# Optimal portfolio strategies under partial information with expert opinions

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# Agenda

# Dynamic Portfolio Optimization

**Initial capital** $x_0 > 0$

**Horizon** $[0, T]$

**Aim** maximize expected utility of terminal wealth

**Problem** find an optimal investment strategy

**How many** shares

of **which** asset

have to be held **at which time** by the portfolio manager **?**

**Market model** continuously tradable assets

drift depends on unobservable finite-state Markov chain

investor only observes stock prices and

**expert opinions**

# Classical Black-Scholes Model of Financial Market

$(\Omega, \mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}, P)$ filtered probability space

**Bond** $S_t^0 = e^{rt}$, $r$ risk-free interest rate

**Stocks** prices $S_t = (S_t^1, \ldots, S_t^n)^\top$, returns $dR_t^i = \dfrac{dS_t^i}{S_t^i}$

$dR_t = \mu\, dt + \sigma\, dW_t$

$\mu \in \mathbb{R}^n$ average stock return, drift

$\sigma \in \mathbb{R}^{n\times n}$ volatility

$W_t$ $n$-dimensional Brownian motion

parameters $\mu$ and $\sigma$ are constant and known

Generalization time-dependent (non-random) parameters $\mu, \sigma, r$

# Portfolio

Initial capital $X_0 = x_0 > 0$

Wealth at time $t$ invested in $X_t = X_t(\underbrace{h_t^0}_{\text{bond}} + \underbrace{h_t^1}_{\text{stock 1}} + \ldots + \underbrace{h_t^n}_{\text{stock n}})$

$h_t^k$ fractions of wealth invested in asset $k$

**Strategy** $\boldsymbol{h_t = (h_t^1, \ldots, h_t^n)^\top}$

Self financing condition (assume $r = 0$ for simplicity) $\Rightarrow$

## Wealth equation

$X_t$ satisfies **linear SDE** with initial value $x_0$

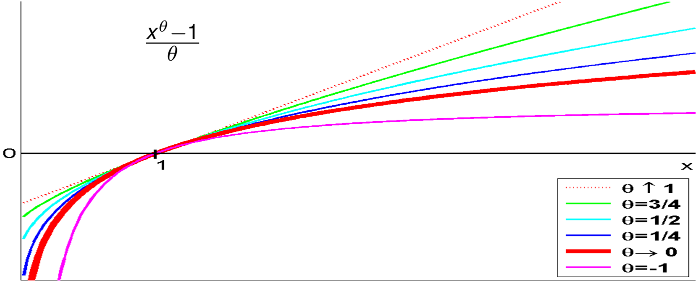
$$\begin{aligned} dX_t^{(h)} &= X_t^{(h)}\, \boldsymbol{h_t^\top}\, (\mu\, dt + \sigma dW_t) \\ X_0^{(h)} &= x_0 \end{aligned}$$

# Utility Function

$U : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ strictly increasing and concave

Inada conditions $\lim_{x \downarrow 0} U'(x) = \infty$ and $\lim_{x \uparrow \infty} U'(x) = 0$

$$U(x) = \begin{cases} \frac{x^\theta}{\theta} & \text{for } \theta \in (-\infty, 1) \setminus \{0\} \quad \text{power utility} \\ \log x & \text{for } \theta = 0 \quad \text{log-utility} \end{cases}$$



# Optimization Problem

**Wealth** $dX_t^{(h)} = X_t^{(h)}\, h_t^\top\ (\mu\, dt + \sigma dW_t), \quad X_0^{(h)} = x_0$

**Admissible Strategies** $\mathcal{H} = \{(h_t)_{t\in[0,T]}\ \big|\ h_t \in \mathbb{R}^n,$

$E\left[\exp\left\{\int_0^T \|h_t\|^2 dt\right\}\right] < \infty\ \}$

**Reward function** $v(t,x,h) = E_{t,x}[\ U(X_T^{(h)})\ ] \qquad \text{for} \quad h \in \mathcal{H}$

**Value function** $V(t,x) \ = \sup_{h\in\mathcal{H}}\ v(t,x,h)$

Find optimal strategy $h^* \in \mathcal{H}$ such that $V(0,x_0) = v(0,x_0,h^*)$

**Solution** optimal fractions of wealth $h_t^* = \dfrac{1}{1-\theta}(\sigma\sigma^\top)^{-1}\mu = \text{const}$

Merton (1969,1973)

using methods from dynamic programming

# Drawbacks of the Merton Strategy

Sensitive dependence of investment strategies on the **drift** $\mu$ of assets

**Drift is hard to estimate empirically**

need data over long time horizons

(other than volatility estimation)

**is not constant**

depends on the state of the economy

**Non-intuitive strategies**

for constant fraction of wealth $\in (0,1) \quad \Longrightarrow$

sell stocks when prices increase

buy stocks when prices decrease

$\Longrightarrow$ **Model drift as stochastic process, not directly observable**

# Models With Partial Information on the Drift

Drift depends on an additional ”source of randomness”

$$\mu = \mu_t = \mu(\boldsymbol{Y_t}) \quad \text{with factor process} \quad Y_t$$

Investor is not informed about factor process $Y_t$, he only observes

**Stock prices** $S_t$ or equivalently stock returns $R_t$

**Expert opinions** news, company reports
recommendations of analysts or rating agencies
own view about future performance

$\Longrightarrow$ Model with **partial information**

**Problem** Investor needs to ”learn” the drift from observable quantities
Find an estimate or **filter** for $\mu(Y_t)$

**Linear Gaussian Model**

Lakner (1998), Nagai, Peng (2002), Brendle (2006)

Drift $\mu(Y_t) = Y_t$ is a mean-reversion process

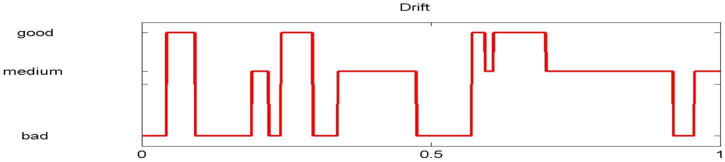
$$dY_t = \alpha(\overline{\mu} - Y_t)dt + \beta dW_t^1$$

where $W_t^1$ is a Brownian motion (in)dependent of $W_t$

# Models With Partial Information on the Drift (cont.)

## Hidden Markov Model (HMM)

Sass, Haussmann (2004), Rieder, Bäuerle (2005), Nagai, Rungaldier (2008)




**Factor process $Y_t$** finite-state Markov chain, independent of $W_t$

state space $\{e_1, \dots, e_d\}$, unit vectors in $\mathbb{R}^d$

states of drift $\mu(Y_t) = MY_t$ where $M = (\mu_1, \dots, \mu_d)$

generator or rate matrix $Q \in \mathbb{R}^{d \times d}$

diagonal: $Q_{kk} = -\lambda_k$ exponential rate of leaving state $k$

conditional transition prob. $P(Y_t = e_l \mid Y_{t-} = k, Y_t \neq Y_{t-}) = Q_{kl}/\lambda_k$

initial distribution $(\pi^1, \dots, \pi^d)^\top$

# HMM Filtering

Returns $dR_t = \frac{dS_t}{S_t} = \mu(Y_t)\,dt + \sigma\,dW_t$ observations

Drift $\mu(Y_t) = M\,Y_t$ non-observable (hidden) state

**Investor Filtration** $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ with $\mathcal{F}_t = \sigma(S_u :\ u \le t) \subset \mathcal{G}_t$

**Filter** $p_t^k := P(Y_t = e_k | \mathcal{F}_t)$

$$\widehat{\mu(Y_t)} := E[\mu(Y_t)|\mathcal{F}_t] = \mu(p_t) = \sum_{j=1}^{d} p_t^j\,\mu_j$$

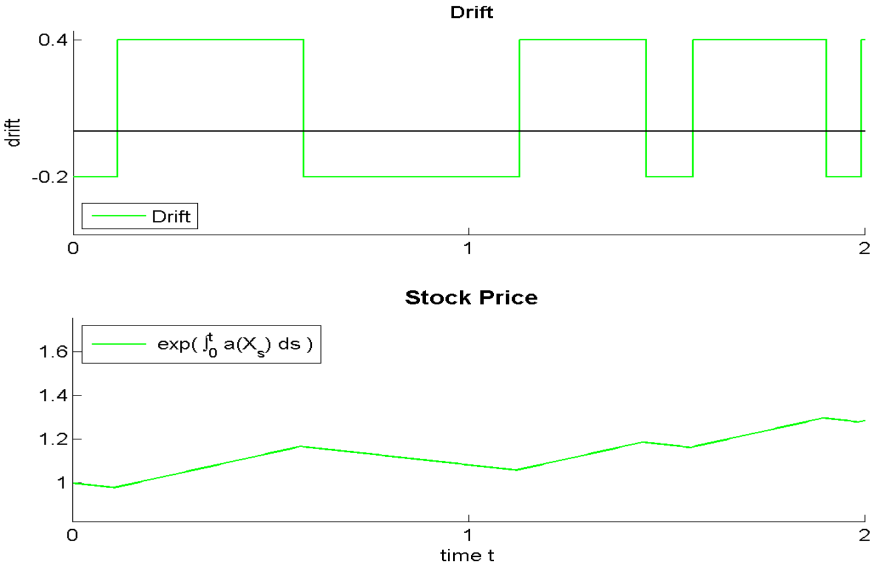
**Innovations process** $B_t := \sigma^{-1}(\ R_t - \int_0^t \widehat{\mu(Y_s)} ds\ )$ is an $\mathbb{F}$-BM

**HMM filter** Liptser, Shiryaev (1974), Wonham (1965), Elliot (1993)

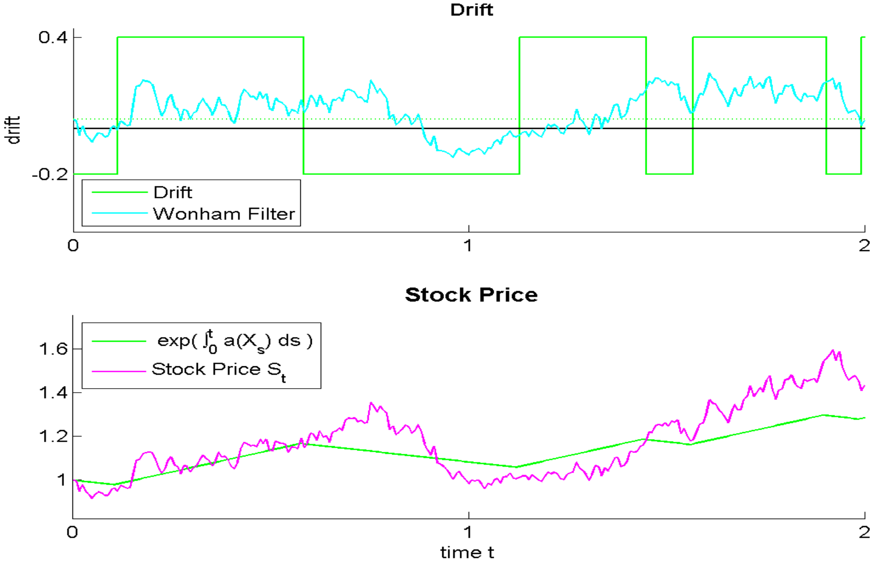
$$\begin{aligned} p_0^k &= \pi^k \\ dp_t^k &= \sum_{j=1}^{d} Q^{jk} p_t^j dt + \beta_k(p_t)^\top dB_t \end{aligned}$$

$$\text{where}\quad \beta_k(p) = p^k \sigma^{-1}\Big(\mu_k - \sum_{j=1}^{d} p^j \mu_j\Big)$$

# HMM Filtering: Example



# HMM Filtering: Example



# Expert Opinions

- **Academic literature:** drift is driven by unobservable factors
  Models with partial information, apply filtering techniques
  - Linear Gaussian models
  - Hidden Markov models
- **Practitioners** use static **Black-Litterman model**
  Apply Bayesian updating to combine
  **subjective views** (such as “asset 1 will grow by 5%”)
  with empirical or implied drift estimates
- Present paper combines the two approaches
  consider **dynamic** models with partial observation
  including **expert opinions**

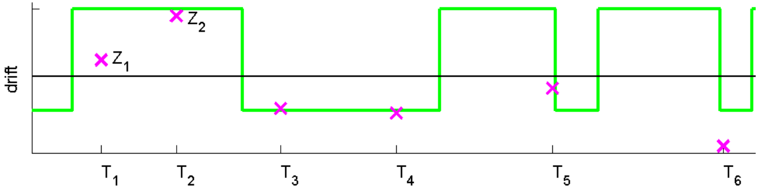
# Expert Opinions

Modelled by marked point process $I = (T_n, Z_n) \sim I(dt, dz)$

- At random points in time $T_n \sim \text{Poi}(\lambda)$ investor observes r.v. $Z_n \in \mathcal{Z}$
- $Z_n$ depends on current state $Y_{T_n}$, density $f(Y_{T_n}, z)$
  $(Z_n)$ cond. independent given $\mathcal{F}_T^Y = \sigma(Y_s : s \in [0, T])$

**Examples**

- Absolute view: $Z_n = \mu(Y_{T_n}) + \sigma_\varepsilon \varepsilon_n$, $(\varepsilon_n)$ i.i.d. $N(0,1)$
  The view "$S$ will grow by 5%" is modelled by $Z_n = 0.05$
  $\sigma_\varepsilon$ models confidence of investor




- Relative view (2 assets): $Z_n = \mu_1(Y_{T_n}) - \mu_2(Y_{T_n}) + \widetilde{\sigma}_\varepsilon \varepsilon_n$

**Investor filtration** $\mathbb{F} = (\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma(S_u : u \le t;\ (T_n, Z_n) : T_n \le t)$

# HMM Filtering - Including Expert Opinions

Extra information has no impact on filter $p_t$ between 'information dates' $T_n$

**Bayesian updating** at $t = T_n$:

$$p^k_{T_n} \;\propto\; p^k_{T_n-}\, f(e_k, Z_n) \qquad \text{recall: } f(Y_{T_n}, z) \text{ is density of } Z_n \text{ given } Y_{T_n}$$

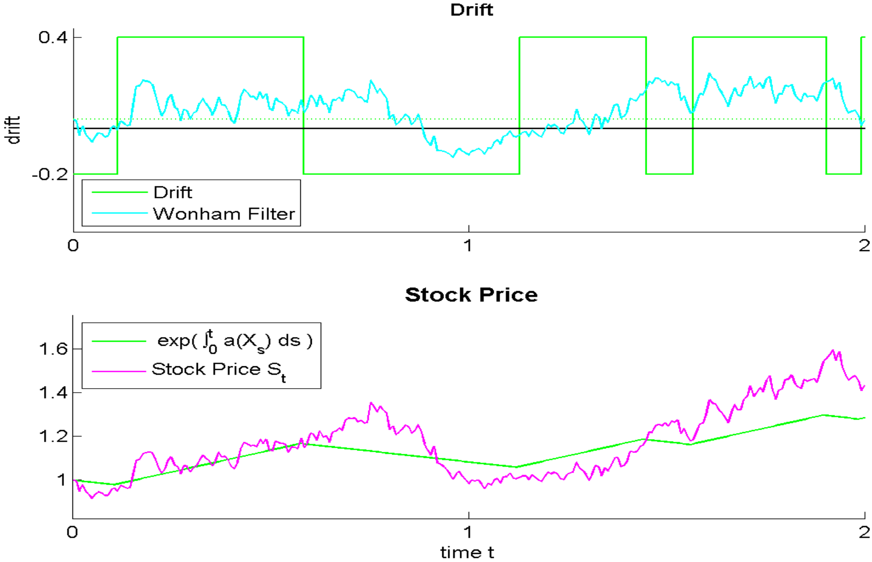
$$\text{with normalizer} \quad \sum_{j=1}^{d} p^j_{T_n-} f(e_j, Z_n) \;=:\; \bar{f}(p_{T_n-}, Z_n)$$

**HMM filter**

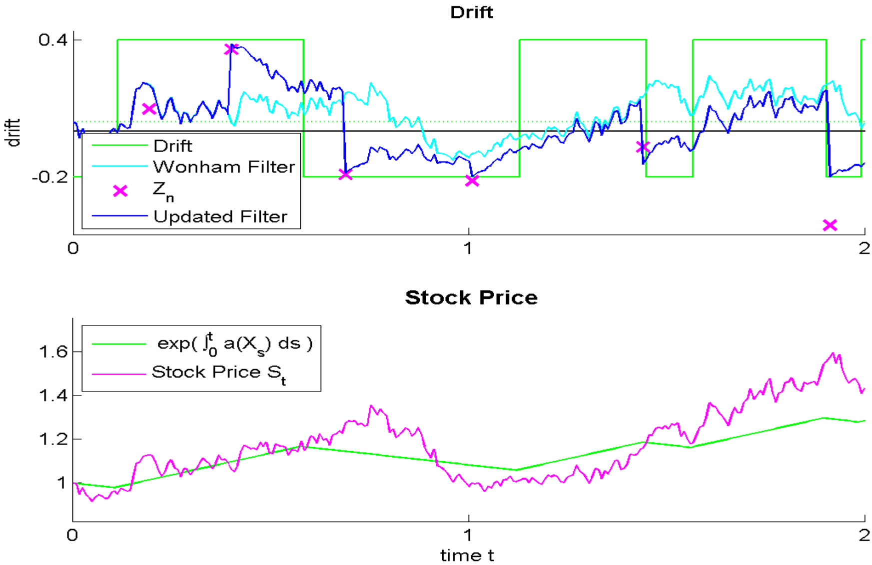
$$\begin{aligned} p^k_0 &= \pi^k \\ dp^k_t &= \sum_{j=1}^{d} Q^{jk} p^j_t dt + \beta_k(p_t)^\top dB_t + \boldsymbol{p^k_{t-} \int_{\mathcal{Z}} \left( \frac{f(e_k,z)}{\bar{f}(p_{t-},z)} - 1 \right) \widetilde{I}(dt \times dz)} \end{aligned}$$

**Compensated measure** $\quad \widetilde{I}(dt \times dz) := I(dt \times dz) - \underbrace{\lambda dt \sum_{k=1}^{d} p^k_{t-} f(e_k, z)\, dz}_{\text{compensator}}$

# Filter: Example



# Filter: Example



# Optimization Problem Under Partial Information

**Wealth** $dX_t^{(h)} = X_t^{(h)} h_t^\top \, (\mu(Y_t)\, dt + \sigma dW_t), \quad X_0^{(h)} = x_0$

**Admissible Strategies** $\mathcal{H} = \{(h_t)_{t\in[0,T]} \mid h_t \in \mathbb{R}^n,$ $h$ is $\mathbb{F}$-adapted and bounded $\}$

**Reward function** $v(t,x,h) = E_{t,x}[\, U(X_T^{(h)})\,]$ for $h \in \mathcal{H}$

**Value function** $V(t,x) = \sup_{h\in\mathcal{H}} v(t,x,h)$

Find optimal strategy $h^* \in \mathcal{H}$ such that $V(0,x_0) = v(0,x_0,h^*)$

# Reduction to an OP Under Full Information

Consider augmented state process $(X_t, p_t)$

**Wealth** $$dX_t^{(h)} = X_t^{(h)}\, h_t^\top \big( \underbrace{\widehat{\mu(Y_t)}}_{=M\, p_t}\, dt + \sigma\, d\boldsymbol{B_t} \big), \qquad X_0^{(h)} = x_0$$

**Filter** $$dp_t^k = \sum_{j=1}^{d} Q^{jk} p_t^j dt + \beta_k(p_t)^\top dB_t$$

$$+p_{t-}^k \int_{\mathcal{Z}} \left( \frac{f(e_k, z)}{\bar{f}(p_{t-}, z)} - 1 \right) \widetilde{I}(dt \times dz), \qquad p_0^k = \pi^k$$

**Reward function** $v(t,x,p,h) = E_{t,x,p}[\, U(X_T^{(h)})\, ]$ for $h \in \mathcal{H}$

**Value function** $$V(t,x,p) = \sup_{h \in \mathcal{H}} v(t,x,p,h)$$

Find $h^* \in \mathcal{H}(0)$ such that $V(0, x_0, \pi) = v(0, x_0, \pi, h^*)$

# Logarithmic Utility

$$U(X_T^{(h)}) = \log(X_T^{(h)}) = \log x_0 + \int_0^T \Big(h_s^\top \widehat{\mu(Y_s)} - \frac{1}{2} h_s^\top \sigma\sigma^\top h_s\Big) ds + \int_0^T h_s^\top \sigma dB_s$$

$$E[U(X_T^{(h)})] = \log x_0 + E\Big[\int_0^T \Big(h_s^\top \widehat{\mu(Y_s)} - \frac{1}{2} h_s^\top \sigma\sigma^\top h_s\Big) ds\Big] + 0$$

**Optimal Strategy**

$$h_t^* = (\sigma\sigma^\top)^{-1}\widehat{\mu(Y_t)}.$$

### Certainty equivalence principle

$h^*$ is obtained by replacing in the optimal strategy under full information

$$h_t^{\text{full}} = (\sigma\sigma^\top)^{-1}\mu(Y_t)$$

the unknown drift $\mu(Y_t)$ by its filter $\widehat{\mu(Y_t)}$

# Solution for Power Utility

**Risk-sensitive control problem**

Nagai & Runggaldier (2008), Davis & Lleo (2011)

Let $\quad Z^h := \exp\left\{\theta \int_0^T h_s^\top \sigma dB_s - \frac{\theta^2}{2}\int_0^T h_s^\top \sigma\sigma^\top h_s ds\right\}$

**Change of measure:** $\quad P^{(h)}(A) = E[Z^{(h)} 1_A] \quad$ for $\quad A \in \mathcal{F}_T$

**Reward function**

$$E_{t,x,p}[U(X_T^{(h)})] \quad = \quad \frac{x^\theta}{\theta} \underbrace{E_{t,p}^{(h)}\left[\exp\left\{-\int_t^T b(p_s, h_s) ds\right\}\right]}_{=:\, v(t,p,h) \text{ independent of } x}$$

$$\text{where} \quad b(p,h) \quad := \quad -\theta\left(h^\top M p - \frac{1-\theta}{2} h^\top \sigma\sigma^\top h\right)$$

**Value function** $\quad V(t,p) = \sup_{h\in\mathcal{H}} \; v(t,p,h) \quad$ for $0 < \theta < 1$

Find $\; h^* \in \mathcal{H} \;$ such that $\quad V(0,\pi) = v(0,\pi,h^*)$

# HJB-Equation

State $\quad dp_t = \alpha(p_t, h_t)dt + \beta^\top(p_t)dB_t + \int_{\mathcal{Z}} \gamma(p_t, z)\widetilde{I}(dt \times dz)$

Generator $\quad \mathcal{L}^h g(p) = \frac{1}{2} tr\left[\beta^\top(p)\beta(p)D^2 g\right] + \alpha^\top(p,h)\nabla g + \lambda \int_{\mathcal{Z}} \{g(p+\gamma(p,z)) - g(p)\}\bar{f}(p,z)dz$

$$V_t(t,p) + \sup_{h\in\mathbb{R}^n} \left\{\mathcal{L}^h V(t,p) - b(p,h)V(t,p)\right\} = 0$$

terminal condition $\quad V(T,p) = 1$

**Candidate for the Optimal Strategy**

$$h^* = h^*(t,p) = \underbrace{\frac{1}{(1-\theta)}(\sigma\sigma^\top)^{-1}}\left\{Mp + \frac{1}{V(t,p)}\sigma\beta(p)\nabla V(t,p)\right\}$$

myopic strategy $+$ correction

Certainty equivalence principle does not hold

# Justification of HJB-Equation

- Standard verification arguments fail, since we cannot guarantee **uniform ellipticity** of the diffusion part: $tr\left[\beta^\top(p)\beta(p)D^2G\right]$

$$\xi^\top \beta^\top(p)\beta(p)\xi \geq c|\xi|^2 \quad \text{for some } c > 0 \text{ and all } \xi \in \mathbb{R}^d$$

satisfiable only if number of assets $n \geq$ number of drift states $d$

- Applying results and techniques from Pham (1998)

$\Longrightarrow$ $V$ is a unique continuous viscosity solution of the HJB-equation

# Regularization of HJB-Equation

- Add a 'small' Gaussian perturbation $\frac{1}{\sqrt{m}} d\widetilde{B}_t$ to the SDE for the first $d-1$ components of the filter
- Consider control problem for the modified dynamics of the filter
- Modified HJB-equation has an additional diffusion term $\frac{1}{2m} \Delta V^m(t,p)$

  $\Longrightarrow$ uniform ellipticity
- Applying results from Davis & Lleo (2011)

  $\Longrightarrow$ classical solution $V^m(t,p)$ to the modified HJB-equation

  Standard verification results can be applied
- Convergence results for $m \to \infty$:

| | |
|---|---|
| optimal strategy | to the modified control problem is an |
| $\varepsilon$-optimal strategy | to the original control problem |

# Approximation of the optimal strategy

$\rightarrow$ Policy Improvement

$\rightarrow$ Numerical solution of HJB equation

- Feynman-Kac formula for linearized HJB equation

# Policy Improvement

Starting approximation is the myopic strategy $h_t^{(0)} = \frac{1}{1-\theta}(\sigma\sigma^\top)^{-1} M p_t$

The corresponding reward function is

$$V^{(0)}(t,p) := v(t,p,h^{(0)}) = E_{t,p}\Big[\exp\Big(-\int_t^T b(p_s^{(h^{(0)})}, h_s^{(0)})ds\Big)\Big]$$

Consider the following optimization problem

$$\max_h \{\mathcal{L}^h V^{(0)}(t,p) - b(p,h) V^{(0)}(t,p)\}$$

with the maximizer

$$h^{(1)}(t,p) = h^{(0)}(t,p) + \frac{1}{(1-\theta)V^{(0)}(t,p)}(\sigma^\top)^{-1}\beta(p)\nabla V^{(0)}(t,p)$$

For the corresponding reward function $V^{(1)}(t,p) := v(t,p,h^{(1)})$ it holds

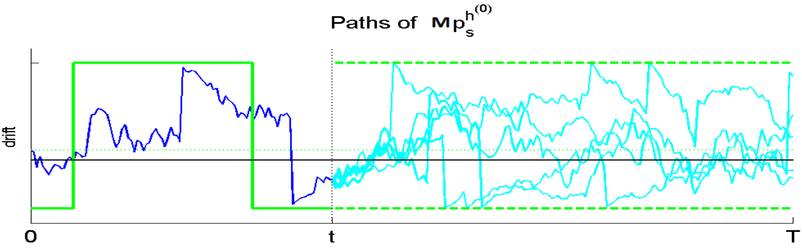
**Lemma** ( $h^{(1)}$ is an improvement of $h^{(0)}$ )

$$V^{(1)}(t,p) \geq V^{(0)}(t,p)$$

# Policy Improvement (cont.)

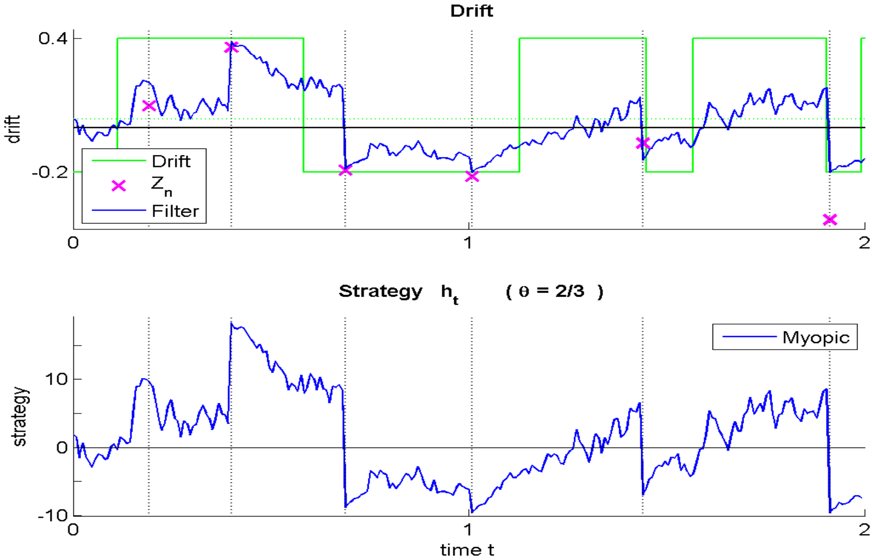
Policy improvement requires Monte-Carlo approximation of reward function

$$V^{(0)}(t,p) = E_{t,p}\Big[\exp\Big(-\int_t^T b(p_s^{(h^{(0)})}, h_s^{(0)})ds\Big)\Big].$$



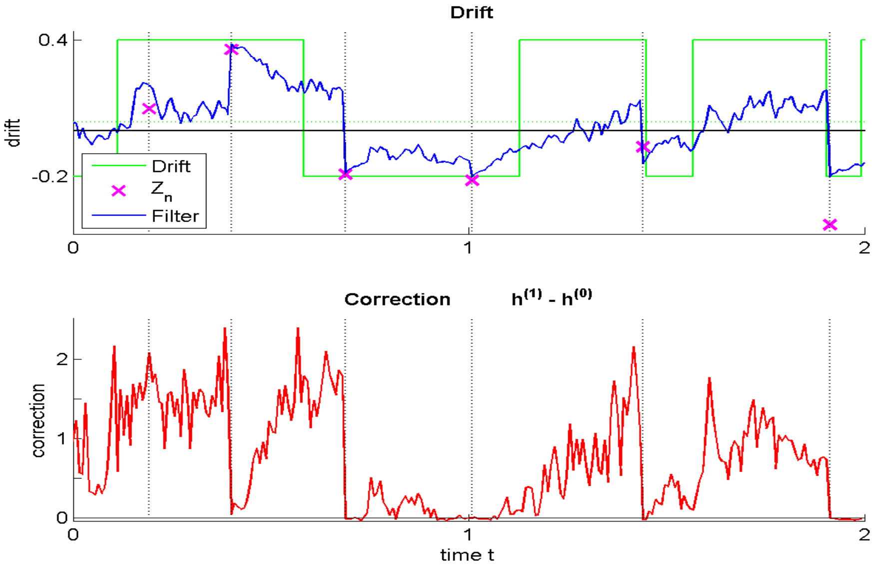

- Generate $N$ paths of $p_s^{h^{(0)}}$ starting at time $t$ with $p = p_t$
- Estimate expectation $E_{t,p}[\cdot]$
- Approximate partial derivatives $V_{p^k}^{(0)}(t,p)$ by finite differences
- Compute first iterate $h^{(1)}$

# Numerical Results



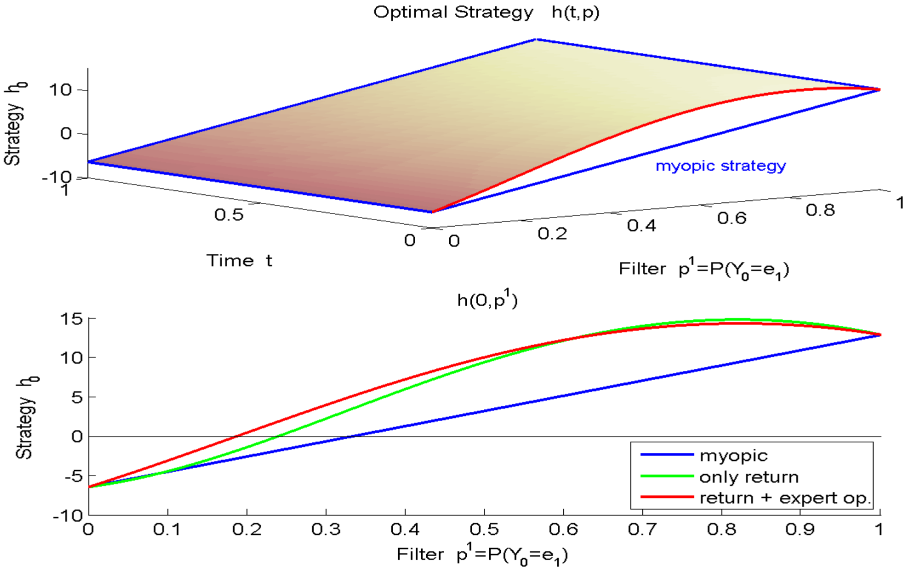

Drift

# Numerical Results




For $t = T_n$ : nearly full information $\Longrightarrow$ correction $\approx 0$

# Numerical solution of HJB equation



# Conclusion

- Portfolio optimization under partial information on the drift
- Investor observes stock prices and expert opinions
- Non-linear HJB-equation with a jump part
- Computation of optimal strategy

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