sigma^{-1} (u - \hat{\mu}) \leq \epsilon^2\}$$

- For a given path of σ, let $\Sigma_t(\omega) = \sigma_t(\omega)\sigma_t(\omega)'$. Then

$$\mu_t(\omega) \in U_\epsilon(\Sigma_t(\omega)) \quad \forall t, \omega$$

- Set of plausible drifts and volatilities is thus

$$\Upsilon := \{(\mu, \sigma) \text{ progr. meas. } \mid \Sigma_t(\omega) \in K,\ \mu_t(\omega) \in U_\epsilon(\Sigma_t(\omega))\}$$

On the choice of the uncertainty sets, I

- The volatility uncertainty set specification is inline with empirical practice, as Σ is the estimated object, and not σ. By Cholesky factorization however there is a one-to-one correspondence

$$\sigma \text{ Lower Triangular }, \sigma_{ii} > 0 \longleftrightarrow \Sigma > 0$$

- Does it make sense to model the drift along a specific realization of volatility?
  Yes. Mean returns estimates are much more subject to imprecision than volatilities.

On the choice of the uncertainty sets, II

- In the dominated setup (σ known), there is a vast literature under different representations of drift ambiguity.
- The *k-ignorance assumption* in Chen and Epstein amounts to a box representation for the drift, $[\underline{\mu}, \overline{\mu}]$-valued (the same in Lin and Riedel).
- Ellipsoidal representation for the ambiguous drifts appears in Goldfarb and G. Iyengar (2003, MOR) and in Garlappi, Uppal and Wang (2007, Rev. Fin. St.) for single period mean-variance optimization.
- Fabozzi et al (Ann. Op. Res. 2010): *The coefficient realizations are assumed to be close to the forecasts... They are more likely to deviate from their (instantaneous) means if their variability (measured by their standard deviation) is higher, so deviations from the mean are scaled by the inverse of the covariance matrix. The parameter ϵ corresponds to the overall amount of scaled deviations of the realized returns from the forecasts against which the investor would like to be protected.*

From uncertain drifts and volatilities to probabilities

- Our set of plausible drifts and volatilities is

$$\Upsilon := \{(\mu, \sigma) \text{ progr. meas. } \mid \Sigma(t, \omega) \in K, \mu_t(\omega) \in U_\epsilon(\Sigma_t(\omega))\}$$

  and Υ_σ is the σ-section.
- A choice of (μ, σ) corresponds to the selection of a P on the canonical space.
- The plausible set $\mathcal{P}$ of probabilities is the set of Ps such that S satisfies the SDE

$$dS_t = Diag(S_t)(\mu_t dt + \sigma_t dW_t^P)$$

  in which W^P denotes an n dimensional P-Brownian motion, for some $(\mu, \sigma) \in \Upsilon$.

More on the family $\mathcal{P}$

- When $K = \{\hat{\Sigma}\}$, then $\mathcal{P}$ is a dominated family.
  Results on SDEs and the Girsanov theorem ensure that there is a one-to-one correspondences between drifts valued in $U_\epsilon(\hat{\Sigma})$ and probabilities on the canonical space, all equivalent to $P^{\hat{\mu},\hat{\sigma}}$.
- When ambiguity on the volatility is considered however, the situation abruptly changes. If $P_1, P_2 \in \mathcal{P}$ correspond to different volatility coefficients σ^1 and σ^2, in the sense that the predictable set

$$A = \{(\omega, t) \mid \Sigma^1_t(\omega) \neq \Sigma^2_t(\omega)\}$$

  has positive measure wrt $dP_1 dt$ and $dP_2 dt$, then P_1 and P_2 are no longer equivalent. When $A = \Omega \times [0, T]$, P_1, P_2 are *orthogonal* to each other, as their supports have reciprocal measure 0.
  In general, $\mathcal{P}$ is nondominated.

Strategies

- Initial wealth $x > 0$.
- Fix $P \in \mathcal{P}$. Then, S under P evolves as

$$dS_t = Diag(S_t)(\mu_t dt + \sigma_t dW_t^P)$$

for some $(\mu, \sigma) \in \Upsilon$.

- θ $n-$dimensional progr. meas. vector of cash allocation in each risky asset;
- c consumption: progr meas, nonnegative, scalar process with $\int_0^t c_s ds < \infty$ - P-a.s.
- wealth X from strategy (θ, c) evolves under P according to

$$dX_t = (rX_t + \theta_t'(\mu_t - r\mathbf{1}) - c_t)dt + \theta_t'\sigma_t dW_t^P$$

classically, (θ, c) is admissibile if X remains nonnegative P-a.s.

Robust admissible strategies

- **Definition.**
  A property holds $\mathcal{P}$-quasi surely (q.s.) if it holds almost surely for all $P \in \mathcal{P}$.
- The strategy (θ, c) is robust admissible if:
  1. c is nonnegative and integrable $\mathcal{P}$-q.s.,
  2. the wealth X is nonnegative $\mathcal{P}$-q.s.
- Namely,

$$\mathcal{A}_{\mathrm{rob}}(x) = \cap_{P \in \mathcal{P}} \{(\theta, c) \mid c \geq 0, \text{int., and } X \geq 0 \, P - a.s.\}$$

- **Remark.** When $\mathcal{P}$ is dominated, and there is a $P \sim P^0$ in $\mathcal{P}$, the robust strategies are simply the P-admissible strategies.

Example of robust strategies: bounded proportion

- Let $\gamma > 0$ and π be two progr meas and bounded processes.
- Any plan of the following type is robust admissible:

$$\theta_t = X_t \pi_t, c_t = X_t \gamma_t,$$

- For a given P, the wealth X satisfies the SDE:

$$dX_t = X_t(r + \pi_t'(\mu_t - r\mathbf{1}) - \gamma_t)dt + X_t \pi_t' \sigma_t dW_t^P$$

- Then, X is a Doléans exponential, P-a.s. nonnegative.
- Same holds for all P.

Expected utility from integrated consumption and terminal wealth

- Let u be a continuous function on $\mathbb{R}^+ \times \mathbb{R}^+$ such that: for fixed t, $u(t, \cdot)$ is a utility function on $\mathbb{R}^+$, concave increasing in the second argument and verifying the Inada condition:

$$\lim_{x \to \infty} u'(t, x) = 0$$

- For a given probability P, consider the expected utility functional

$$E^P \left[\int_0^T u(t, c_t) dt + U(X_T) \right]$$

in which U is the utility from terminal wealth.

- This formulation covers both finite & infinite horizon planning (set $U = 0$ when $T = \infty$)

The robust utility maximization

- The investor is uncertainty averse, and faces the following robust Merton problem:

$$V(0,x) := \sup_{(\theta,c)\in\mathcal{A}_{\mathrm{rob}}(x)} \inf_{P\in\mathcal{P}} \mathbb{E}^P\left[\int_0^T u(t,c_t)dt + U(X_T)\right]$$

- More conservative portfolio choices are made when the uncertainty set Υ, and thus the family $\mathcal{P}$, is larger.
- An ambiguity-neutral investor sets ϵ equal to zero and $\Sigma = \hat{\Sigma}$ $\Rightarrow$ classical Merton problem.
- Lemma: the value $V(0,x)$ is increasing and concave in x.

Martingale Principle of Optimal Control

- Robust verification theorem. Suppose that:
  1. there exists a function $v : [0, T] \times \mathbb{R}^+ \to \mathbb{R}$, continuous on $[0, T] \times \mathbb{R}^+$ and $C^{1,2}$ on $[0, T) \times \mathbb{R}^+$, verifying $v(T, \cdot) = U(\cdot)$;
  2. for any $(\theta, c) \in \mathcal{A}_{\rm rob}(x)$ there exists an optimal $P^{(\theta,c)} \in \mathcal{P}$ of the inner minimization, such that

$$Y_t = Y_t^{(\theta,c)} \equiv v(t, X_t) + \int_0^t u(s, c_s) ds$$

     is a $P^{(\theta,c)}$-supermartingale;
  3. there exist some $(\bar{\theta}, \bar{c}) \in \mathcal{A}_{\rm rob}(x)$ such that the corresponding $\overline{Y}$ is a $P^{(\bar{\theta},\bar{c})}$- martingale.
- Then $(\bar{\theta}, \bar{c}, P^{(\bar{\theta},\bar{c})})$ is an optimizer for the robust Merton problem and $v(0, x) = V(0, x)$.

HJB-Isaacs equation for the candidate value function

- Under $P \in \mathcal{P}$,

$$dY_t = \left(u(t, c_t) + v_t + v_x(rx + \theta_t'(\mu_t - r\mathbf{1}) - c_t) + \frac{1}{2}\theta_t'\Sigma_t\theta_t v_{xx} \right) dt + v_x\theta_t'\sigma$$

- By Ito's Lemma, we derive a drift condition:

the sup over the agent's controls
of the inf over Nature's controls
of Y's drift must be zero.

- PDE of HJBI type:

$$\sup_{(\theta,c)} \inf_{(\Sigma,\mu)} \{u(t,c) + v_t + v_x(rx + \theta'(\mu - r\mathbf{1}) - c) + \frac{1}{2}\theta'\Sigma\theta v_{xx}\} = 0$$

where $(\theta, c) \in \mathbb{R}^n \times \mathbb{R}_+$, and $\Sigma \in K, \mu \in U_\epsilon(\Sigma)$.

- HJBI equations arise naturally in game theory. For more on these equations, see Evans & Souganidis (1984, Indiana Univ. Math. J.)

The HJBI PDE

- Proposition. Under the assumptions on u and K, the supremum and the infimum in the HJBI equation

$$\sup_{(\theta,c)\in\mathbb{R}^n\times\mathbb{R}_+}\ \inf_{\Sigma\in K,\mu\in U_\epsilon(\Sigma)}\{u(t,c)+v_t+v_x(rx+\theta'(\mu-r\mathbf{1})-c)+\frac{1}{2}\theta'\Sigma\theta v_{xx}\}=0$$

are attained for any $v\in C^{1,2}$ on $(0,T)\times\mathbb{R}^+$ with $v_x>0, v_{xx}<0$.

Inner minimization, I

- Minimize first over $\mu \in U_\epsilon(\Sigma)$ for Σ fixed.
- This amounts to the minimization of the linear function

$$v_x \theta' \mu$$

over the ellipsoid, and is just an exercise in constrained optimization.
- The optimizer is unique when $\theta \neq 0$

$$\mu(\theta) := \hat{\mu} - \epsilon \frac{\Sigma\theta}{\sqrt{\theta'\Sigma\theta}}$$

Inner minimization, II

- Minimize then over K:

$$\inf_{\Sigma \in K} \left[-\epsilon v_x \sqrt{\theta' \Sigma \theta} + \frac{1}{2} v_{xx} \theta' \Sigma \theta \right]$$

- Set $t = \sqrt{\theta' \Sigma \theta}$. The RHS is the restriction of the concave parabola

$$y(t) = -\epsilon v_x t + \frac{1}{2} v_{xx} t^2$$

  to a compact subset of the positive axis. Since the vertex has negative abscissa, the minimum is reached for the maximum t.
- Therefore, the optimizers are those $\Sigma \in K$ for which $\theta' \Sigma \theta$ is maximal.
- Call $M(\theta) := \max_K \theta' \Sigma \theta$ — continuous function of θ.

Outer maximization

- The last step is the maximization in the HJBI equation:

$$\sup_{(\theta,c)} u(t,c)+v_t+v_x(rx+\theta'(\hat{\mu}-r\mathbf{1})-\epsilon\sqrt{M(\theta)}-c)+\frac{1}{2}M(\theta)v_{xx}=0 \tag{1}$$

- The maximization can be split into the sum of:
  1. $\sup_{c\in\mathbb{R}^+}(u(t,c)+v_t-cv_x)$
     Concavity of u - $v_x>0$ and the Inada on u imply $\lim_{c\to+\infty}[u(t,c)+v_t-cv_x]=-\infty$. By continuity, sup is a max.
  2. $\sup_{\theta\in\mathbb{R}^n}\left(v_x(\theta'(\hat{\mu}-r\mathbf{1})-\epsilon\sqrt{M(\theta)})+\frac{1}{2}M(\theta)v_{xx}\right)$

$$v_x(\theta'(\hat{\mu}-r\mathbf{1})-\epsilon\sqrt{M(\theta)})+\frac{1}{2}M(\theta)v_{xx}\leq$$

$$\leq v_x(\theta'(\hat{\mu}-r\mathbf{1})-\epsilon h\|\theta\|)+\frac{1}{2}h^2\|\theta\|^2v_{xx}$$

Coercivity when $\|\theta\|\to\infty$. The sup is then attained by some $\bar{\theta}$, since the objective function is continuous.

Specifying to the CRRA utility case

- Assume $u(t, x)$ is either a power utility, or a log utility, in x.
- Make educated guess on the form of the value function
- Solve the above HJBI, find the optimal controls and verify.
- Infinite horizon T here (in the paper also $T < \infty$).
- Fix the time impatience constant $\rho > 0$. Either take:
  1. $u(t, x) = e^{-\rho t}\frac{x^{1-R}}{1-R}$, with $R > 0, R \neq 1$, or
  2. $u(t, x) = e^{-\rho t} \ln x$
- Structural properties give as candidates:
  1. $v(t, x) = (\gamma_\epsilon)^{-R} e^{-\rho t}\frac{x^{1-R}}{1-R}$, or
  2. $v(t, x) = e^{-\rho t}(\frac{\ln x}{\rho} + k_\epsilon)$

  in which γ_ϵ and k_ϵ are suitable constants.

Some relevant constants

- Let

$$\overline{K} := \mathrm{argmin}_{\Sigma \in K} \left((\hat{\mu} - r\mathbf{1})' \Sigma^{-1} (\hat{\mu} - r\mathbf{1}) \right)$$

and $\overline{H}$ denote the square root of the minimum above, namely the Sharpe Ratio

$$\overline{H} := \sqrt{(\hat{\mu} - r\mathbf{1})' \overline{\Sigma}^{-1} (\hat{\mu} - r\mathbf{1})}, \text{ for any } \overline{\Sigma} \in \overline{K}$$

- Let also $\overline{H}_\epsilon^+ = \max(\overline{H} - \epsilon, 0)$.
- A simple calculation shows that $\overline{H}_\epsilon^+$ is the minimum Sharpe Ratio in the uncertainty set:

$$\overline{H}_\epsilon^+ = \min_{\Sigma \in K, \mu \in U_\epsilon(\Sigma)} \sqrt{(\mu - r\mathbf{1})' \Sigma^{-1} (\mu - r\mathbf{1})}$$

Theorem (power case)

- The infinite-horizon robust Merton problem, with power utility and initial endowment $x > 0$:

$$V(0,x) = \sup_{(\theta,c)\in\mathcal{A}_{\mathrm{rob}}(x)} \inf_{P\in\mathcal{P}} E^P \left[\int_0^\infty e^{-\rho t} \frac{c_t^{1-R}}{1-R} dt \right],$$

is finite valued if and only if

$$\overline{\gamma}_\epsilon = \frac{\rho + (R-1)(r + \frac{1}{2}\frac{(\overline{H}_\epsilon^+)^2}{R})}{R} > 0.$$

Theorem (continued)

In case $\overline{\gamma}_\epsilon > 0$:

- The optimal value is

$$V(0,x) = \overline{\gamma}_\epsilon^{-R} \frac{x^{1-R}}{1-R}$$

- Optimal adverse control is any $\overline{P}$ under which S evolves with constant instantaneous covariance $\overline{\Sigma} \in \overline{K}$ and drift

$$\bar{\mu} := \begin{cases} \hat{\mu} - \epsilon \frac{\overline{\Sigma}}{\sqrt{\overline{\pi}_\epsilon' \overline{\Sigma} \overline{\pi}_\epsilon}} \overline{\pi}_\epsilon & \text{if } r\mathbf{1} \notin U_\epsilon(\overline{\Sigma}) \\ r\mathbf{1} & \text{otherwise} \end{cases}$$

- Optimal agent controls are:

$$\bar{\theta}_t = \overline{\pi}_\epsilon \bar{X}_t \,,\ \bar{c}_t = \overline{\gamma}_\epsilon \bar{X}_t,$$

with optimal portfolio proportions vector given by

$$\overline{\pi}_\epsilon := \frac{\overline{H}_\epsilon^+}{R\overline{H}} \overline{\Sigma}^{-1} (\hat{\mu} - r\mathbf{1})$$

Theorem (continued)

- The optimal wealth process has $\overline{P}$ dynamics given by

$$\bar{X}_t = x \exp\left(\overline{\pi}_\epsilon \overline{\sigma} W_t^{\bar{P}} + (r + \frac{(\overline{H}_\epsilon^+)^2(2R-1)}{2R^2} - \overline{\gamma}_\epsilon)t\right)$$

  when $\overline{\pi}_\epsilon \neq 0$, and otherwise is $\bar{X}_t = x \exp\left((r - \overline{\gamma}_\epsilon)t\right)$.
- the equalities

$$V(0,x) = \inf_{P \in \mathcal{P}} \sup_{(\theta,c) \in \mathcal{A}_{\mathrm{rob}}(x)} E^P\left[\int_0^\infty u(t,c_t)dt\right]$$

  hold and $((\bar{\theta},\bar{c}),\bar{P})$ is a saddle point of the agent-adverse market game;
- any optimal $(\overline{\Sigma},\bar{\mu})$ pointwisely realizes the worst SR among the models $\mathcal{P}$:

$$\min_{P \in \mathcal{P}} \sqrt{(\mu_t - r\mathbf{1})'\Sigma_t^{-1}(\omega)(\mu_t - r\mathbf{1})} = const = \overline{H}_\epsilon^+$$

Theorem (Log case)

The infinite-horizon robust Merton problem, with logarithmic utility and initial endowment $x > 0$:

$$V(0,x) = \sup_{(\theta,c)\in\mathcal{A}_{\mathrm{rob}}(x)} \inf_{P\in\mathcal{P}} E^P\left[\int_0^\infty e^{-\rho t}\ln c_t\, dt\right]$$

has value

$$V(0,x) = \frac{\ln x}{\rho} + \overline{k_\epsilon},$$

in which

$$\overline{k_\epsilon} = \frac{1}{\rho^2}\left[\rho\ln\rho + r - \rho + \frac{(\overline{H_\epsilon}^+)^2}{2}\right].$$

- The optimal controls are

  $$\bar{\theta}_t = \overline{\pi}_\epsilon \bar{X}_t\,,\ \ \bar{c}_t = \rho\bar{X}_t,$$

  with optimal portfolio proportions vector given by

  $$\overline{\pi}_\epsilon := \frac{\overline{H_\epsilon}^+}{\overline{H}}\overline{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1}).$$

Theorem (Log case, continued)

- Any optimal adverse control is given by a probability $\bar{P}$ under which S evolves with any constant instantaneous covariance $\overline{\Sigma}$ and drift

$$\bar{\mu} := \begin{cases} \hat{\mu} - \epsilon \frac{\overline{\Sigma}}{\sqrt{\overline{\pi}_\epsilon' \overline{\Sigma} \overline{\pi}_\epsilon}} \overline{\pi}_\epsilon & \text{if } r\mathbf{1} \notin U_\epsilon(\overline{\Sigma}) \\ r\mathbf{1} & \text{otherwise} \end{cases}$$

- The optimal wealth process has $\overline{P}$ dynamics given by

$$\bar{X}_t = x \exp\left(\overline{\pi}_\epsilon \sigma W_t^{\overline{P}} + (r + (H_\epsilon^+)^2 - \rho)t\right). \tag{2}$$

when $\overline{\pi}_\epsilon \neq 0$, and is the deterministic $\bar{X}_t = x \exp((r - \rho)t)$ otherwise.

- The minimax equality holds:

$$V(0,x) = \inf_{P \in \mathcal{P}} \sup_{(\theta,c) \in \mathcal{A}_{\mathrm{rob}}(x)} E^P\left[\int_0^\infty e^{-\rho t} \ln c_t \, dt\right]$$

and $((\bar{\theta}, \bar{c}), \overline{P})$ is a saddle point for the agent-market game.

Some comments

- When $\epsilon = 0$ and no uncertainty on Σ we obtain exactly the Merton's controls.
- The agent does not participate to the stock market ($\overline{\pi}_\epsilon = 0,$) if and only if ambiguity is too high, in the sense that $r\mathbf{1}$ is a plausible drift under a worst case $\overline{\Sigma}$.
- The verification is by comparison with the reduced ambiguity problem with Σ fixed and constant, and varying drift. Then, minimizing over $\Sigma \in K$.

Equity premium puzzle and effect of robustness on decisions

- Mehra and Prescott (JMonEcon, 1985) the high levels of historical equity premium and the simultaneous moderate equity demand seem to be implied by unreasonable levels of risk aversion.
- Cecchetti, Lam and Mark (AER 2000) relax hp of perfect knowledge of P. In a discrete economy, robustness in the decisions lowers the optimal demand on equity.
- Theoretical basis for a possible explanation of the equity premium puzzle.
- The optimal relative allocation of wealth depends on risk and ambiguity aversion:

$$\frac{1}{R}\frac{\max\{\overline{H}-\epsilon,0\}}{\overline{H}}.$$

- In the extreme case $\epsilon \geq \overline{H}$, ambiguity aversion leads to non-participation in the risky assets.

Mutual Fund Theorem for the CRRA class

- Mutual Fund Theorem: *independently of the agent's utility, the optimal portfolio consists of an allocation between two fixed mutual funds, namely the riskless asset and a fund of risky assets.*
  Tobin (RES 1958); Merton JET 1971 & Econom. 1973; Cass& Stiglitz (JET 1970); Chamberlain (Econom. 1988); Schachermayer, Sirbu and Taflin (FinaSto 2009)
- Our result proves that the MFT holds in the robust case for CRRA utilities, the risky mutual fund being the constant

$$\overline{\Sigma}^{-1}(\hat{\mu} - r\mathbf{1})$$

Final Remarks

- Robust problem observationally equivalent to CRRA utility maximization with $\overline{P}$ under which S evolves with the *worst SR.*
- Results in accordance with the existing literature. Our contribution differs in that we do not require any specific condition on the volatility structure as Lin-Riedel, nor convexity of K or compactness of the strategies as in Neufeld-Nutz.
- From a computational point of view, one only needs to find the worst cases matrices set $\bar{K}$. Easy when K is *convex*, but convexity is not required for our theoretical results to hold.
- Simple case: K has a maximal element Σ^M with respect to the positive ordering of symmetric matrices:

  $$x'\Sigma^M x \geq x'\Sigma x \quad \text{for all } x \in \mathbb{R}^n, \Sigma \in K$$

  then Σ^M will be a worst case matrix, as it happens in the two examples below.

Example I

- Estimated $\hat{\Sigma}$ constant (n stocks, uncorrelated).
- Constraints on all eigenvalues of Σ. Compact set K given by

$$K = \{\Sigma \mid \Sigma \in Diag, \text{ and } 0 < \underline{\sigma}_i^2 \leq \sigma_{ii} \leq \overline{\sigma}_i^2, i = 1 \dots n\}$$

- The worst case Σs for given θ are the maximizers of

$$\theta' \Sigma \theta$$

- Independently of θ, $\overline{\Sigma} = Diag(\overline{\sigma}_1^2, \dots, \overline{\sigma}_n^2)$, optimal strategy

$$\overline{\theta} = \frac{x}{R} \frac{\overline{H}_\epsilon^+}{\overline{H}} \begin{pmatrix} \frac{1}{\overline{\sigma}_1^2} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \frac{1}{\overline{\sigma}_n^2} \end{pmatrix} (\hat{\mu} - r\mathbf{1})$$

- Analog of the case in Lin & Riedel.

Example II

- Variation of the previous example: constraints on the VarCov formulated via the quadratic form $\theta'\Sigma\theta$.
- Example:

$$K = \{\Sigma \mid 0 < h^2 \leq x'\Sigma x \leq \lambda^2 \text{ on the unit sphere}\}$$

  e.g. maximum eigenvalue smaller than $\lambda^2 > 0$, minimum bigger than h^2.
- The worst case Σs for given θ are obtained as the maximizers of

$$\theta'\Sigma\theta$$

- So, the optimal constant $\overline{\Sigma} = \lambda^2 I_n$ and the optimal strategy is

$$\overline{\theta} = \frac{x}{R}\frac{\overline{H}_\epsilon^+}{\overline{H}}\frac{1}{\lambda^2}(\hat{\mu} - r\mathbf{1}) = \frac{x}{R\lambda^2}\frac{\max\{\|\hat{\mu} - r\mathbf{1}\| - \epsilon\lambda, 0\}}{\|\hat{\mu} - r\mathbf{1}\|}(\hat{\mu} - r\mathbf{1})$$

Thank you!