Stochastic Orders, Multi-Utility Representations and Central Regions.

A Set Optimization Perspective.

Andreas H. Hamel

Free University Bozen

WU Wien, October 2015

Outline.

- A motivating example: second order stochastic dominance
- A general framework: set relations via scalar families
- Examples
- The set optimization approach to preference optimization

First.

- Second order stochastic dominance

Second order stochastic dominance.

- $\mathcal{M}_{1,1}(\mathrm{I\!R}, \mathcal{B})$ set of all (Borel) probability measures on $\mathrm{I\!R}$ with finite mean

Motivating example.

Second order stochastic dominance.

- $\mathcal{M}_{1,1}(\mathbb{R}, \mathcal{B})$ set of all (Borel) probability measures on $\mathbb{R}$ with finite mean
- $\mathcal{U}$ set of all (strictly) increasing, (strictly) concave functions $u \colon \mathbb{R} \to \mathbb{R}$

Second order stochastic dominance.

- $\mathcal{M}_{1,1}(\mathrm{I\!R}, \mathcal{B})$ set of all (Borel) probability measures on $\mathrm{I\!R}$ with finite mean
- $\mathcal{U}$ set of all (strictly) increasing, (strictly) concave functions $u \colon \mathrm{I\!R} \to \mathrm{I\!R}$
- for $\mu, \nu \in \mathcal{M}_{1,1}(\mathrm{I\!R}, \mathcal{B})$

$$\mu \succeq_{SSD} \nu \quad :\Leftrightarrow \quad \forall u \in \mathcal{U} \colon \int u \, \mu(dx) \geq \int u \, \nu(dx),$$

i.e., every "rational" (= risk averse) decision maker prefers μ over ν.

Stochastic dominance.

$$\mu \succeq_{SSD} \nu \quad :\Leftrightarrow \quad \forall u \in \mathcal{U} \colon \int u \, \mu(dx) \geq \int u \, \nu(dx),$$

- a reflexive, transitive, antisymmetric order relation on $\mathcal{M}_{1,1}(\mathrm{I\!R}, \mathcal{B})$, i.e. a partial order

Motivating example.

Stochastic dominance.

$$\mu \succeq_{SSD} \nu \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}\colon \int u \, \mu(dx) \geq \int u \, \nu(dx),$$

- a reflexive, transitive, antisymmetric order relation on $\mathcal{M}_{1,1}(\mathbb{R}, \mathcal{B})$, i.e. a partial order
- defined through a family of real-valued functions on $\mathcal{M}_{1,1}(\mathbb{R}, \mathcal{B})$,

$$\mu \mapsto \int u \, \mu(dx).$$

Motivating example.

Stochastic dominance.

$$\mu \succeq_{SSD} \nu \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}\colon \int u\,\mu(dx) \geq \int u\,\nu(dx),$$

- a reflexive, transitive, antisymmetric order relation on $\mathcal{M}_{1,1}(\mathbb{R}, \mathcal{B})$, i.e. a partial order
- defined through a family of real-valued functions on $\mathcal{M}_{1,1}(\mathbb{R}, \mathcal{B})$,

$$\mu \mapsto \int u\,\mu(dx).$$

- "hard to maximize," i.e. it is difficult to identify a "best" element in a set $\mathcal{N} \subseteq \mathcal{M}_{1,1}(\mathbb{R}, \mathcal{B})$ w.r.t. $\succeq_{SSD}$.

Stochastic dominance.

$$\mu \succeq_{SSD} \nu \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}\colon \int u\,\mu(dx) \geq \int u\,\nu(dx),$$

- kind of annoying on a random variable level: the relation

$$X \succeq_{SSD} Y \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}\colon \int u(X)\,dP \geq \int u(Y)\,dP$$

for $X, Y \in L^1(\Omega, \mathcal{F}, P)$ is no longer antisymmetric, and it is not a vector order on L^1.

Motivating example.

Stochastic dominance.

$$\mu \succeq_{SSD} \nu \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}\colon \int u\, \mu(dx) \geq \int u\, \nu(dx),$$

- kind of annoying on a random variable level: the relation

$$X \succeq_{SSD} Y \quad :\Leftrightarrow \quad \forall u \in \mathcal{U}\colon \int u(X)\, dP \geq \int u(Y)\, dP$$

  for $X, Y \in L^1(\Omega, \mathcal{F}, P)$ is no longer antisymmetric, and it is not a vector order on L^1.

Questions. Improve the order structure? How to maximize w.r.t. $\succeq_{SSD}$?

Note. Stochastic dominance of second order is just one instance of an *integral stochastic order*; in turn, integral stochastic orders are just examples for order relations defined through *families of scalar functions*.

Note. Stochastic dominance of second order is just one instance of an *integral stochastic order*; in turn, integral stochastic orders are just examples for order relations defined through *families of scalar functions*.

Note. Such orders are usually **non-total**, i.e. they do not satisfy the vNM axioms.

Bacic question.

Note. Stochastic dominance of second order is just one instance of an *integral stochastic order*; in turn, integral stochastic orders are just examples for order relations defined through *families of scalar functions*.

Note. Such orders are usually **non-total**, i.e. they do not satisfy the vNM axioms.

Basic question. How to deal with non-total preferences, in particular how to maximize/minimize w.r.t. such orders?

Basic answer.

Today's answer.

Turn the problem into a complete lattice-valued one and use set optimization concepts.

Next.

- Set relations via scalar families.

Preorders via extended real-valued functions.

- $(Z, \preceq)$ a preordered set, i.e. $\preceq$ is reflexive and transitive

[1] It is already a not-so-apparent idea to ask this question.

Set relations via scalar families.

Preorders via extended real-valued functions.

- $(Z, \preceq)$ a preordered set, i.e. $\preceq$ is reflexive and transitive
- Ψ collection of functions $\psi \colon Z \to \mathrm{I\!R} \cup \{\pm\infty\}$ satisfying

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \forall \psi \in \Psi \colon \psi(z_1) \leq \psi(z_2)$$

for $z_1, z_2 \in Z$.

[1] It is already a not-so-apparent idea to ask this question.

Preorders via extended real-valued functions.

- $(Z, \preceq)$ a preordered set, i.e. $\preceq$ is reflexive and transitive
- Ψ collection of functions $\psi \colon Z \to \mathrm{IR} \cup \{\pm\infty\}$ satisfying

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \forall \psi \in \Psi \colon \psi(z_1) \leq \psi(z_2)$$

for $z_1, z_2 \in Z$.

Question.[1] How can Ψ be used to define an order on

$$\mathcal{P}(Z) = \{A \mid A \subseteq Z\}?$$

[1] It is already a not-so-apparent idea to ask this question.

Inf-extension of $\psi \in \Psi$ is $\psi^{\vartriangle} \colon \mathcal{P}(Z) \to \mathrm{I\!R} \cup \{\pm\infty\}$ defined by

$$\psi^{\vartriangle}(D) = \inf_{z \in D} \psi(z).$$

Set relations via scalar families.

Inf-extension of $\psi \in \Psi$ is $\psi^{\triangle} \colon \mathcal{P}(Z) \to \mathrm{I\!R} \cup \{\pm\infty\}$ defined by

$$\psi^{\triangle}(D) = \inf_{z \in D} \psi(z).$$

Inf-extension of $\preceq$ to $\mathcal{P}(Z)$ is $\preceq_{\Psi}$ defined by

$$D_1 \preceq_{\Psi} D_2 \quad \Leftrightarrow \quad \forall \psi \in \Psi \colon \psi^{\triangle}(D_1) \leq \psi^{\triangle}(D_2).$$

Set relations via scalar families.

Inf-extension of $\psi \in \Psi$ is $\psi^{\triangle} \colon \mathcal{P}(Z) \to \mathrm{I\!R} \cup \{\pm\infty\}$ defined by

$$\psi^{\triangle}(D) = \inf_{z \in D} \psi(z).$$

Inf-extension of $\preceq$ to $\mathcal{P}(Z)$ is $\preceq_{\Psi}$ defined by

$$D_1 \preceq_{\Psi} D_2 \quad \Leftrightarrow \quad \forall \psi \in \Psi \colon \psi^{\triangle}(D_1) \leq \psi^{\triangle}(D_2).$$

The relation $\preceq_{\Psi}$ is a preorder on $\mathcal{P}(Z)$, a new "set relation!" It extends $\preceq$ from Z to $\mathcal{P}(Z)$ since by "$\Leftrightarrow$"

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \{z_1\} \preceq_{\Psi} \{z_2\}.$$

Set relations via scalar families.

A closure operator associated with $\preceq_\Psi$: For $D \subseteq Z$,

$$\mathrm{cl}_\Psi D = \bigcap_{\psi \in \Psi} \left\{z \in Z \mid \psi^\triangle(D) \leq \psi(z)\right\}.$$

Set relations via scalar families.

A closure operator associated with $\preceq_\Psi$: For $D \subseteq Z$,

$$\mathrm{cl}_\Psi D = \bigcap_{\psi \in \Psi} \left\{ z \in Z \mid \psi^\triangle(D) \leq \psi(z) \right\}.$$

Proposition

For all $D \in \mathcal{P}(Z)$,

(i) $D \subseteq \mathrm{cl}_\Psi D$,

(ii) $\mathrm{cl}_\Psi D = \mathrm{cl}_\Psi(\mathrm{cl}_\Psi D)$,

(iii) $C \subseteq D \Rightarrow \mathrm{cl}_\Psi C \subseteq \mathrm{cl}_\Psi D$.

Set relations via scalar families.

A closure operator associated with $\preceq_\Psi$: For $D \subseteq Z$,

$$\mathrm{cl}_\Psi D = \bigcap_{\psi \in \Psi} \left\{ z \in Z \mid \psi^\Delta(D) \leq \psi(z) \right\}.$$

Proposition

For all $D \in \mathcal{P}(Z)$,

(i) $D \subseteq \mathrm{cl}_\Psi D$,

(ii) $\mathrm{cl}_\Psi D = \mathrm{cl}_\Psi(\mathrm{cl}_\Psi D)$,

(iii) $C \subseteq D \Rightarrow \mathrm{cl}_\Psi C \subseteq \mathrm{cl}_\Psi D$.

Note. This means that $D \mapsto \mathrm{cl}_\Psi D$ is a closure (hull) operator.

Set relations via scalar families.

The previous proposition guarantees that the following set is well-defined:

$$\mathcal{P}(Z, \Psi) = \{D \in \mathcal{P}(Z) \mid D = \mathrm{cl}_{\Psi} D\}.$$

Set relations via scalar families.

The previous proposition guarantees that the following set is well-defined:

$$\mathcal{P}(Z, \Psi) = \{D \in \mathcal{P}(Z) \mid D = \mathrm{cl}_{\Psi} D\}.$$

Proposition

On $\mathcal{P}(Z, \Psi)$, the relation $\preceq_{\Psi}$ coincides with $\supseteq$. The pair $(\mathcal{P}(Z, \Psi), \supseteq)$ is a complete lattice, and for $\mathfrak{A} \subseteq \mathcal{P}(Z, \Psi)$

$$\inf \mathfrak{A} = \mathrm{cl}_{\Psi} \bigcup_{A \in \mathfrak{A}} A \quad \textit{and} \quad \sup \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A$$

where $\inf \mathfrak{A} = \emptyset$ and $\sup \mathfrak{A} = Z$ whenever $\mathfrak{A} = \emptyset$. The greatest element in $(\mathcal{P}(Z, \Psi), \supseteq)$ is $\emptyset$, the least element is Z.

Set relations via scalar families.

The previous proposition guarantees that the following set is well-defined:

$$\mathcal{P}(Z, \Psi) = \{D \in \mathcal{P}(Z) \mid D = \mathrm{cl}_{\Psi} D\}.$$

Proposition

On $\mathcal{P}(Z, \Psi)$, the relation $\preceq_{\Psi}$ coincides with $\supseteq$. The pair $(\mathcal{P}(Z, \Psi), \supseteq)$ is a complete lattice, and for $\mathfrak{A} \subseteq \mathcal{P}(Z, \Psi)$

$$\inf \mathfrak{A} = \mathrm{cl}_{\Psi} \bigcup_{A \in \mathfrak{A}} A \quad \textit{and} \quad \sup \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A$$

where $\inf \mathfrak{A} = \emptyset$ and $\sup \mathfrak{A} = Z$ whenever $\mathfrak{A} = \emptyset$. The greatest element in $(\mathcal{P}(Z, \Psi), \supseteq)$ is $\emptyset$, the least element is Z.

Note. This is true without further assumptions to $\preceq$.

Inf-stability.

Proposition

Let $\mathfrak{A} \subseteq \mathcal{P}(Z, \Psi)$. *Then*

$$\forall \psi \in \Psi \colon \inf_{A \in \mathfrak{A}} \psi^{\triangle}(A) = \psi^{\triangle}\left(\inf_{A \in \mathfrak{A}} A\right).$$

Set relations via scalar families.

Inf-stability.

Proposition

Let $\mathfrak{A} \subseteq \mathcal{P}(Z, \Psi)$. *Then*

$$\forall \psi \in \Psi \colon \inf_{A \in \mathfrak{A}} \psi^{\triangle}(A) = \psi^{\triangle}\left(\inf_{A \in \mathfrak{A}} A\right).$$

Note. "Inf on the left" in $\mathrm{I\!R} \cup \{\pm\infty\}$, "inf on the right" in $(\mathcal{P}(Z, \Psi), \supseteq)$.

Set relations via scalar families.

Inf-stability.

Proposition

Let $\mathfrak{A} \subseteq \mathcal{P}(Z, \Psi)$. *Then*

$$\forall \psi \in \Psi \colon \inf_{A \in \mathfrak{A}} \psi^{\triangle}(A) = \psi^{\triangle}\left(\inf_{A \in \mathfrak{A}} A\right).$$

Note. "Inf on the left" in $\mathbb{R} \cup \{\pm\infty\}$, "inf on the right" in $(\mathcal{P}(Z, \Psi), \supseteq)$.

Note. "Sup-stability" not true in general. But one can start with

$$\psi^{\triangledown}(D) = \sup_{z \in D} \psi(z), \quad \operatorname{cl}^{\Psi} D = \bigcap_{\psi \in \Psi} \left\{z \in Z \mid \psi(z) \leq \psi^{\triangledown}(D)\right\}.$$

Set relations via scalar families.

Embedding $(Z, \preceq) \hookrightarrow (\mathcal{P}(Z, \Psi), \supseteq)$. Define $a \colon Z \to \mathcal{P}(Z, \Psi)$ by

$$a(z) = \operatorname{cl}_{\Psi} \{z\} = \bigcap_{\psi \in \Psi} \{y \in Z \mid \psi(z) \leq \psi(y)\}.$$

Embedding $(Z, \preceq) \hookrightarrow (\mathcal{P}(Z, \Psi), \supseteq)$**.** Define $a \colon Z \to \mathcal{P}(Z, \Psi)$ by

$$a(z) = \mathrm{cl}_{\Psi}\{z\} = \bigcap_{\psi \in \Psi} \{y \in Z \mid \psi(z) \leq \psi(y)\}.$$

Then, for all $z \in Z$
(i) $z \in a(z)$,
(ii) $a(z) \in \mathcal{P}(Z, \Psi)$ and

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \{z_1\} \preceq_{\Psi} \{z_2\} \quad \Leftrightarrow \quad a(z_1) \supseteq a(z_2),$$

(iii) $\psi(z) = \psi^{\triangle}(a(z))$.

Set optimization. Let $F \colon X \to Z$ be a function. Instead of

$$\text{minimize} \quad F(x) \quad \text{over} \quad X \quad \text{w.r.t.} \ \preceq$$

solve the complete lattice-valued problem

$$\text{minimize} \quad (a \circ F)(x) = a(F(x)) \quad \text{over} \quad X \quad \text{w.r.t.} \ \supseteq$$

with $a \circ F \colon X \to \mathcal{P}(Z, \Psi)$.

Set relations via scalar families.

Set optimization. Let $F\colon X \to Z$ be a function. Instead of

$$\text{minimize} \quad F(x) \quad \text{over} \quad X \quad \text{w.r.t.} \ \preceq$$

solve the complete lattice-valued problem

$$\text{minimize} \quad (a \circ F)(x) = a(F(x)) \quad \text{over} \quad X \quad \text{w.r.t.} \ \supseteq$$

with $a \circ F\colon X \to \mathcal{P}(Z, \Psi)$.

Questions. How can we do this? Optimality, (Lagrange) duality, algorithms? And why should we do this?

Set relations via scalar families.

Summary:

- "Set relations" can be defined as (canonical) extensions of preorders given by a family of scalar functions.

Set relations via scalar families.

Summary:

- "Set relations" can be defined as (canonical) extensions of preorders given by a family of scalar functions.
- Finding "best" decisions/alternatives becomes a complete lattice-valued set optimization problem: **a new paradigm**.

Set relations via scalar families.

Summary:

- "Set relations" can be defined as (canonical) extensions of preorders given by a family of scalar functions.
- Finding "best" decisions/alternatives becomes a complete lattice-valued set optimization problem: **a new paradigm**.
- All depends on tractability of

$$\mathrm{cl}_{\Psi} D = \bigcap_{\psi \in \Psi} \left\{ z \in Z \mid \psi^{\Delta}(D) \leq \psi(z) \right\}$$

as $(\mathcal{P}(Z, \Psi) = \left\{ D \in \mathcal{P}(Z) \mid D = \mathrm{cl}_{\Psi} D \right\}, \supseteq)$ is a complete lattice, and on the properties of $\psi^{\Delta}(D) = \inf_{z \in D} \psi(z)$.

Next.

- Examples

Examples.

Multi-utility representations. Let $(Z, \preceq)$ be a preordered set.

Multi-utility representations. Let $(Z, \preceq)$ be a preordered set.

A multi-loss representation of $\preceq$ on Z is a family $\Psi = \mathcal{L}$ of functions $\ell \colon Z \to \mathrm{I\!R}$ satisfying

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \forall \ell \in \mathcal{L} \colon \ell(z_1) \leq \ell(z_2).$$

Examples.

Multi-utility representations. Let $(Z, \preceq)$ be a preordered set.

A multi-loss representation of $\preceq$ on Z is a family $\Psi = \mathcal{L}$ of functions $\ell \colon Z \to \mathrm{I\!R}$ satisfying

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \forall \ell \in \mathcal{L} \colon \ell(z_1) \leq \ell(z_2).$$

Of course. Negative loss = utility.

Question. Does a given preorder have a multi-loss representation? $\longrightarrow$ Evren, Ok etc.

Examples.

Indicator functions of level sets. For $z \in Z$, denote

$$L(z) = \{y \in Z \mid y \preceq z\} \quad \text{and} \quad U(z) = \{y \in Z \mid z \preceq y\}.$$

Examples.

Indicator functions of level sets. For $z \in Z$, denote

$$L(z) = \{y \in Z \mid y \preceq z\} \quad \text{and} \quad U(z) = \{y \in Z \mid z \preceq y\}.$$

For $A \subseteq Z$, let $I_A \colon Z \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$I_A(z) = \begin{cases} 0 & : \quad z \in A \\ +\infty & : \quad z \notin A \end{cases}$$

Examples.

Indicator functions of level sets. For $z \in Z$, denote

$$L(z) = \{y \in Z \mid y \preceq z\} \quad \text{and} \quad U(z) = \{y \in Z \mid z \preceq y\}.$$

For $A \subseteq Z$, let $I_A \colon Z \to \mathrm{I\!R} \cup \{+\infty\}$ be the function defined by

$$I_A(z) = \begin{cases} 0 & : \quad z \in A \\ +\infty & : \quad z \notin A \end{cases}$$

Result 1. $\mathcal{I} = \{I_L(z)\}_{z \in Z}$ represents $\preceq$, i.e. for $z_1, z_2 \in Z$,

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \forall z \in Z \colon I_{L(z)}(z_1) \leq I_{L(z)}(z_2).$$

Examples.

Indicator functions of level sets. For $z \in Z$, denote

$$L(z) = \{y \in Z \mid y \preceq z\} \quad \text{and} \quad U(z) = \{y \in Z \mid z \preceq y\}.$$

For $A \subseteq Z$, let $I_A \colon Z \to \mathrm{I\!R} \cup \{+\infty\}$ be the function defined by

$$I_A(z) = \begin{cases} 0 & : \quad z \in A \\ +\infty & : \quad z \notin A \end{cases}$$

Result 1. $\mathcal{I} = \{I_{L}(z)\}_{z \in Z}$ represents $\preceq$, i.e. for $z_1, z_2 \in Z$,

$$z_1 \preceq z_2 \quad \Leftrightarrow \quad \forall z \in Z \colon I_{L(z)}(z_1) \leq I_{L(z)}(z_2).$$

Result 2. For $D \subseteq Z$, $a(z) = U(z)$ and

$$\mathrm{cl}_{\mathcal{I}} D = \{y \in Z \mid \exists d \in D \colon d \preceq y\} = \{y \in Z \mid D \cap L(y) \neq \emptyset\}.$$

Second order stochastic dominance re-visited.

- $(\Omega, \mathcal{F}, P)$ a probability space, $Z = L^1 = L^1(\Omega, \mathcal{F}, P)$

Examples.

Second order stochastic dominance re-visited.

- $(\Omega, \mathcal{F}, P)$ a probability space, $Z = L^1 = L^1(\Omega, \mathcal{F}, P)$
- $X, Y \in L^1$ are in relation w.r.t. 2nd order stochastic dominance, i.e. $X \succeq_{SSD} Y$ if, and only if,

$$\forall \alpha \in (0,1] \colon AV@R_\alpha(X) \leq AV@R_\alpha(Y).$$

Examples.

Second order stochastic dominance re-visited.

- $(\Omega, \mathcal{F}, P)$ a probability space, $Z = L^1 = L^1(\Omega, \mathcal{F}, P)$
- $X, Y \in L^1$ are in relation w.r.t. 2nd order stochastic dominance, i.e. $X \succeq_{SSD} Y$ if, and only if,

$$\forall \alpha \in (0,1] \colon AV@R_\alpha(X) \leq AV@R_\alpha(Y).$$

- The function $X \mapsto AV@R_\alpha(X)$ is called the average value at risk defined by either one of:

$$AV@R_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha V@R_\beta(X)d\beta = \inf_{r \in \mathbb{R}} \left\{ \frac{1}{\alpha} E[(r-x)^+] - r \right\}.$$

Examples.

Second order stochastic dominance re-visited.

- $(\Omega, \mathcal{F}, P)$ a probability space, $Z = L^1 = L^1(\Omega, \mathcal{F}, P)$
- $X, Y \in L^1$ are in relation w.r.t. 2nd order stochastic dominance, i.e. $X \succeq_{SSD} Y$ if, and only if,

$$\forall \alpha \in (0,1] \colon AV@R_\alpha(X) \leq AV@R_\alpha(Y).$$

- The function $X \mapsto AV@R_\alpha(X)$ is called the average value at risk defined by either one of:

$$AV@R_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha V@R_\beta(X)d\beta = \inf_{r\in\mathbb{R}}\left\{\frac{1}{\alpha}E[(r-x)^+] - r\right\}.$$

- So, $\Psi = \{AV@R_\alpha\}_{\alpha\in(0,1]}$ on $Z = L^1$ works as well for SSD.

Examples.

Second order stochastic dominance re-visited.

- $(\Omega, \mathcal{F}, P)$ a probability space, $Z = L^1 = L^1(\Omega, \mathcal{F}, P)$
- $X, Y \in L^1$ are in relation w.r.t. 2nd order stochastic dominance, i.e. $X \succeq_{SSD} Y$ if, and only if,

$$\forall \alpha \in (0,1] \colon AV@R_\alpha(X) \leq AV@R_\alpha(Y).$$

- The function $X \mapsto AV@R_\alpha(X)$ is called the average value at risk defined by either one of:

$$AV@R_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha V@R_\beta(X)d\beta = \inf_{r \in \mathbb{R}} \left\{ \frac{1}{\alpha} E[(r - x)^+] - r \right\}.$$

- So, $\Psi = \{AV@R_\alpha\}_{\alpha \in (0,1]}$ on $Z = L^1$ works as well for SSD.

Question. cl_Ψ and $\mathcal{P}(Z, \Psi)$?

Examples.

Vector orders.

- Z a locally convex, real linear space, $C \subseteq Z$ a convex cone with $0 \in C$, $\operatorname{cl} C \neq Z$;

  $$z_1 \leq_C z_2 \quad \Leftrightarrow \quad z_2 - z_1 \in C \quad \Leftrightarrow \quad z_1 + C \supseteq z_2 + C$$

  defines a vector preorder $\leq_C$ on Z.

Examples.

Vector orders.

- Z a locally convex, real linear space, $C \subseteq Z$ a convex cone with $0 \in C$, $\operatorname{cl} C \neq Z$;

$$z_1 \leq_C z_2 \quad \Leftrightarrow \quad z_2 - z_1 \in C \quad \Leftrightarrow \quad z_1 + C \supseteq z_2 + C$$

  defines a vector preorder $\leq_C$ on Z.

- $\Psi = C^+$ where

$$C^+ = \{z^* \in Z^* \mid \forall z \in C \colon z^*(z) \geq 0\}$$

Vector orders.

- Z a locally convex, real linear space, $C \subseteq Z$ a convex cone with $0 \in C$, $\operatorname{cl} C \neq Z$;

$$z_1 \leq_C z_2 \quad \Leftrightarrow \quad z_2 - z_1 \in C \quad \Leftrightarrow \quad z_1 + C \supseteq z_2 + C$$

  defines a vector preorder $\leq_C$ on Z.
- $\Psi = C^+$ where

$$C^+ = \{z^* \in Z^* \mid \forall z \in C \colon z^*(z) \geq 0\}$$

- for $D \subseteq Z$, $\psi \in C^+$

$$\psi^{\triangle}(D) = \inf_{z \in D} z^*(z)$$

  is just the negative of the support function of D at $-z^*$.

Examples.

Vector orders.

- Z a locally convex, real linear space, $C \subseteq Z$ a convex cone with $0 \in C$, $\operatorname{cl} C \neq Z$;

$$z_1 \leq_C z_2 \quad \Leftrightarrow \quad z_2 - z_1 \in C \quad \Leftrightarrow \quad z_1 + C \supseteq z_2 + C$$

  defines a vector preorder $\leq_C$ on Z.
- $\Psi = C^+$ where

$$C^+ = \{z^* \in Z^* \mid \forall z \in C \colon z^*(z) \geq 0\}$$

- for $D \subseteq Z$, $\psi \in C^+$

$$\psi^\triangle(D) = \inf_{z \in D} z^*(z)$$

  is just the negative of the support function of D at $-z^*$.
- By separation,

$$\operatorname{cl}_{C^+} D = \operatorname{cl}\operatorname{co}(D + C),$$

$$\mathcal{P}(Z, C^+) = \{D \in \mathcal{P}(Z) \mid D = \operatorname{cl}\operatorname{co}(D + C)\} = \mathcal{G}(Z, C).$$

Examples.

Conclusion. $(\mathcal{G}(Z,C), \supseteq)$ is a complete lattice with

$$\inf \mathfrak{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathfrak{A}} A \quad \text{and} \quad \sup \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A$$

for $\mathfrak{A} \subseteq \mathcal{G}(Z,C)$.

Examples.

Conclusion. $(\mathcal{G}(Z, C), \supseteq)$ is a complete lattice with

$$\inf \mathfrak{A} = \operatorname{cl} \operatorname{co} \bigcup_{A \in \mathfrak{A}} A \quad \text{and} \quad \sup \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A$$

for $\mathfrak{A} \subseteq \mathcal{G}(Z, C)$.

The embedding function $a \colon Z \to \mathcal{G}(Z, C)$ is given by $a(z) = z + \operatorname{cl} C$.

Examples.

Conclusion. $(\mathcal{G}(Z, C), \supseteq)$ is a complete lattice with

$$\inf \mathfrak{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathfrak{A}} A \quad \text{and} \quad \sup \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A$$

for $\mathfrak{A} \subseteq \mathcal{G}(Z, C)$.

The embedding function $a \colon Z \to \mathcal{G}(Z, C)$ is given by $a(z) = z + \operatorname{cl} C$.

For $C = \{0\}$ the class of all closed convex sets is obtained with $\Psi = C^{+} = Z^{*}$.

Conclusion. $(\mathcal{G}(Z,C), \supseteq)$ is a complete lattice with

$$\inf \mathfrak{A} = \operatorname{cl}\operatorname{co} \bigcup_{A \in \mathfrak{A}} A \quad \text{and} \quad \sup \mathfrak{A} = \bigcap_{A \in \mathfrak{A}} A$$

for $\mathfrak{A} \subseteq \mathcal{G}(Z,C)$.

The embedding function $a \colon Z \to \mathcal{G}(Z,C)$ is given by $a(z) = z + \operatorname{cl} C$.

For $C = \{0\}$ the class of all closed convex sets is obtained with $\Psi = C^+ = Z^*$.

See Hamel et al. (70+ pp., 230+ references): *Set Optimization - A Rather Short Introduction*, ArXiv, 2014.

Examples.

Multi-probability representations. (Bewley preferences)

- $(\Omega, \mathcal{F})$ measureable space, $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathcal{F})$ linear space of measurable functions $X \colon \Omega \to \mathrm{I\!R}$, Π set of probability measures on $(\Omega, \mathcal{F})$.

Examples.

Multi-probability representations. (Bewley preferences)

- $(\Omega, \mathcal{F})$ measureable space, $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathcal{F})$ linear space of measurable functions $X \colon \Omega \to \mathrm{I\!R}$, Π set of probability measures on $(\Omega, \mathcal{F})$.
- $u \colon \mathrm{I\!R} \to \mathrm{I\!R} \cup \{-\infty\}$ upper semicontinuous, concave, strictly increasing and bounded (from above).

Examples.

Multi-probability representations. (Bewley preferences)

- $(\Omega, \mathcal{F})$ measureable space, $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathcal{F})$ linear space of measurable functions $X \colon \Omega \to \mathrm{I\!R}$, Π set of probability measures on $(\Omega, \mathcal{F})$.
- $u \colon \mathrm{I\!R} \to \mathrm{I\!R} \cup \{-\infty\}$ upper semicontinuous, concave, strictly increasing and bounded (from above).
- Define $\psi \colon \mathcal{L}^0 \to \mathrm{I\!R} \cup \{-\infty\}$ by

$$\psi(Z) = \mathbb{E}^P\left[u(Z)\right] =: (\mathbb{E}^P \circ u)(Z).$$

Examples.

Multi-probability representations. (Bewley preferences)

- $(\Omega, \mathcal{F})$ measureable space, $\mathcal{L}^0 = \mathcal{L}^0(\Omega, \mathcal{F})$ linear space of measurable functions $X \colon \Omega \to \mathrm{I\!R}$, Π set of probability measures on $(\Omega, \mathcal{F})$.
- $u \colon \mathrm{I\!R} \to \mathrm{I\!R} \cup \{-\infty\}$ upper semicontinuous, concave, strictly increasing and bounded (from above).
- Define $\psi \colon \mathcal{L}^0 \to \mathrm{I\!R} \cup \{-\infty\}$ by

$$\psi(Z) = \mathbb{E}^P\left[u(Z)\right] =: (\mathbb{E}^P \circ u)(Z).$$

- A preorder is defined via

$$Z_1 \preceq Z_2 \quad \Leftrightarrow \quad \forall P \in \Pi \colon \mathbb{E}^P\left[u(Z_1)\right] \leq \mathbb{E}^P\left[u(Z_2)\right].$$

Examples.

The preorder here is defined via

$$Z_1 \preceq Z_2 \quad \Leftrightarrow \quad \forall P \in \Pi \colon \mathbb{E}^P \left[u(Z_1)\right] \leq \mathbb{E}^P \left[u(Z_2)\right],$$

and one wants to maximize!

Examples.

The preorder here is defined via

$$Z_1 \preceq Z_2 \quad \Leftrightarrow \quad \forall P \in \Pi\colon \mathbb{E}^P\left[u(Z_1)\right] \leq \mathbb{E}^P\left[u(Z_2)\right],$$

and one wants to maximize!

Therefore, the *sup-extension* is needed:

$$\psi^{\triangledown}(D) = \sup_{Z \in D} \mathbb{E}^P\left[u(Z)\right]$$

as well as the corresponding closure operator.

Examples.

- The family

$$\mathcal{E} = \left\{\mathbb{E}^P \circ u\right\}_{P \in \Pi}$$

represents $\preceq$; for $D \subseteq \mathcal{L}^0(\Omega, \mathcal{F})$

$$\mathrm{cl}^{\mathcal{E}} D = \bigcap_{P \in \Pi} \left\{ Y \in \mathcal{L}^0 \mid \mathbb{E}^P\left[u(Y)\right] \leq \sup_{Z \in D} \mathbb{E}^P\left[u(Z)\right] \right\}.$$

Examples.

- The family

$$\mathcal{E} = \left\{\mathbb{E}^P \circ u\right\}_{P \in \Pi}$$

represents $\preceq$; for $D \subseteq \mathcal{L}^0(\Omega, \mathcal{F})$

$$\mathrm{cl}^{\mathcal{E}} D = \bigcap_{P \in \Pi} \left\{Y \in \mathcal{L}^0 \mid \mathbb{E}^P\left[u(Y)\right] \leq \sup_{Z \in D} \mathbb{E}^P\left[u(Z)\right]\right\}.$$

- In particular, $b(Z) = \mathrm{cl}^{\mathcal{E}}\{Z\}$ is

$$b(Z) = \bigcap_{P \in \Pi} \left\{Y \in \mathcal{L}^0 \mid \mathbb{E}^P\left[u(Y)\right] \leq \mathbb{E}^P\left[u(Z)\right]\right\}$$

$$= \left\{Y \in \mathcal{L}^0 \mid \sup_{P \in \Pi} \left\{\mathbb{E}^P\left[u(Y)\right] - \mathbb{E}^P\left[u(Z)\right]\right\} \leq 0\right\}.$$

The utility maximization problem under uncertainty. In $\big(\mathcal{P}(\mathcal{L}^0, \subseteq)\big)$ solve

$$\text{maximize} \quad b(Z) \quad \text{over} \quad \mathcal{Z} \subseteq \mathcal{L}^0$$

where

$$b(Z) = \left\{ Y \in \mathcal{L}^0 \mid \sup_{P \in \Pi} \left\{ \mathbb{E}^P \left[u(Y) \right] - \mathbb{E}^P \left[u(Z) \right] \right\} \leq 0 \right\},$$

and $\mathcal{Z}$ is a set of admissible elements.

Examples.

The utility maximization problem under uncertainty. In $\left(\mathcal{P}(\mathcal{L}^0, \supseteq)\right)$ solve

$$\text{maximize} \quad b(Z) \quad \text{over} \quad \mathcal{Z} \subseteq \mathcal{L}^0$$

where

$$b(Z) = \left\{ Y \in \mathcal{L}^0 \mid \sup_{P \in \Pi} \left\{ \mathbb{E}^P \left[u(Y) \right] - \mathbb{E}^P \left[u(Z) \right] \right\} \leq 0 \right\},$$

and $\mathcal{Z}$ is a set of admissible elements.

Note. A totally new problem.

Finally.

- The set optimization approach to preference optimization.

The set optimization approach.

Problem. Given $(Z, \preceq, \Psi)$, $F\colon X \to Z$ and $\mathcal{X} \subseteq X$, find

$$\text{'best'}\, \{F(x) \mid x \in \mathcal{X}\}\,.$$

The set optimization approach.

Problem. Given $(Z, \preceq, \Psi)$, $F\colon X \to Z$ and $\mathcal{X} \subseteq X$, find

$$\text{`best'} \left\{F(x) \mid x \in \mathcal{X}\right\}.$$

Set-valued extension. Define $f\colon X \to \mathcal{P}(Z, \Psi)$ by

$$f(x) = (a \circ F)(x) = a(F(x)) = \operatorname{cl}_{\Psi} \{F(x)\}$$

and look for

$$\inf \{f(x) \mid x \in \mathcal{X}\}$$

along with appropriate "solutions."

The set optimization approach.

Question.

What is a solution of a set optimization problem? In particular, of a $\mathcal{P}(Z, \Psi)$-valued problem?

The set optimization approach.

Definition

Let $(L, \leq)$ be a complete lattice, $X \neq \emptyset$ a set and $f\colon X \to L$.

Definition

Let $(L, \leq)$ be a complete lattice, $X \neq \emptyset$ a set and $f\colon X \to L$. A set $M \subseteq L$ is called an *infimizer* for f if

$$\inf\{f(x) \mid x \in M\} = \inf\{f(x) \mid x \in X\}.$$

The set optimization approach.

Definition

Let $(L, \leq)$ be a complete lattice, $X \neq \emptyset$ a set and $f \colon X \to L$. A set $M \subseteq L$ is called an *infimizer* for f if

$$\inf \{f(x) \mid x \in M\} = \inf \{f(x) \mid x \in X\}.$$

An element $\bar{x} \in X$ is called a *minimizer* for f if

$$x \in X,\ f(x) \leq f(\bar{x}) \ \Rightarrow\ f(x) = f(\bar{x}).$$

The set optimization approach.

Definition

Let $(L, \leq)$ be a complete lattice, $X \neq \emptyset$ a set and $f \colon X \to L$.
A set $M \subseteq L$ is called an *infimizer* for f if

$$\inf \{f(x) \mid x \in M\} = \inf \{f(x) \mid x \in X\}.$$

An element $\bar{x} \in X$ is called a *minimizer* for f if

$$x \in X,\ f(x) \leq f(\bar{x}) \ \Rightarrow\ f(x) = f(\bar{x}).$$

A set $M \subseteq X$ is called a *solution* for the problem

$$\text{minimize} \quad f \quad \text{over} \quad X$$

if M is an infimizer and each $m \in M$ is a minimizer for f.

Comments.

- Due to Heyde/Löhne 2011 with precursor Hamel 2004.

Comments.

- Due to Heyde/Löhne 2011 with precursor Hamel 2004.
- Infimum attainment and minimality do not coincide (anymore).

Comments.

- Due to Heyde/Löhne 2011 with precursor Hamel 2004.
- Infimum attainment and minimality do not coincide (anymore).
- Clear-cut concept with no further assumptions to $\preceq$.

Comments.

- Due to Heyde/Löhne 2011 with precursor Hamel 2004.
- Infimum attainment and minimality do not coincide (anymore).
- Clear-cut concept with no further assumptions to $\preceq$.
- Different from (and more general than) 'vector optimization' and 'multi-criteria decision making' approaches.

Comments.

- Due to Heyde/Löhne 2011 with precursor Hamel 2004.
- Infimum attainment and minimality do not coincide (anymore).
- Clear-cut concept with no further assumptions to $\preceq$.
- Different from (and more general than) 'vector optimization' and 'multi-criteria decision making' approaches.

However, no link to Ψ. Not even if the complete lattice is $(L, \leq) = (\mathcal{P}(Z, \Psi), \supseteq)$. Another definition is required.

The set optimization approach.

Definition (Hamel, Schrage 2015)

$(Z, \preceq, \Psi)$ as before, $f\colon X \to (\mathcal{P}(Z, \Psi), \supseteq)$.

The set optimization approach.

Definition (Hamel, Schrage 2015)

$(Z, \preceq, \Psi)$ as before, $f \colon X \to (\mathcal{P}(Z, \Psi), \supseteq)$.
A set $M \subseteq X$ is called a *Ψ-infimizer* for f if

$$\forall \psi \in \Psi \colon \psi^{\triangle}\left(\inf_{x \in M} f(x)\right) = \psi^{\triangle}\left(\inf_{x \in X} f(x)\right).$$

The set optimization approach.

Definition (Hamel, Schrage 2015)

$(Z, \preceq, \Psi)$ as before, $f\colon X \to (\mathcal{P}(Z, \Psi), \supseteq)$.
A set $M \subseteq X$ is called a *Ψ-infimizer* for f if

$$\forall \psi \in \Psi\colon \psi^{\triangle}\left(\inf_{x \in M} f(x)\right) = \psi^{\triangle}\left(\inf_{x \in X} f(x)\right).$$

An element $\bar{x} \in X$ is called a *Ψ-minimizer* for f if

$$\exists \psi \in \Psi\colon \psi^{\triangle}(f(\bar{x})) = \psi^{\triangle}\left(\inf_{x \in X} f(x)\right).$$

The set optimization approach.

Definition (Hamel, Schrage 2015)

$(Z, \preceq, \Psi)$ as before, $f \colon X \to (\mathcal{P}(Z, \Psi), \supseteq)$.
A set $M \subseteq X$ is called a *Ψ-infimizer* for f if

$$\forall \psi \in \Psi \colon \psi^{\triangle}\left(\inf_{x \in M} f(x)\right) = \psi^{\triangle}\left(\inf_{x \in X} f(x)\right).$$

An element $\bar{x} \in X$ is called a *Ψ-minimizer* for f if

$$\exists \psi \in \Psi \colon \psi^{\triangle}(f(\bar{x})) = \psi^{\triangle}\left(\inf_{x \in X} f(x)\right).$$

A set $M \subseteq X$ is called a *Ψ-solution* for the problem

$$\text{minimize} \quad f \quad \text{over} \quad X$$

if M is a Ψ-infimizer and each $m \in M$ is a Ψ-minimizer for f.

The set optimization approach.

Comments.

- Need to know $\inf_{x \in X} f(x)$–this is a new feature.

Comments.

- Need to know $\inf_{x \in X} f(x)$–this is a new feature.
- Ψ-minimizers can be found by solving scalar problems:

$$\text{minimize} \quad \psi^{\triangle}(f(x)) \quad \text{over} \quad X.$$

Comments.

- Need to know $\inf_{x \in X} f(x)$–this is a new feature.
- Ψ-minimizers can be found by solving scalar problems:

$$\text{minimize} \quad \psi^{\vartriangle}(f(x)) \quad \text{over} \quad X.$$

- So, it is important to study functions of the type

$$(\psi \circ f)(x) = \inf_{z \in f(x)} \psi(z) = \psi^{\vartriangle}(f(x)).$$

If $\psi = z^*$ is continuous linear, this is a version of the support function of the set $f(x)$.

The set optimization approach.

Another comment. The mathematical fun starts here:

- Existence (and uniqueness) of solutions,
- optimality conditions,
- duality,
- numerical representation of set-valued functions and algorithms etc.

The set optimization approach.

Another comment. The mathematical fun starts here:

- Existence (and uniqueness) of solutions,
- optimality conditions,
- duality,
- numerical representation of set-valued functions and algorithms etc.

Good news. In a "linear space set-up," most of the above already exists.

Program. Do this ($= \psi^{\triangle}$, cl_{Ψ}, a, optimization) for

a) stochastic orders,

b) multi-utility representations,

c) Bewley preferences,

d) a merge of b) and c)

e) for multi-variate random variables, vector lotteries etc.

Last slide.

Thank you for the dance.