Hedging with temporary price impact

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joint work (partially in progress) with

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October 21, 2016

Quadratic Hedging

- $H \in L^2(\mathscr{F}_T)$ contingent claim to be hedged
- $S \in \mathscr{M}^2$ price evolution of a tradable asset with local variance $\sigma_t^2 = d\langle S \rangle_t / dt$
- Föllmer and Sondermann: Minimize quadratic hedging error

$$\xi^H = \arg\min \mathbb{E}\left[\left(H - \int_0^T \xi_t dS_t)\right)^2\right]$$

$\rightsquigarrow$ ξ^H is replicating strategy if H is attainable

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What if market frictions force us to follow an alternative strategy X instead of ξ^H?

Quadratic Hedging with frictions

Minimize quadratic hedging error

$$
\begin{aligned}
X^* &= \arg\min \mathbb{E}\left[\left(H - \int_0^T X_t dS_t)\right)^2\right] \\
&= \arg\min \mathbb{E}\left[\left(H - \int_0^T \xi_t^H dS_t)\right)^2\right] + \mathbb{E}\left[\left(\int_0^T (\xi_t^H - X_t) dS_t\right)^2\right] \\
&= \arg\min \mathbb{E}\left[\int_0^T (\xi_t^H - X_t)^2 \sigma_t^2 dt\right]
\end{aligned}
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$\rightsquigarrow$ We should try to track $\xi \triangleq \xi^H$ as close as possible. . .

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\end{aligned}
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$\rightsquigarrow$ We should try to track $\xi \triangleq \xi^H$ as close as possible...
...subject to constraint by expected transaction costs:

$$\mathbb{E}\int_0^T \kappa_t u_t^2\, dt \leq c$$

where $u_t = \dot{X}_t$ measures trading speed and

$$\text{position at time } t = X_t = x + \int_0^t u_s\, ds$$

Quadratic tracking problem

Mathematical optimization problem

For a given predictable $\xi \in L^2(\mathbb{P} \otimes dt)$ and given $x \in \mathbb{R}$, find an absolutely continuous, adapted process $X_t = x + \int_0^t u_s ds$ with $u \in L^2(\mathbb{P} \otimes ds)$, which minimizes

$$J(u) \triangleq \mathbb{E}\left[\int_0^T (\xi_t - X_t)^2 \sigma_t^2 \, dt + \int_0^T \kappa_t u_t^2 \, dt\right]$$

for given progressively measurable, strictly positive processes σ, κ.

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for given progressively measurable, strictly positive processes σ, κ.
Possible additional constraint on terminal position:

$$X_T = \Xi_T \text{ for some given } \Xi_T \in L^2.$$

Closely related references from Mathematical Finance

Rogers & Singh (2010) , Naujokat & Westray (2011), Frei & Westray (2013), Schied (2013), Horst & Naujokat (2014), Almgren & Li (2014), Cartea & Jaimungal (2015), Cai et al. (2015, 2016), . . .

Constant coefficients in the unconstrained case

Theorem

If σ and κ are constant and there is no constraint on the terminal position, it is optimal to always trade towards

$$\hat{\xi}_t = \frac{\operatorname{sech}(\frac{T-t}{\sqrt{\lambda}})}{\sqrt{\lambda}} \mathbb{E}\left[\int_t^T \xi_s \cosh(\frac{T-s}{\sqrt{\lambda}})\, ds \,\middle|\, \mathscr{F}_t \right]$$

according to

$$dX_t^* = \frac{\tanh(\frac{T-t}{\sqrt{\lambda}})}{\sqrt{\lambda}} \left(\hat{\xi}_t - X_t^* \right) dt$$

where $\lambda \triangleq \kappa/\sigma^2$.

Rather than towards the current target ξ_t, one should trade towards its expected future $\hat{\xi}_t$; cf. Garleanu & Pedersen (2014).

Constant coefficients in the constrained case

Theorem

If σ and κ are constant and the terminal position has to be Ξ_T, it is optimal to always trade towards

$$\hat{\xi}_t = \frac{1}{\cosh(\frac{T-t}{\sqrt{\lambda}})} \mathbb{E}\left[\Xi_T \mid \mathscr{F}_t\right] + \left(1 - \frac{1}{\cosh(\frac{T-t}{\sqrt{\lambda}})}\right) \mathbb{E}\left[\int_t^T \xi_s \frac{\sinh(\frac{T-s}{\sqrt{\lambda}})}{\sqrt{\lambda}(\cosh(\frac{T-t}{\sqrt{\lambda}}) - 1)} \,\middle|\, \mathscr{F}_t\right]$$

according to

$$dX_t^* = \frac{\coth(\frac{T-t}{\sqrt{\lambda}})}{\sqrt{\lambda}} \left(\hat{\xi}_t - X_t^*\right) dt$$

where $\lambda \triangleq \kappa/\sigma^2$.

As $t \uparrow T$ we have to trade towards $\hat{\xi}$ (and thus towards Ξ_T) with higher and higher urgency.

Illustration: Frictionless hedge with jump midway

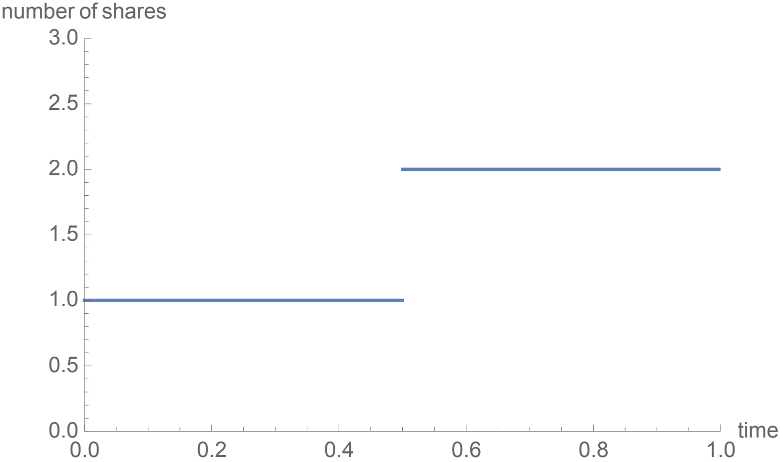

Figure: Target strategy ξ with a jump at $t = T/2$ (blue)

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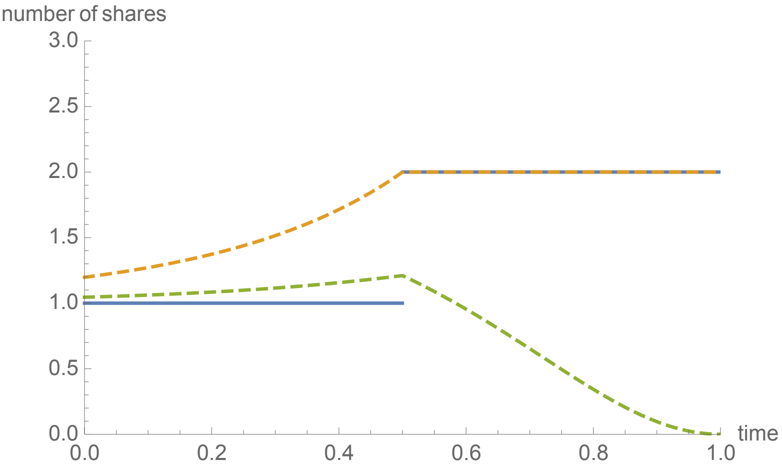

Figure: Target strategy ξ with a jump at $t = T/2$ (blue), unconstrained (orange, dashed) and constrained (green, dashed) target

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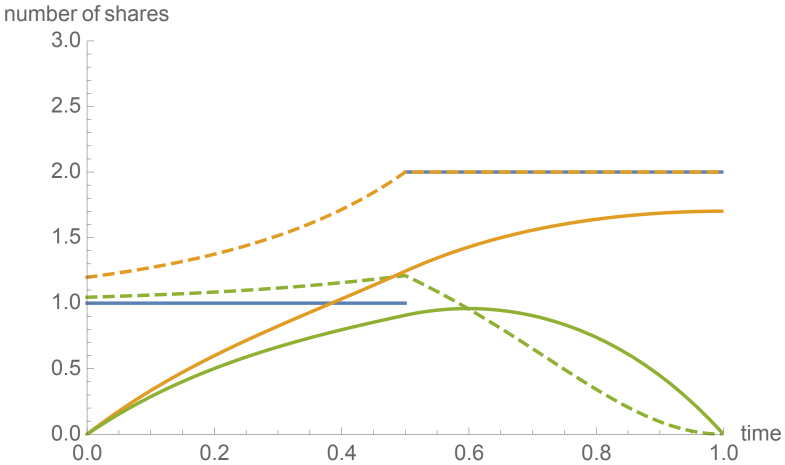

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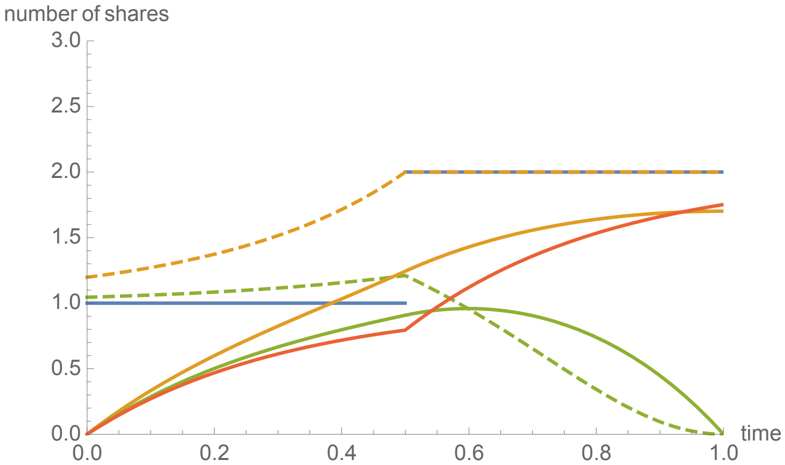

Figure: Target strategy ξ with a jump at $t = T/2$ (blue), unconstrained (orange, dashed) and constrained (green, dashed) target, corresponding unconstrained (orange) and constrained (green) frictional hedge, and directly targeting strategy (red)

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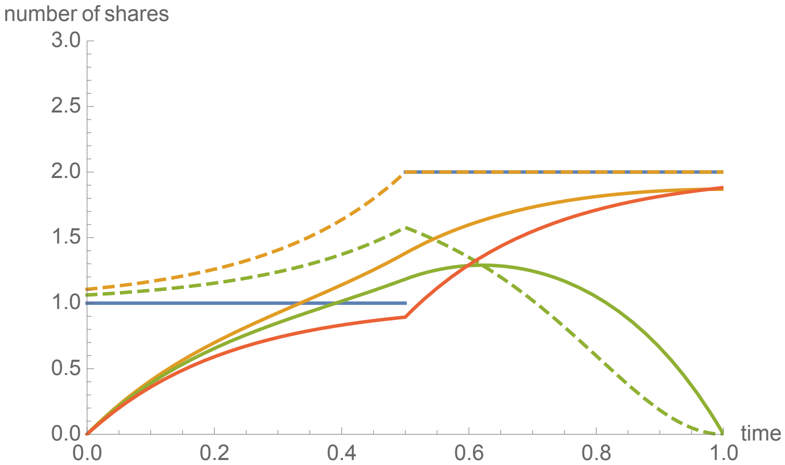

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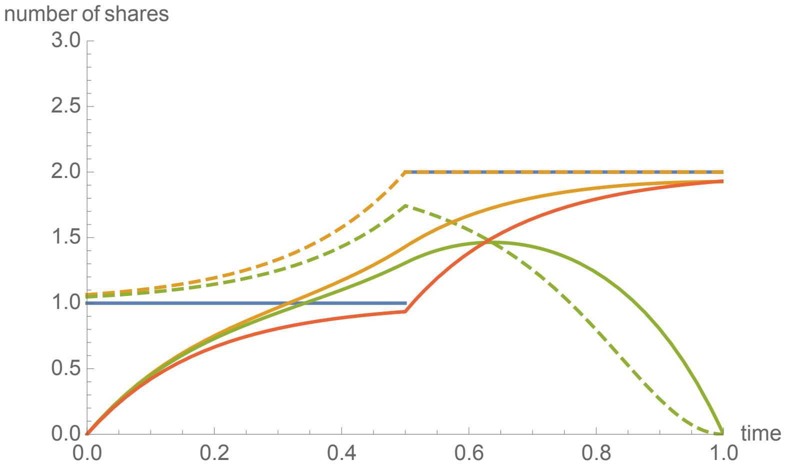

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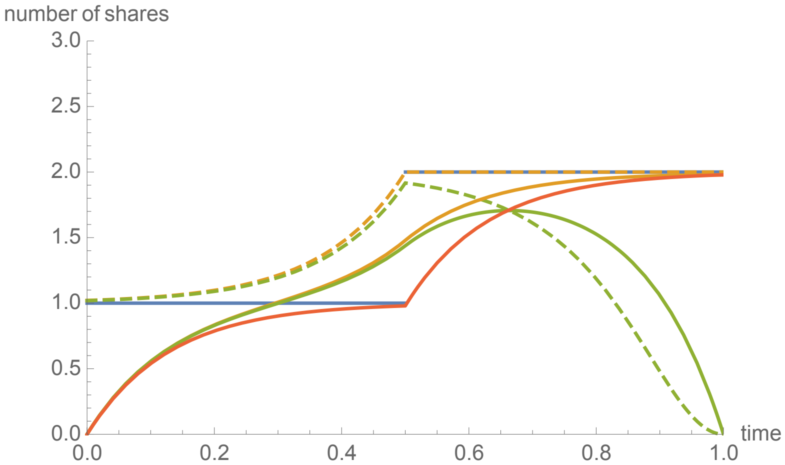

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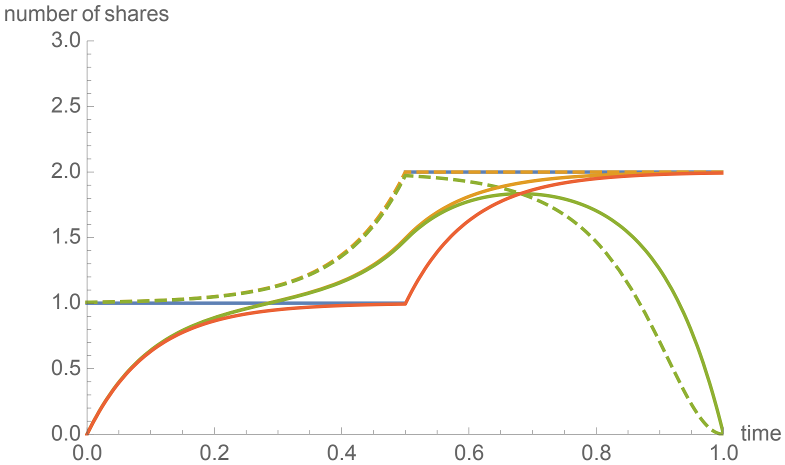

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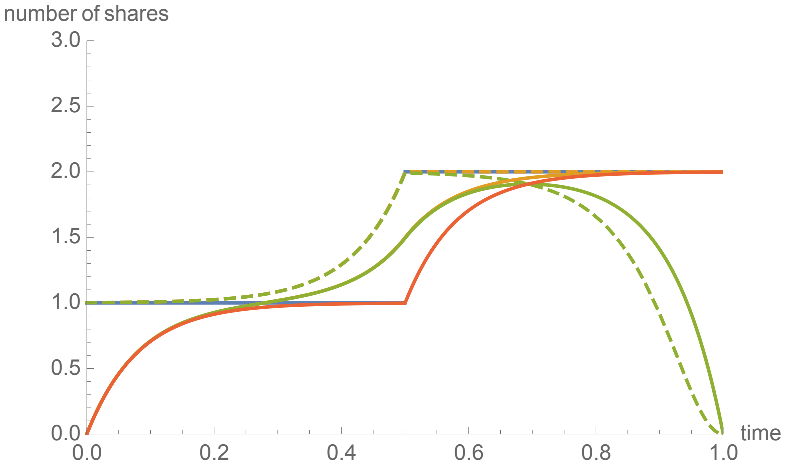

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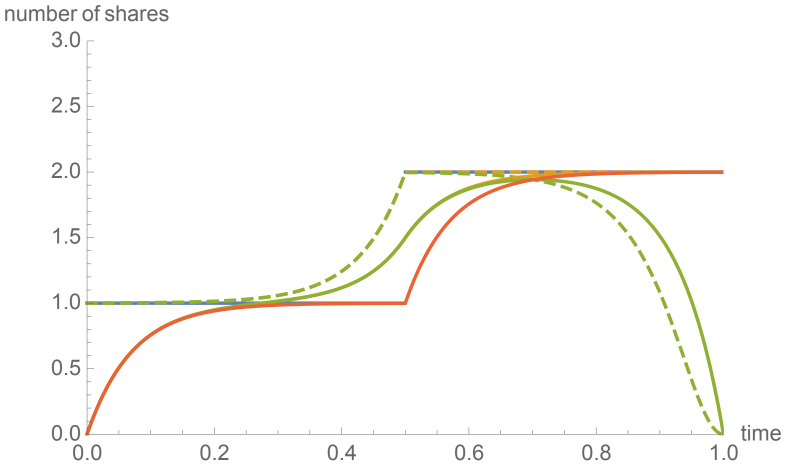

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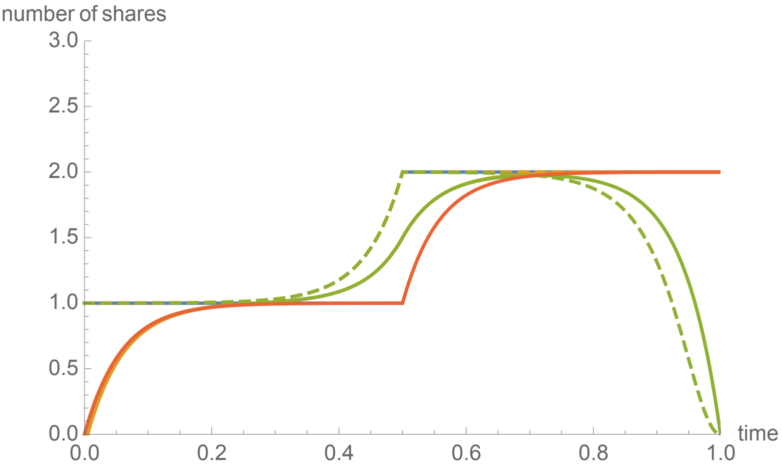

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Illustration: Discretely monitored Asian option

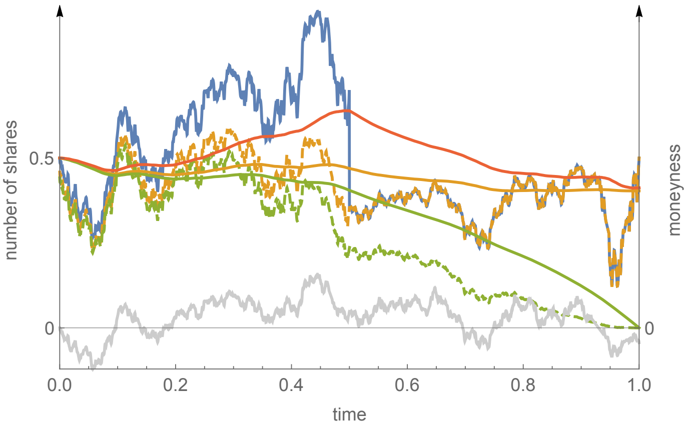


Figure: Target strategy ξ of "Asian option" $(\frac{1}{2}(S_{T/2} + S_T) - K)^+$ (blue), unconstrained (orange, dashed) and constrained (green, dashed) target, corresponding unconstrained (orange) and constrained (green) frictional hedge, and directly targeting strategy (red)

Illustration: Call option with physical delivery

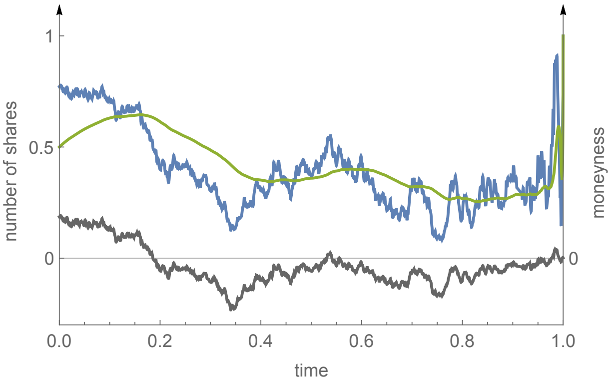
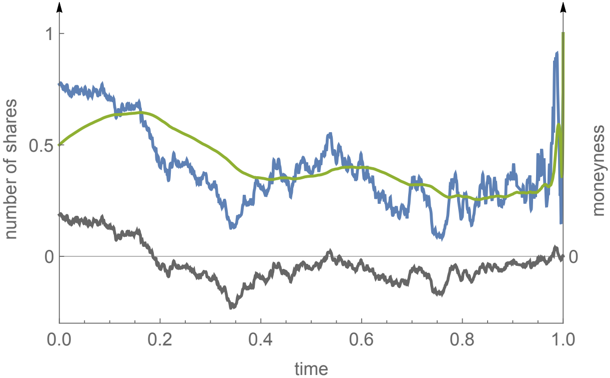


Illustration: Call option with physical delivery ???



Lemma

A terminal position Ξ_T can be attained at finite expected costs if and only if

$$\int_0^T \frac{\mathbb{E}[(\Xi_T - \Xi_t)^2]}{(T-t)^2}\, dt < \infty \text{ where } \Xi_t = \mathbb{E}\left[\Xi_T \mid \mathscr{F}_t\right].$$

General case with stochastic coefficients

For a given predictable $\xi \in L^2(\mathbb{P} \otimes dt)$ and given $x \in \mathbb{R}$, find an absolutely continuous, adapted process $X = x + \int_0^{\cdot} u_t dt$ with $u \in L^2(\mathbb{P} \otimes dt)$, which minimizes

$$\mathbb{E}\left[\int_0^T (\xi_t - X_t)^2 \sigma_t^2 \, dt + \int_0^T \kappa_t u_t^2 \, dt + \eta(\Xi_T - X_T)^2\right]$$

with σ, κ progressively measurable, strictly positive, bounded processes, nonnegative η and $\Xi_T \in \mathscr{F}_T$.

General case with stochastic coefficients

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with σ, κ progressively measurable, strictly positive, bounded processes, nonnegative η and $\Xi_T \in \mathscr{F}_T$.

Also allow for $\eta = +\infty$ with positive probability:

- $\rightsquigarrow$ imposes implicitly the terminal state constraint $X_T = \Xi_T$ on $\{\eta = +\infty\}$ (constrained problem)
- $\rightsquigarrow$ we have to be careful with $\eta(\Xi_T - X_T)^2$ if $\eta = \infty$ and $\Xi_T = X_T$: "truncation in space" vs. "truncation in time".

Bounded penalization

Kohlmann and Tang (2002) : for $\eta \geq 0$ bounded, consider

$$\text{(BSRDE)} \quad dc_t = \frac{c_t^2}{\kappa_t} dt - \sigma_t^2 dt - dM_t \quad (0 \leq t \leq T), \quad c_T = \eta.$$

Theorem

The **optimal tracking** *strategy* X^* *is given by*

$$dX_t^* = \frac{c_t}{\kappa_t} \left(\hat{\xi}_t - X_t^* \right) dt$$

where

$$\hat{\xi}_t \triangleq w_t \, \mathbb{E}_{\mathbb{Q}}[\Xi_T | \mathscr{F}_t] + (1 - w_t) \, \mathbb{E} \left[\int_t^T \xi_r \frac{e^{-\int_t^r \frac{c_u}{\kappa_u} du}}{(1 - w_t) c_t} \sigma_r^2 dr \middle| \mathscr{F}_t \right]$$

with the supermartingale $L_t \triangleq c_t e^{-\int_0^t \frac{c_u}{\kappa_u} du} \geq 0$ *yielding*

weights $w_t \triangleq \dfrac{\mathbb{E}[L_T | \mathscr{F}_t]}{L_t}$ *and the probability* $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = \dfrac{L_T}{\mathbb{E}[L_T]}.$

Solution to optimal liquidation problem

In case where $\mathbb{P}[\eta = +\infty] > 0$, but targeting $\xi \equiv 0$, $\Xi_T = 0$:

Theorem (Kruse & Popier (2015))

Let $\xi_t \equiv 0$ *and* $\Xi_T = 0$ $\mathbb{P}$*-a.s. Consider solution* $(c_t)_{0 \leq t \leq T}$ *of*

$$\text{(BSRDE)} \quad dc_t = \frac{c_t^2}{\kappa_t} dt - \sigma_t^2 dt - dM_t \quad (0 \leq t < T), \quad c_T = \eta.$$

Then the **optimal liquidation** *strategy* X^0 *is given by*

$$dX_t^0 = -\frac{c_t}{\kappa_t} X_t^0 dt$$

and satisfies $\lim_{t \uparrow T} X_t^0 = 0$ *on* $\{\eta = +\infty\}$*.*
The minimal costs are given by

$$J(X^0) = c_0 x^2.$$

General result

Suppose:

- integrable coefficients: $\int_0^T (\sigma_t^2 + \kappa_t^{-1})\, dt < \infty$ a.s.
- There is a unique semimartingale $c = (c_t)_{0 \leq t < T} > 0$ with

$$\text{(BSRDE)} \quad dc_t = \frac{c_t^2}{\kappa_t} dt - \sigma_t^2 dt - dM_t \quad (0 \leq t < T), \quad \lim_{t \uparrow T} c_t = \eta$$

  such that

$$\mathbb{E} \sup_{s<t} (c_s^2 + M_s^2) < \infty \text{ for any } t < T$$

  and

$$\int_{[0,T)} \frac{d[c]_t}{c_{t-}^2} < \infty \text{ on } \{\eta = +\infty\}.$$

- integrable targets: $\xi_t \in L^1(\mathbb{P} \otimes \sigma_t^2\, dt)$, $\Xi_T L_T \in L^1(\mathbb{P})$

General result (ctd)

Then:

- The signal process

$$\hat{\xi}_t \triangleq \frac{1}{L_t}\mathbb{E}\left[\Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u}du}\sigma_r^2 dr \,\middle|\, \mathscr{F}_t\right] \quad (0 \le t < T)$$

  is well defined and satisfies $\lim_{t\uparrow T}\hat{\xi}_t = \Xi_T$ on $\{\eta > 0\}$.

- The target functional

$$J(u) \triangleq \limsup_{\tau\uparrow T}\mathbb{E}\left[\int_0^\tau (X_t^u - \xi_t)^2\sigma_t^2 dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau(X_\tau^u - \hat{\xi}_\tau)^2\right]$$

  has nonempty domain $\mathrm{dom}\, J \triangleq \{u \mid J(u) < \infty\}$ iff

$$\mathbb{E}\left[\int_0^T \hat{\xi}_t^2\sigma_t^2 dt\right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\int_{[0,T)} c_t d[\hat{\xi}]_t\right] < +\infty.$$

General result (ctd)

- If $\operatorname{dom} J \neq \emptyset$, the optimal control u^* is given in feedback form with $X^* \triangleq X^{u^*}$ via

$$u^*_t = \frac{c_t}{\kappa_t}(\hat{\xi}_t - X^*_t), \quad 0 \leq t < T.$$

- The minimal costs decompose as

$$J(u^*) = c_0(x - \hat{\xi}_0)^2 + \mathbb{E}\left[\int_0^T (\xi_t - \hat{\xi}_t)^2 \sigma_t^2 dt\right] + \mathbb{E}\left[\int_{[0,T)} c_t d[\hat{\xi}]_t\right]$$

into costs due to suboptimal starting position, to the (lack of) regularity and compatibility of the targets ξ, Ξ_T, and to the signal's variability given new information on problem data.

Key insights for proof

A lengthy calculation reveals that

$$\begin{aligned}
&\int_0^\tau (X_t^u - \xi_t)^2 \sigma_t^2 dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u - \hat{\xi}_\tau)^2 \\
=& c_0 (x - \hat{\xi}_0)^2 + \int_0^\tau (\xi_t - \hat{\xi}_t)^2 \sigma_t^2 dt + \int_0^\tau c_t d[\hat{\xi}]_t \\
&+ \int_0^\tau \kappa_t \left(u_t - \frac{c_t}{\kappa_t} \left(\hat{\xi}_t - X_t^u \right) \right)^2 dt \\
&+ \text{local martingale}_\tau \,.
\end{aligned}$$

Carefully taking expectations and letting $\tau \uparrow T$ reveals optimality of given $\hat{u}$ along with necessary and sufficient conditions for $\operatorname{dom} J \neq \emptyset$.

Conclusions

- quadratic hedging with quadratic transaction costs from temporary price impact
- explicit solution for constant coefficients: trade towards expected average future position of suitable frictionless optimum
- ... possibly combined with weighted expectation of ultimate target position
- characterization of ultimate positions which are attainable with finite expected costs
- closed-form hedging recipes also for frictionless reference hedges which have singularities
- very general optimal control with stochastic coefficients solved in terms of (singular) backward stochastic Riccati equation
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