# Variance reduction by conditioning in the pricing problem where the underlying is a continuous-time finite state Markov process

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[1] based on joint work with J.M. Montes and V.Prezioso

- Asset price evolutions are generally described as a geometric Brownian motion or an exponential Levy.
- The evolution of other quantities, in particular rates, but occasionally also asset prices, may more conveniently be modeled as a continuous time Markov chain (CTMC).
- Levy processes include the case of CTMC, but for this case a direct approach may be more convenient computationally.

A full theory of financial markets based on CTMC (*prices, rates or, more generally, factors) is given in Norberg (2003).*

- Assume given the following:
  i) An underlying factor process $X_t \in \{x^1, \cdots, x^N\}$ *(could simply be the short rate itself)* with a time homogeneous transition intensity matrix $Q$
  ii) A simple claim of the form $H(X_T) = H_0 := [H(x^1), \cdots, H(x^N)]'$.
  iii) Assume furthermore that for the short rate one has $r_t = r^i$ if $X_t = x^i$ *(obvious if $X_t = r_t$)*.

- The price $\Pi_i(t)$ at time $t$ when $X_t = x^i$ is then given by

$$\Pi_i(t) = [\exp\{(Q - R)\,(T - t)\}\, H_0]_i$$

where $[z]_i$ denotes the $i-$th component of the vector $z$ and $R$ is the diagonal matrix with elements $r^i$ $(i = 1, \cdots, N)$.

- The previous explicit formula may not be of much use if:

i) The evolution of the underlying factors is not time homogeneous;

ii) the derivative is path dependent.

*In all these more involved cases a full Monte Carlo (MC) simulation is always possible:*

- For the CTMC $X_t$ simulate the successive jump times $\tau_n$ and the values $X_n$ of $X_t$ at $\tau_n$.
- For an intensity matrix $Q = \{q_{i,j}\}$, putting $q_i = \sum_{i \neq j} q_{i,j}$ one has that, if $X_{\tau_n} = x^i$, the inter-jump times $\tau_{n+1} - \tau_n$ are exponentially distributed with parameter $q_i$ and the probability for $X_{\tau_{n+1}} = x^j \neq x^i$ is $p_{i,j} = \frac{q_{i,j}}{q_i}$.

$\longrightarrow$ In addition to a possibly large variance, plain MC may lead to biased results *(Quasi-Montecarlo may allow to better explore the various regions of possible trajectories)*.

# Purpose

We show first that, conditionally on the number $\nu_{t,T}$ of jumps of $X_t$ in a given interval $[t, T]$, one can obtain an explicitly computable expression also for exotic derivatives and when the underlying is multivariate and/or has a time non homogeneous evolution.

Since

$$\Pi_i(t) = E^{\tilde{P}}\left\{e^{-\int_t^T r_s ds} H(X_T) \mid X_t = i\right\}$$

$$= E^{\tilde{P}}\left\{E^{\tilde{P}}\left\{e^{-\int_t^T r_s ds} H(X_T) \mid \nu_{t,T}, X_t = i\right\} \mid X_t = i\right\}$$

where $\tilde{P}$ is a (calibrated) martingale measure, then, given that the inner expression allows for an explicit computation, one needs to simulate only the r.v. $\nu_{t,T}$.

- With respect to a full MC this allows to reduce the variance (*variance reduction by conditioning*).
- Allows also to reduce a possible bias.

$\longrightarrow$ *Shall show how to compute the inner expression in various more general cases*

# Outline

- For simplicity of exposition we first present the procedure for the case of a simple claim on a time homogeneous underlying $X_t$ given by a CTMC.
- *Successively we show the extensions/changes for the more general case.*
- Finally we present numerical results and comparisons.

# The model (simple case first)

$X_t$ a ***CTMC*** under a martingale measure $\tilde{P}$

- state space $E = \{x^1, x^2, \ldots, x^N\}$, $N \in \mathbb{N}$ *(identify $x^i$ with $i$)*
- $Q = (q_{i,j})_{1 \le i,j \le N}$ the transition intensity matrix, homogeneous w.r. to time
- $q_i := \sum_{\substack{j=1 \\ j \neq i}}^{N} q_{i,j}, \ i = 1, \ldots, N$ the intensities associated with the states $x^i$.

- $\tau_n$ : random time at which the $n^{th}$ jump occurs,
- $X_n := X_{\tau_n}$ and $X_s \equiv X_n$ for $s \in [\tau_n, \tau_{n+1})$
- $r_{\tau_n} = r^i$ if $X_{\tau_n} = x^i$ $(i = 1, \cdots, N)$
  *(write $r_n := r_{\tau_n}$; $r_s = r_n$ for $s \in [\tau_n, \tau_{n+1})$)*
- $\left(\tau_{n+1} - \tau_n \mid X_{\tau_n} = x^i\right) \sim \mathcal{E}xp(q_i)$

- $\nu_t := \sup\{n \mid \tau_n \leq t\}$ (#of jumps up to time $t$) ; $\nu_{t,T} := \nu_T - \nu_t$.

$$\begin{array}{cccccccccc} & & t & & & & T & & & \\ - & -|- & -|- & -|- & ---- & -|- & -|- & -|- & \longrightarrow \\ & \tau_{\nu_t} & & \tau_{\nu_t+1} & & \tau_{\nu_T} & & \tau_{\nu_T+1} & \end{array}$$

- Pricing a derivative

$$\Pi(t) = E^{\tilde{P}}\left\{e^{-\int_t^T r_s ds} H(X_T) \mid \mathcal{F}_t\right\}$$
$$= \sum_{i=1}^{N} E^{\tilde{P}}\left\{e^{-\int_t^T r_s ds} H(X_T) \mid X_t = i\right\} \mathbf{1}_{\{X_t = i\}}$$
$$\Downarrow$$

$$\Pi_i(t) = E^{\tilde{P}}\Big\{\exp[r_t(t - \tau_{\nu_t}]$$
$$\exp\left[-\sum_{i=\nu_t}^{\nu_T - 1} r_i(\tau_{i+1} - \tau_i) - r_T(T - \tau_{\nu_T})\right] H(X_T) \mid X_t = i\Big\}$$
$$= \exp[r_t(t - \tau_{\nu_t}]$$
$$E^{\tilde{P}}\left\{exp\left[-\sum_{i=\nu_t}^{\nu_T - 1} r_i(\tau_{i+1} - \tau_i) - r_T(T - \tau_{\nu_T})\right] H(X_T) \mid X_t = i\right\}$$

$\longrightarrow$ *Not restrictive to assume* $t = T_{\nu_t}$

# Prototype product (analogue to Arrow-Debreu prices)

- Its price at time $t < T$ is

$$V_{H_0,t,T}(X_t) =$$

$$= E^{\tilde{P}}\left\{\exp\left[-\sum_{i=\nu_t}^{\nu_T-1} r_i(\tau_{i+1}-\tau_i) - r_{\nu_T}(T-\tau_{\nu_T})\right] H_0(X_T) \mid X_t\right\}$$

with

$$H_0(\cdot) = \sum_{i=1}^{N} w_i^0 \mathbf{1}_{\{\cdot = x^i\}}, \; x^i \in E, \;\; w_i^0 \in \mathbb{R}$$

- In the calculations to follow, in order to determine the explicit analytical expression conditional on $\nu_{t,T}$, we shall *(except for the case of Asian options)* drop the last factor: it is in general a small quantity but we shall take it into account in the MC simulations anyway *(the MC simulations will be performed to determine $\nu_{t,T}$ and thus also $\nu_T = \nu_t + \nu_{t,T}$)*.
- *Various interest rate derivatives can be obtained as particular cases or as linear combinations of prototype products with underlying the short rate.*

- For given $n \in \mathbb{N}$ consider the recursions

$$\begin{cases} H_0(X_{\nu_t+n}) \text{ given by the } \textbf{\textit{Prototype payoff}} \quad \left(H_0(\cdot) = \sum_{i=1}^{N} w_i^0 \mathbf{1}_{\{\cdot = x^i\}}\right) \\ H_h(X_{\nu_t+n-h}) = E^{\tilde{P}} \left\{ e^{-r_{\nu_t+n-h}(\tau_{\nu_t+n-h+1} - \tau_{\nu_t+n-h})} H_{h-1}(X_{\nu_t+n-h+1}) \mid X_{\nu_t+n-h} \right\} \\ \qquad \forall h = 1, \ldots, n \end{cases}$$

**Proposition:** The price of the Prototype product can be computed as

$$V_{H_0,t,T}(X_t) = E^{\tilde{P}} \left\{ H_{\nu_{t,T}}(X_t) \mid X_t \right\} = \sum_{n=0}^{+\infty} H_n(X_t) \tilde{P}(\nu_{t,T} = n \mid X_t)$$

where

- $\nu_{t,T} = \nu_T - \nu_t$ (number of jumps between $t$ and $T$)
- $H_n(X_t) = H_n(X_{\nu_t})$ is as obtained recursively above.

- Setting $\underline{x} = [x^1, \dots, x^N]'$ we have the representations

$$H_0(\underline{x}) := [w_1^0, \dots, w_N^0]' \;\rightarrow\; H_n(\underline{x}) := [w_1^n, \dots, w_N^n]'$$

Putting, furthermore,

$$\widetilde{Q} = \left(\tilde{q}_{i,j}\right)_{1 \le i,j \le N} \quad \text{with} \quad \tilde{q}_{i,j} = \begin{cases} \frac{q_{i,j}}{r^i + q_i} & i \ne j \\ 0 & i = j \end{cases}$$

one obtains, at the generic $\tau_n$, the following one-step evolution of $H_n$,

$$H_n(\underline{x}) = \widetilde{Q}\, H_{n-1}(\underline{x}).$$

$\longrightarrow$ In the time homogeneous case it follows that $H_n(\underline{x}) = \widetilde{Q}^n H_0(\underline{x})$ by putting $\widetilde{Q}^0 = I_N$.

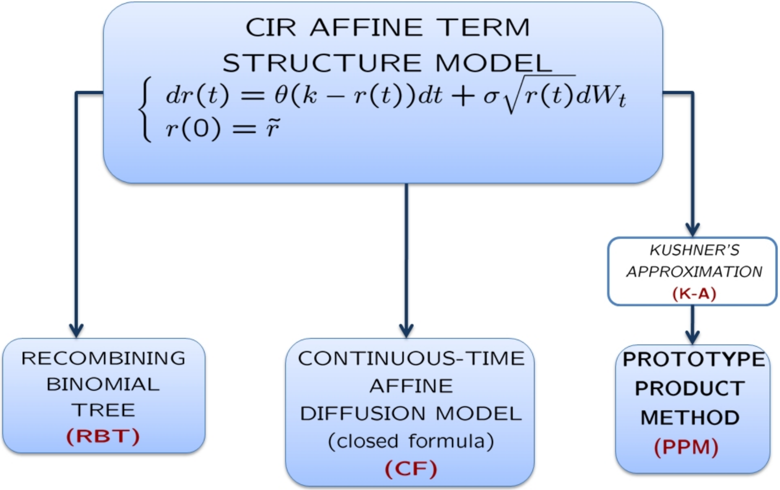
- The actual derivative price is then given by

$$\begin{aligned}\Pi_i(t) &= V_{H_0,t,T}(X_t)_{|X_t=x^i} \\ &= \sum_{n=0}^{\infty} \left[\widetilde{Q}^n H_0(\underline{x})\right]_i \tilde{P}\left(\nu_{t,T} = n \mid X_t = x^i\right) \\ &= E^{\tilde{P}}\left\{\left[\widetilde{Q}^{\nu_{t,T}} H_0(\underline{x})\right]_i \mid X_t = x^i\right\}\end{aligned}$$

($[z]_i$ is the $i-$th component of the vector $z$).

*From here two possibilities for actual computation:*

- *Explicit numerical computation (middle term)*
- *MC simulation by simulating just $\nu_{t,T}$ (rightmost term), i.e. MC simulation by conditioning.*


CIR AFFINE TERM STRUCTURE MODEL
$$\begin{cases} dr(t) = \theta(k - r(t))dt + \sigma\sqrt{r(t)}dW_t \\ r(0) = \tilde{r} \end{cases}$$
KUSHNER'S APPROXIMATION (K-A)
RECOMBINING BINOMIAL TREE (RBT)
CONTINUOUS-TIME AFFINE DIFFUSION MODEL (closed formula) (CF)
PROTOTYPE PRODUCT METHOD (PPM)

**Bond prices** with **CF**, **RBT**, **PPM(MC1)+K-A** and **PPM(MC2)+K-A** (***stepsMC=stepsRBT***=500)

MC1: MC with conditioning MC2: full MC

| $T$(years) | 0.5 | 0.5 | 0.5 | 0.5 |
|---|---|---|---|---|
| $\widetilde{r}(= r^i)$ | 0.01 | 0.02 | 0.03 | 0.02 |
| $k$ | 0.8 | 0.5 | 1.1 | 1.2 |
| $\theta$ | 0.01 | 0.02 | 0.03 | 0.02 |
| $\sigma$ | 0.1 | 0.05 | 0.1 | 0.1 |
| **CF** | 0.995014 | 0.990051 | 0.985116 | 0.990052 |
| **RBT** | 0.995042 | 0.99007 | 0.985146 | 0.990072 |
| **PPM(MC1)+K-A** | **0.995024** | 0.990143 | **0.985128** | **0.990059** |
| **PPM(MC2)+K-A** | **0.994988** | 0.989963 | 0.984903 | **0.990049** |

## **Bond prices** with **CF**, **RBT**, **PPM(MC1)+K-A** and **PPM(EF)+K-A** (***stepsMC=stepsRBT***=500)

| $T$(years) | 0.5 | 0.5 | 0.5 | 0.5 |
|---|---|---|---|---|
| $\widetilde{r}(=r^i)$ | 0.1 | 0.1 | 0.2 | 0.3 |
| $k$ | 0.1 | 0.1 | 0.2 | 0.3 |
| $\theta$ | 0.1 | 0.4 | 0.2 | 0.3 |
| $\sigma$ | 0.1 | 0.05 | 0.2 | 0.3 |
| **CF** | 0.95124806 | 0.95123369 | 0.90497717 | 0.86113958 |
| **RBT** | 0.951343 | 0.951329 | 0.905157 | 0.861394 |
| **PPM(MC1)+K-A** | 0.951022 | 0.950859 | 0.905229 | 0.861104 |
| **PPM(EF)+K-A** | **0.951324** | 0.951723 | **0.905012** | 0.861756 |

# Extensions

- $X_t$ (scalar) but time inhomogeneous

  $\longrightarrow$ *Barrier options*
  (may include credit risky derivatives)

- Path dependent derivatives/claims with $X_t$ multivariate:
  - *lookback options*
  - *Asian options*

# Time inhomogeneous case

- Generalize $Q$ as

$$Q \quad \longrightarrow \quad Q(n) = \{q^n_{i,j}\}_{i,j=1,\cdots,N}$$

so that also

$$\widetilde{Q} \quad \longrightarrow \quad \widetilde{Q}(n) = \left\{ \frac{q^n_{i,j}}{r^i + q^n_i} \right\}_{i,j=1,\cdots,N}$$

Then

$$H_n(\underline{x}) = \widetilde{Q}(n) H_{n-1}(\underline{x})$$

or, equivalently,

$$H_n(\underline{x}) = \widetilde{Q}(n)\widetilde{Q}(n-1)\cdots\widetilde{Q}(0)H_0(\underline{x})$$

# The multivariate (bivariate) case

- Consider e.g. $(X_t, Y_t)$ with

$$X_t \in \{x^1, \cdots, x^N\} \quad \text{and} \quad Y_t \in \{y^1, \cdots, y^M\}$$

and put

$$r_{\tau_n} = r^{i,h} \quad \text{if} \quad (X_{\tau_n}, Y_{\tau_n}) = (x^i, y^h)$$

$\rightarrow$ *(in the more general time-inhomogeneous case)*

$$Q(n) = \left\{ q^n_{(i,h),(j,k)} \right\} \left[ \begin{array}{ccc} i,j & = & 1, \cdots, N \\ h,k & = & 1, \cdots, M \end{array} \right]$$

# The multivariate case

- With $\underline{z} = (\underline{x}, \underline{y})'$ where $\underline{x} = (x^1, \cdots, x^N)$, $\underline{y} = (y^1, \cdots, y^M)$ and

$$H_0(\underline{z}) = H_0(\underline{x}, \underline{y}) = [w_1, \cdots, w_{N \cdot M}]'$$

also

$$H_n(\underline{z}) = \tilde{Q}(n) H_{n-1}(\underline{z})$$

where

$$\tilde{Q}(n) = \left\{ \frac{q^n_{(i,h),(j,k)}}{r^{i,h} + q^n_{i,h}} \right\} \left[ \begin{array}{ccc} i,j & = & 1, \cdots, N \\ h,k & = & 1, \cdots, M \end{array} \right]$$

with $q^n_{i,h} = \sum_{j \neq i, k \neq h} q^n_{(i,h),(j,k)}$.

# The multivariate case

## Application to defaultable bond pricing

- With $\tau$ denoting the default time and $\lambda_t$ the default intensity, for the price of a defaultable bond we have

$$\Pi(t) = \mathbf{1}_{\{\tau>t\}} E^{\tilde{P}} \left\{ \exp\left[ - \int_t^T (r_s + \lambda_s)\, ds \right] \mid \mathcal{F}_t \right\}$$

$\longrightarrow$ $r_t$ and $\lambda_t$ may form two different CTMC, i.e. $X_t = r_t, , Y_t = \lambda_t$

$\longrightarrow$ *They may also be driven by a common factor process $Z_t$ evolving as a CTMC, i.e.* $r_t = r(t, Z_t),\ \lambda_t = \lambda(t, Z_t)$.

# Lookback options

## Lookback call options

- For an underlying CTMC $X_t$ consider a claim of the form

$$H_T = \left(X_T - g(X_0^T)\right)^+$$

- Put $Y_t := g(X_0^t)$ which takes a given finite number of values.
  - → For $t \leq T$, the process $Y_t$ then takes a finite number of values *(w.l.of g. we can identify them with $h = 1, \cdots, M$)*
  - → it jumps only at jump times of $X_t$.

# Lookback options

- Assume, furthermore,

$$g(X_0^{\tau_n}) = G(X_{\tau_n}, g(X_0^{\tau_{n-1}})) \quad \text{for some measurable } G(\cdot,\cdot)$$

$\longrightarrow$ $(X_t, Y_t)$ is a CTMC and $H_T = (X_T - Y_T)^+$.

$\longrightarrow$ Need only to derive the $Q-$matrix for $(X_t, Y_t)$.

# Lookback options

- Recall that, if for a scalar CTMC $X_t$ the $Q-$matrix is $Q = \{q_{i,j}\}$, then the transition probabilities of the embedded chain $X_n$ are

$$p_{i,j} = \frac{q_{i,j}}{q_i} \quad \text{with} \quad q_i = \sum_{j \neq i} q_{i,j} \quad (q_{i,i} = p_{i,i} = 0)$$

- *Viceversa, given $p_{i,j}$, there are various possible $q_{i,j}$ that lead to the same $p_{i,j}$. They differ by the choice of $q_i$ since we have $q_{i,j} = q_i p_{i,j}$.*

# Lookback options

- Since in our case $Y_t$ jumps exactly when $X_t$ does, we may put

$$q_{(i,h)} \quad \left( = \sum_{j,k} q_{(i,h),(j,k)} \right) = q_i \quad \forall h = 1, \cdots, M$$

where $q_i$ is the intensity of leaving state $i$ for the chain $X_t$. *(At a generic $\tau_n$ the process $X_t$ actually leaves the current state, while $Y_t$ may jump to itself)*

$\longrightarrow$ *Start thus from constructing $p_{(i,h),(j,k)}$.*

# Lookback options

We have (recall $X_n = X_{\tau_n}$, $Y_n = Y_{\tau_n}$)

$$p_{(i,h),(j,k)} := P\{X_{n+1} = j, Y_{n+1} = k \mid X_n = i, Y_n = h\}$$

$$= P\{X_{n+1} = j, G(X_{n+1}, Y_n) = k \mid X_n = i, Y_n = h\}$$

$$= P\{G(X_{n+1}, Y_n) = k \mid X_{n+1} = j, X_n = i, Y_n = h\} \cdot P\{X_{n+1} = j \mid X_n = i, Y_n = h\}$$

$$= \mathbf{1}_{\{G(j,h)=k\}} P\{X_{n+1} = j \mid X_n = i\} = \mathbf{1}_{\{G(j,h)=k\}} p_{i,j}$$

$$\rightarrow \quad q_{(i,h),(j,k)} = p_{(i,h),(j,k)} \cdot q_i = q_{i,j} \mathbf{1}_{\{G(j,h)=k\}}$$

# Example

- Let $\quad Y_t = g(X_0^t) := \min_{s \leq t} X_s$
  *($Y_t$ has the same finite number of possible values as $X_t$)*

$$\rightarrow \quad G(X_{\tau_n}, g(X_0^{\tau_{n-1}})) = \min \left[ X_{\tau_n}, \min_{s \leq \tau_{n-1}} X_s \right] = g(X_0^{\tau_n})$$

# Example

- In this case *(states in increasing order of magnitude)*

$$p_{(i,h),(j,k)} = \mathbf{1}_{\{G(j,h)=k\}} p_{i,j} = \mathbf{1}_{\{\min\{j,h\}=k\}} p_{i,j}$$

which implies that

$$p_{(i,h),(j,k)} = \begin{cases} p_{ik} & \text{if} \quad k < h \\ p_{ij} & \text{if} \quad k = h,\ j \geq k \\ 0 & \text{if} \quad k > h \end{cases} = \begin{cases} \frac{q_{ik}}{q_i} & \text{if} \quad k < h \\ \frac{q_{ij}}{q_i} & \text{if} \quad k = h,\ j \geq k \\ 0 & \text{if} \quad k > h \end{cases}$$

and, consequently,

$$q_{(i,h),(j,k)} = p_{(i,h),(j,k)} \cdot q_i = \begin{cases} q_{ik} & \text{if} \quad k < h \\ q_{ij} & \text{if} \quad k = h,\ j \geq k \\ 0 & \text{if} \quad k < h \end{cases}$$

Comparing Plain MC and MC + Variance Reduction for Lookback Call pricing.

$E = [0.8, 0.9, 1.0, 1.1, 1.2]$, $x_0 = 3$, $T = 2$ years, time unit: 1 day

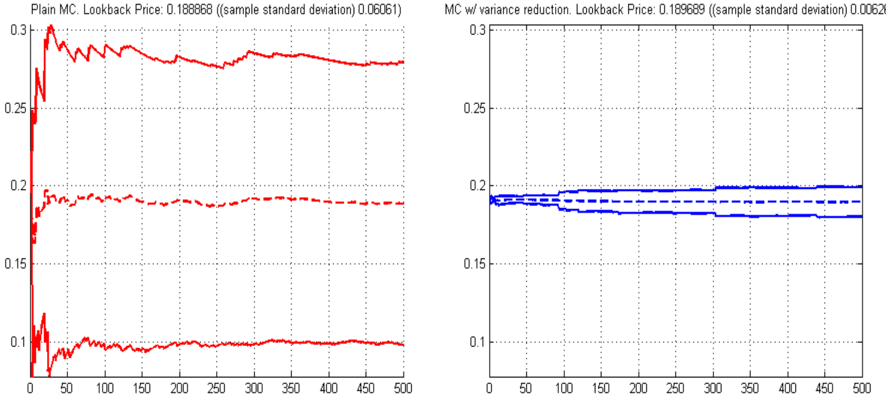
- $Q$-matrix for Test 1

$$Q = \begin{bmatrix} -1200 & 300 & 300 & 300 & 300 \\ 0.6 & -2.4 & 0.6 & 0.6 & 0.6 \\ 6 & 6 & -24.0 & 6 & 6 \\ 21 & 21 & 21 & -84 & 21 \\ 400 & 400 & 400 & 400 & -1600 \end{bmatrix}$$

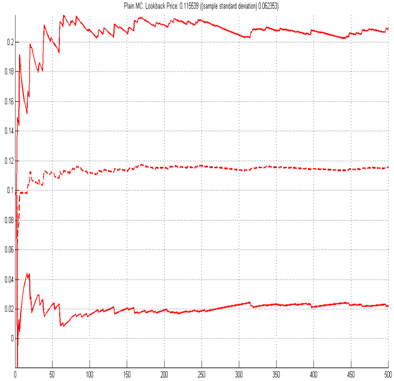
- $Q$-matrix for Test 2

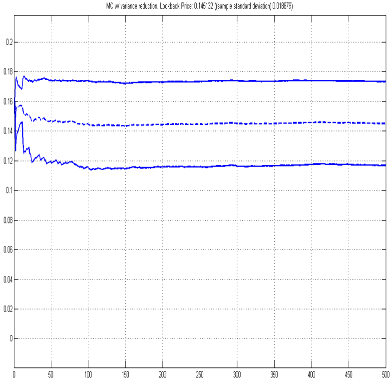
$$Q = \begin{bmatrix} -0.12 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.3 & -1.2 & 0.3 & 0.3 & 0.3 \\ 0.6 & 0.6 & -2.3 & 0.5 & 0.6 \\ 0.9 & 0.8 & 1 & -3.7 & 1 \\ 1.1 & 1 & 0.9 & 0.8 & -3.8 \end{bmatrix}$$

Running Mean of Price vs. Iteration Number (Test 1)
(Red) Plain MC; (Blue ) MC+Variance Reduction
*Diagram Width = 3 empirical standard deviations*

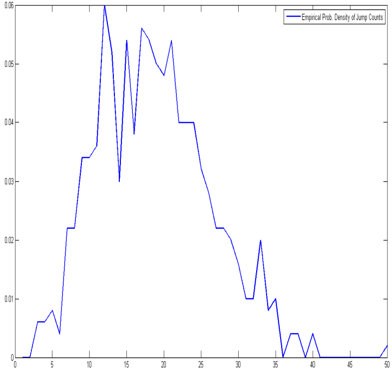


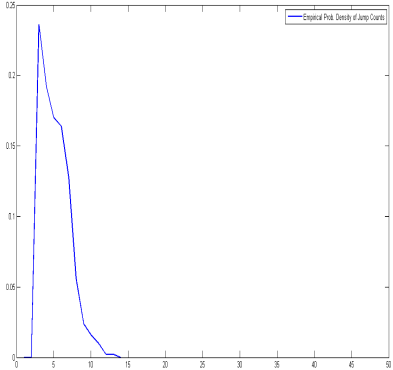
Running Mean of Price vs. Iteration Number(Test 2)
(Red) Plain MC;  (Blue ) MC+Variance Reduction
*Diagram Width = 3 empirical standard deviations*



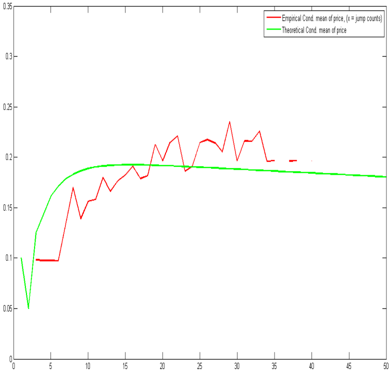
## (Left) Empirical Distribution of Jump Counts for Test 1 samples
## (Right) Empirical Distibution of Jump Counts for Test 2 samples

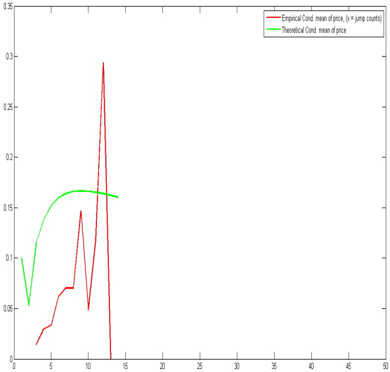


Price vs. Jump Count:
Test 1 samples (Left); Test 2 samples (Right)

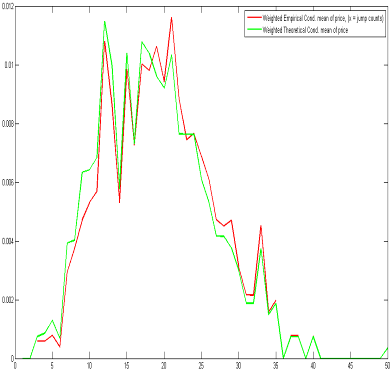
red - sample price; green- theoretical price

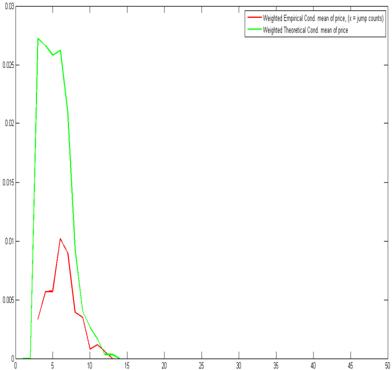


Weighted Price vs. Jump Count:
Test 1 samples (Left);  Test 2 samples (Right)

red - sample price;  green- theoretical price



# Asian options

- For Asian options consider the two processes

$$\begin{cases} X_t & \text{a CTMC, and} \\ Y_t := \int_0^t X_s ds = \sum_{\tau_n \le t} X_{\tau_{n-1}}(\tau_n - \tau_{n-1}) + X_{\tau_n}(t - \tau_n) \end{cases}$$

and write $X_n$ and $Y_n$ for $X_{\tau_n}$ and $Y_{\tau_n}$ respectively.

- The claim of a standard Asian option can then be represented as

$$H_T = \left( \frac{1}{T-t} \int_t^T X_s ds - K \right)^+ = \left( \frac{1}{T-t} (Y_T - Y_t) - K \right)^+$$

# Asian options

- $X_t$ is finite-state, while $Y_t$ is continuous-valued

  $\longrightarrow$ *Want also $Y_t$ to become finite-state in order to have $(X_t, Y_t)$ finite-state Markov*

  $\longrightarrow$ Discretization of the values of $Y_t$.

# Asian options

- Assuming that $X_t \in \left\{x^1, \cdots, x^N\right\}$, *(in increasing order of magnitude)* the range for the values of $Y_t$ is $[0,\ T \max_{t \leq T} X_t] = \left[0, Tx^N\right]$ (one may denote the states of $X_t$ by $i = 1, \cdots, N$.)
- Partition now the interval $\left[0, Tx^N\right]$ into intervals of equal length $\Delta$ assuming that $Tx^N = K\Delta$ for a suitable positive integer $K$. The generic $k-$th interval of the partition is then

$$A^k = [a^{k-1}, a^k) = [(k-1)\Delta, k\Delta), \quad k = 1, \cdots, K$$

- Denote by $y^k$ the midpoint of $A^k$ (*other choices are possible*) and let $Y_t = y^k$ if $Y_t \in A^k$ *(in what follows denote this value simply by k)*. Since $Y_0 = 0$, we have also to allow for the value $y = 0$ that we may consider as corresponding to $k = 0$.

# Asian options

- At the generic jump time $\tau_n \leq T$ of the chain $X_t$ we then have

*i*) If $\tau_{n+1} \leq T$ then

$$\begin{aligned} Y_{n+1} = y^k \quad \leftrightarrow \quad & Y_n + X_n(\tau_{n+1} - \tau_n) \in A^k \\ \leftrightarrow (k-1)\Delta \leq & Y_n + X_n(\tau_{n+1} - \tau_n) < k\Delta \end{aligned}$$

*ii*) If $\tau_{n+1} > T$ then

$$\begin{aligned} Y_T = y^k \quad \leftrightarrow \quad & Y_n + X_n(T - \tau_n) \in A^k \\ \leftrightarrow (k-1)\Delta \leq & Y_n + X_n(T - \tau_n) < k\Delta \end{aligned}$$

*iii*) For $\tau_0 = 0$ we put $Y_0 = 0$.

# Asian options

- From the previous relations one can see that, in order to have Markovianity, the pair $(X_n, Y_n)$ alone does not suffice, one has to include also $\tau_n$.
- Again, as for $Y_t$, also $\tau_n$ is continuous-valued *(recall that the distribution of $\tau_{n+1} - \tau_n$, given $X_n = x^i$, is exponential with parameter $q_i$)* and so to obtain a finite-state Markov chain one has to discretize also $\tau_n$.

# Asian options

- Partition the interval $[0, T]$ into intervals of equal length $\delta > 0$ assuming that $T = L\delta$ for a suitable integer $L$. The generic interval of the partition is then

$$B^{\ell} = [b^{\ell-1}, b^{\ell}) = [(\ell - 1)\delta, \ell\delta)$$

- *Denote by $t_\ell$ the midpoint of $B^\ell$ and let $\tau_n = t_\ell$ if $\tau_n \in B^\ell$ (again, in what follows, we may denote this value simply by $\ell$ with $\ell = 1, \cdots, L$).* We have also to allow for $\ell = 0$ that corresponds to $\tau_0 = 0$.
- If $\tau_{n+1} > T$ then we shall assign it the value $(L + 1)\delta$ and denote it simply by $L + 1$.

# Asian options

- *We may now consider the 3−dimensional chain $(X_n, Y_n, \tau_n)$, for which we have to derive the corresponding Q−matrix $\{q_{(i,h,m),(j,k,\ell)}\}$.*
- Again, the entire chain jumps only when $X_n$ jumps *(with a last jump when $\tau_{n+1} > T$)* and so we have for the intensities that

$$\sum_{j\neq i,k\neq h,\ell\neq m} q_{(i,h,m),(j,k,\ell)} = q_{(i,h,m)} = q_i$$

→ *It thus suffices to determine $p_{(i,h,m),(j,k,\ell)}$ from which then*

$$q_{(i,h,m),(j,k,\ell)} = q_i \, p_{(i,h,m),(j,k,\ell)}$$

# Asian options

- Given the definition of the process $Y_t$ as

$$Y_t = \sum_{\tau_n \leq t} X_{\tau_{n-1}}(\tau_n - \tau_{n-1}) + X_{\tau_n}(t - \tau_n)$$

at the generic jump time $\tau_n$ we have to restrict the possible values of the triple $(X_n, Y_n, \tau_n)$ to those triples $(i, h, m)$ with $i = 1, \cdots, N$; $m = 0, 1, \cdots, L$ for which $h \in \{0, 1, \cdots, K\}$ is such that

$$y^h \leq x^N t_m \quad \text{(in fact, } x^N > x^i \text{ for } i < N\text{)}$$

$\longrightarrow$ *We have now the following relations for the transition probabilities* $p_{(i,h,m),(j,k,\ell)}$:

# Asian options

$$p_{(i,h,m),(j,k,\ell)} := P\{X_{n+1} = j, Y_{n+1} = k, \tau_{n+1} = \ell \mid X_n = i, Y_n = h, \tau_n = m\}$$

$$\begin{aligned} &= P\{X_{n+1}=j, Y_{n+1}=k, \tau_{n+1}=\ell, \tau_{n+1} \leq T | X_n=i, Y_n=h, \tau_n=m\} \\ &\quad + P\{X_{n+1}=j, Y_{n+1}=k, \tau_{n+1}=L+1, \tau_{n+1} > T | X_n=i, Y_n=h, \tau_n=m\} \\ &= P\{X_{n+1}=j, Y_{n+1}=k, \tau_{n+1}=\ell | X_n=i, Y_n=h, \tau_n=m\} \mathbf{1}_{\{\tau_{n+1} \leq T\}} \mathbf{1}_{\{\ell \leq L\}} \\ &\quad + P\{X_{n+1}=j, Y_{n+1}=k, \tau_{n+1}=L+1 | X_n=i, Y_n=h, \tau_n=m\} \mathbf{1}_{\{\tau_{n+1} > T\}} \end{aligned}$$

where we have used the fact that, for $\ell \leq L$,

$$\{\tau_{n+1} = \ell\} \cap \{\tau_{n+1} \leq T\} = \{\tau_{n+1} \in B^{\ell}\} \cap \{\tau_{n+1} \leq T\} = \{\tau_{n+1} \in B^{\ell}\}$$

and, analogously, for $\ell = L + 1$,

$$\{\tau_{n+1} = L + 1\} \cap \{\tau_{n+1} > T\} = \{\tau_{n+1} > T\}$$

# Asian options

The first term, i.e. relative to the event $\tau_{n+1} \leq T$, can be continued as

$$P\{X_{n+1} = j \mid Y_{n+1} = k, \tau_{n+1} = \ell, X_n = i, Y_n = h, \tau_n = m\}$$

$$\cdot P\{Y_{n+1} = k \mid \tau_{n+1} = \ell, X_n = i, Y_n = h, \tau_n = m\}$$

$$\cdot P\{\tau_{n+1} = \ell \mid X_n = i, Y_n = h, \tau_n = m\}$$

$$= p_{i,j} \mathbf{1}_{\{(k-1)\Delta \leq y^h + x^i(\tau_{n+1} - t_m) < k\Delta\}} \mathbf{1}_{\{(\ell-1)\delta \leq \tau_{n+1} < \ell\delta\}}$$

$$\cdot \left[ \mathbf{1}_{\{t_m \notin [(\ell-1)\delta, \ell\delta)\}} \int_{(\ell-1)\delta - t_m}^{\ell\delta - t_m} q_i e^{-q_i t} dt + \mathbf{1}_{\{t_m \in [(\ell-1)\delta, \ell\delta)\}} \int_{t_m}^{\ell\delta} q_i e^{-q_i t} dt \right]$$

where we have used the fact that $\cdots$

# Asian options

- we have $Y_{n+1} = k$ $(k = 1, \cdots, K)$, i.e. $Y_{n+1} \in A^k$ under the condition $\tau_{n+1} = \ell$ $(\ell = 1, \cdots, L), X_n = i, Y_n = h, \tau_n = m$ if and only if

  $$(k-1)\Delta \leq y^h + x^i(\tau_{n+1} - t_m) < k\Delta \text{ with } \tau_{n+1} \in [(\ell-1)\delta, \ell\delta)$$

- Furthermore, given $X_n = i$, $\tau_n = m$, the random variable $\tau_{n+1} - t_m$ has the exponential density $q_i e^{-q_i t}$.

# Asian options

Analogously, on the event $\tau_{n+1} > T$, the second term can be continued as

$$
\begin{aligned}
& P\{X_{n+1} = j \mid Y_{n+1} = k, \tau_{n+1} = L+1, X_n = i, Y_n = h, \tau_n = m\} \\
& \quad \cdot P\{Y_{n+1} = k \mid \tau_{n+1} = L+1, X_n = i, Y_n = h, \tau_n = m\} \\
& \quad \cdot P\{\tau_{n+1} = L+1 \mid X_n = i, Y_n = h, \tau_n = m\}
\end{aligned}
$$

$$
= \delta_{i,j} \mathbf{1}_{\{(k-1)\Delta \le y^h + x^i (T - t_m) < k\Delta\}} \int_{T-t_m}^{\infty} q_i e^{-q_i t} dt
$$

where $\delta_{i,j}$ is the Kronecker symbol due to the fact that, on the event $\tau_{n+1} > T$, the chain $X_t$ stops. On the other hand $Y_t$ moves as far as it can on the time window $[0, T]$. Furthermore, as before, the r.v. $\tau_{n+1} - t_m$ has the exponential density $q_i e^{-q_i t}$, given that $X_n = i$ and $\tau_n = m$.

# Conclusions

- We have considered a specific market model where the underlying evolves as a continuous time finite state Markov chain (CTMC)
- For those cases where an explicit analytic pricing formula is not available *(i.e. most of the cases)* we have presented a hybrid MC simulation method which, with respect to a plain MC allows to:
    i) reduce the variance
    ii) obtain more precise results
- We have presented numerical results and comparisons for the case of lookback call and Asian options.

*Thank you for your attention*

# Barrier options

- Let an option be knocked out when the underlying $X_t$ reaches or falls below a level $L$
- Assume also that for the background (not knocked out) option we have

$$\bar{H}_0(\cdot) = \sum_{i=1}^{N} \bar{w}_i^0 \mathbf{1}_{\{\cdot = x^i\}}$$

# Barrier options

- Assuming the values $x^i$ are in increasing order of magnitude, put

$$\ell := \min\{i \in \{1, \cdots, N\} \mid x^i > L\}$$

- For the knock-out option we may then start from

$$H_0(X_T) = \sum_{i=1}^{N} \bar{w}_i^0 \mathbf{1}_{\{X_T = x^i, i \geq \ell\}} := \sum_{i=1}^{N} w_i^0 \mathbf{1}_{\{X_T = x^i\}}$$

having put $w_i^0 := \bar{w}_i^0 \mathbf{1}_{\{i \geq \ell\}}$.

$\longrightarrow$ Want to obtain also here a relation of the form

$$H_n(\underline{x}) = \widetilde{Q}(n) H_{n-1}(\underline{x})$$

for a suitable $\widetilde{Q}(n)$.

# Barrier options

Proposition: Starting from

$$H_0(\cdot) = \sum_{i=1}^{N} \bar{w}_i^0 \mathbf{1}_{\{\cdot = x^i, i \geq \ell\}} := \sum_{i=1}^{N} w_i^0 \mathbf{1}_{\{\cdot = x^i\}}$$

with $w_i^0 := \bar{w}_i^0 \mathbf{1}_{\{i \geq \ell\}}$ we have, for $n \leq \nu_T$ *(recall that we compute the price without the last term, i.e. as if $T = \tau_{\nu_T}$)*, $H_n(\cdot) = \sum_{i=1}^{N} w_i^n \mathbf{1}_{\{\cdot = x^i\}}$, where $w^n = [w_1^n, \cdots, w_N^n]'$ is given recursively by

$$w^n = I_\ell \widetilde{Q}(n) w^{n-1}$$

with $I_\ell$ a unit matrix having the first $\ell$ rows equal to zero and, as before, $\widetilde{Q}(n) = \left\{ \frac{q_{i,j}^n}{r_i + q_i^n} \right\}_{i,j=1,\cdots,N}$

# Barrier options

- As a consequence of the Proposition, we may restrict consideration to an $(N-\ell)-$vector $\tilde{w}^n$ for which

$$w_i^0 := \bar{w}_i^0 \mathbf{1}_{\{i \geq \ell\}} \quad \text{and} \quad \tilde{w}^n = \widetilde{Q}_\ell(n) \tilde{w}^{n-1}$$

where $\widetilde{Q}_\ell(n)$ is the $(N-\ell) \times (N-\ell)$ sub matrix of $\widetilde{Q}^n$ formed by the last $N-\ell$ towns and columns.

$\longrightarrow$ *We have the equivalent representations*

$$H_n(X_{\nu_T - n}) = \sum_{i=1}^{N} w_i^n \mathbf{1}_{\{X_{\nu_T - n} = x^i\}} = \sum_{i=1}^{N-\ell} \tilde{w}_i^n \mathbf{1}_{\{X_{\nu_T - n} = x^i\}}$$

# Explicit numerical computation

- $\widetilde{Q}$ may also be viewed as a mapping acting as follows

$$\widetilde{Q}H(v) = E_v^{\tilde{P}}\left\{e^{-v\mathcal{I}}H(u)\right\} \quad \text{with} \quad \mathcal{I} \sim Exp(q(v))$$

It is a contraction mapping with fixed point zero and contraction constant

$$\gamma := \max_{i \le N} \frac{q_i}{r^i + q_i} < 1$$

# Price of Prototype product: explicit formula

Consequently we have that the price of the Prototype product assur
$X_t = x^i$ for a fixed $x^i \in E$, is

$$V_{H_0,t,T}(X_t)_{|X_t=x^i} = \sum_{n=0}^{n_\epsilon} [\widetilde{Q}^n \cdot H_0(\underline{x})]_i \tilde{P}(\nu_{t,T} = n \mid X_t = x^i)$$

with

- $\widetilde{Q} = (\widetilde{q}_{i,j})_{1 \le i,j \le N}$ where $\widetilde{q}_{i,j} = \begin{cases} \frac{q_{i,j}}{r^i + q_i} & i \neq j \\ 0 & i = j \end{cases}$
- $[v]_i$ is the $i^{th}$ component of a general vector $v$
- $H_0(\underline{x}) := [w_1^0, \cdots, w_N^0]'$ whose components are given by the Prototype payoff $H_0(\cdot) = \sum_{i=1}^N w_i \mathbf{1}_{\{\cdot = x^i\}}$

$\longrightarrow$ *A specific form when $\widetilde{Q}$ is diagonalizable.*