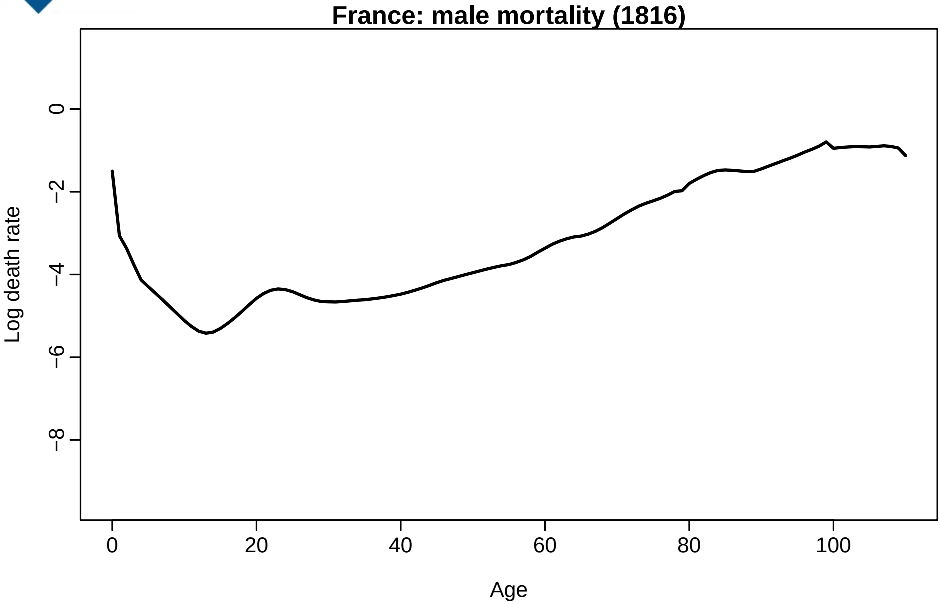
Demographic forecasting using functional data analysis

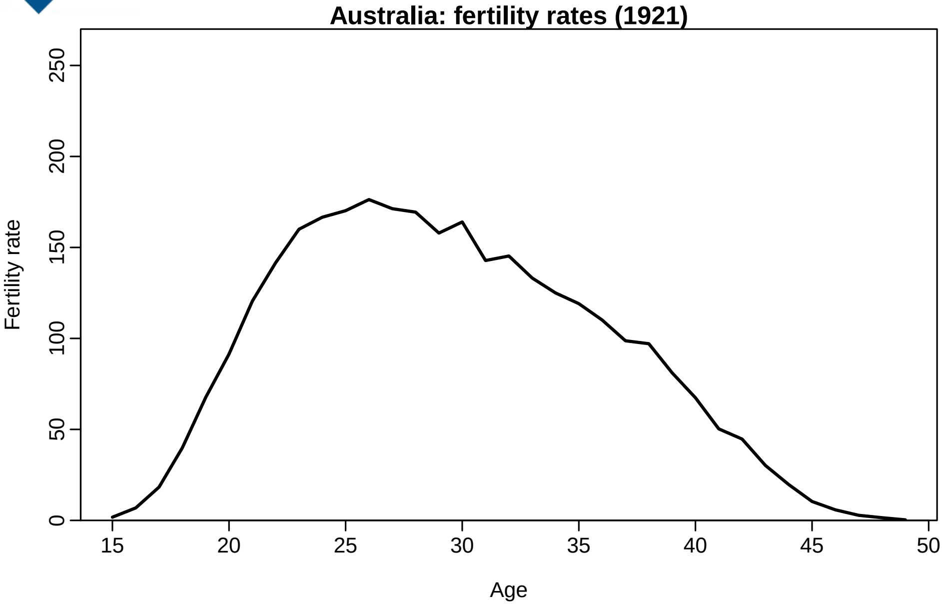
Rob J Hyndman

Joint work with: Heather Booth, Han Lin Shang, Shahid Ullah, Farah Yasmeen.

Mortality rates



Fertility rates



Outline

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Some notation

Let $y_{t,x}$ be the observed (smoothed) data in period t at age x, $t = 1, \dots, n$.

$$y_{t,x} = f_t(x) + \sigma_t(x)\varepsilon_{t,x}$$

$$f_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)$$

- Estimate $f_t(x)$ using penalized regression splines.
- Estimate $\mu(x)$ as me(di)an $f_t(x)$ across years.
- Estimate $\beta_{t,k}$ and $\phi_k(x)$ using (robust) functional principal components.
- $\varepsilon_{t,x} \overset{iid}{\sim} N(0,1)$ and $e_t(x) \overset{iid}{\sim} N(0, v(x))$.

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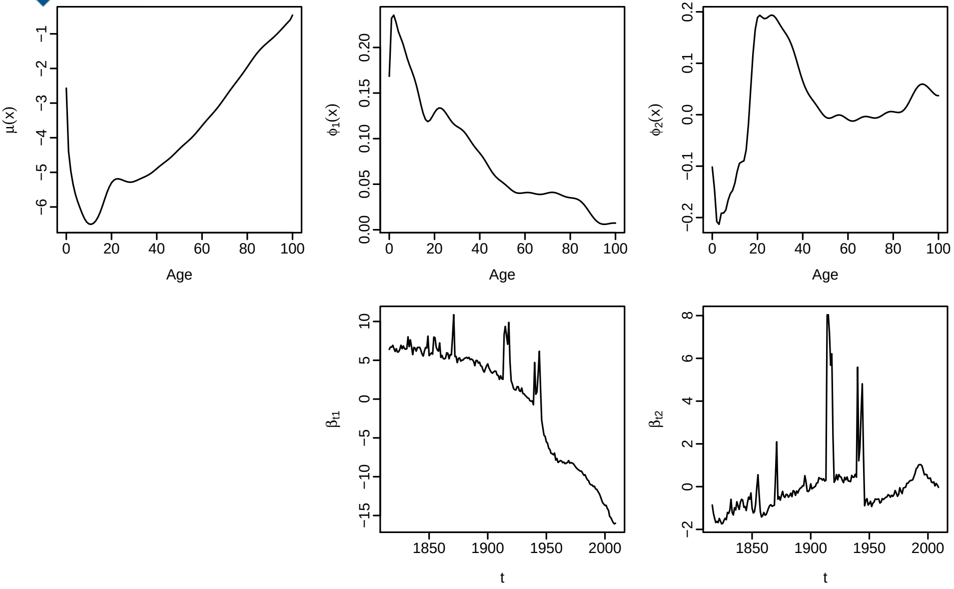
French mortality components



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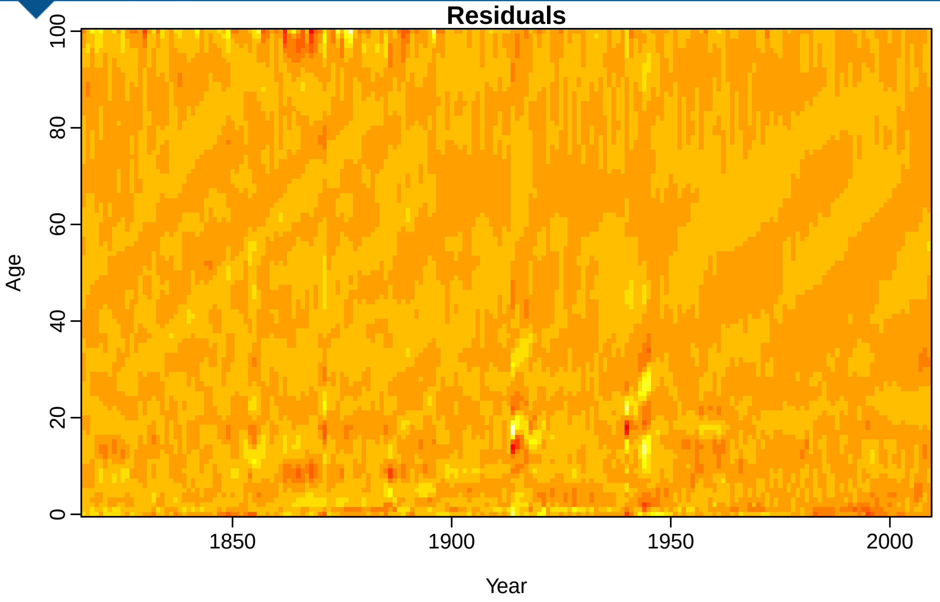

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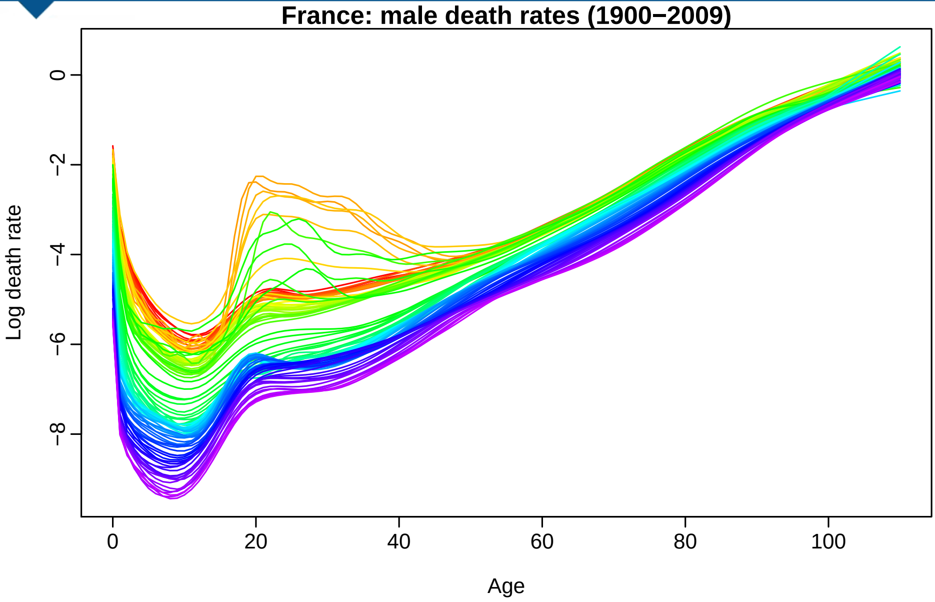
French male mortality rates



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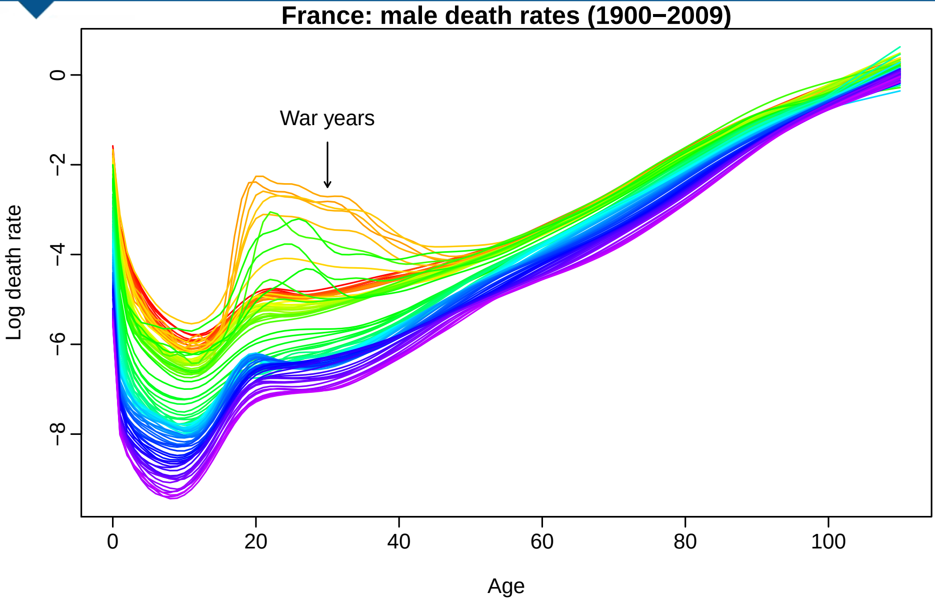

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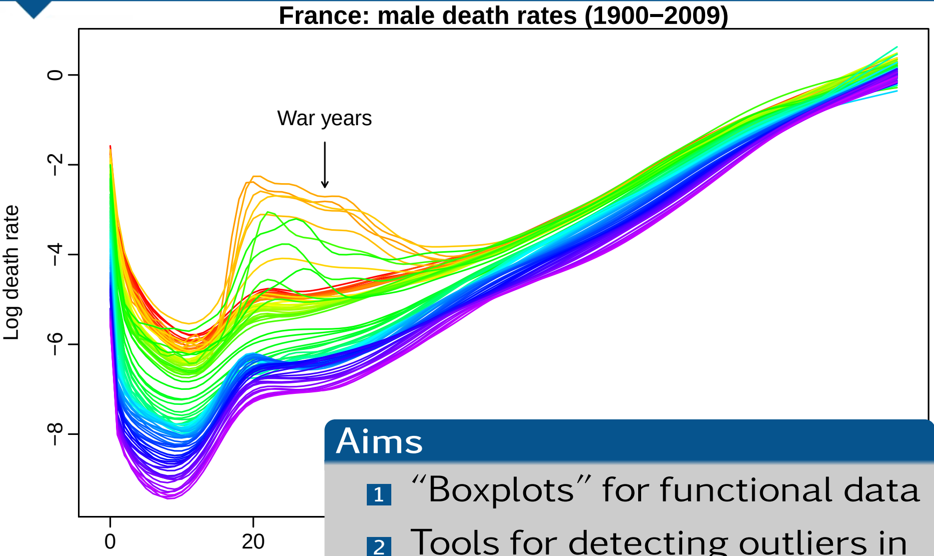

Aims

1. "Boxplots" for functional data
2. Tools for detecting outliers in functional data

Robust principal components

Let $\{f_t(x)\}$, $t = 1, \dots, n$, be a set of curves.

1. Apply a robust principal component algorithm

$$f_t(x) = \mu(x) + \sum_{k=1}^{n-1} \beta_{t,k} \phi_k(x)$$

   - $\mu(x)$ is median curve

2. Plot $\beta_{t,2}$ vs $\beta_{t,1}$

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   - Each point in scatterplot represents one curve.
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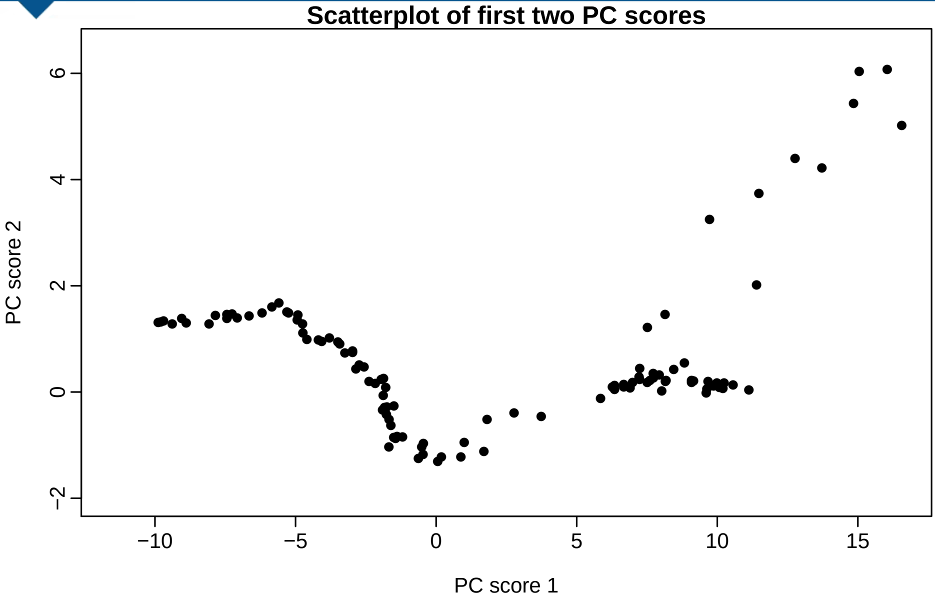

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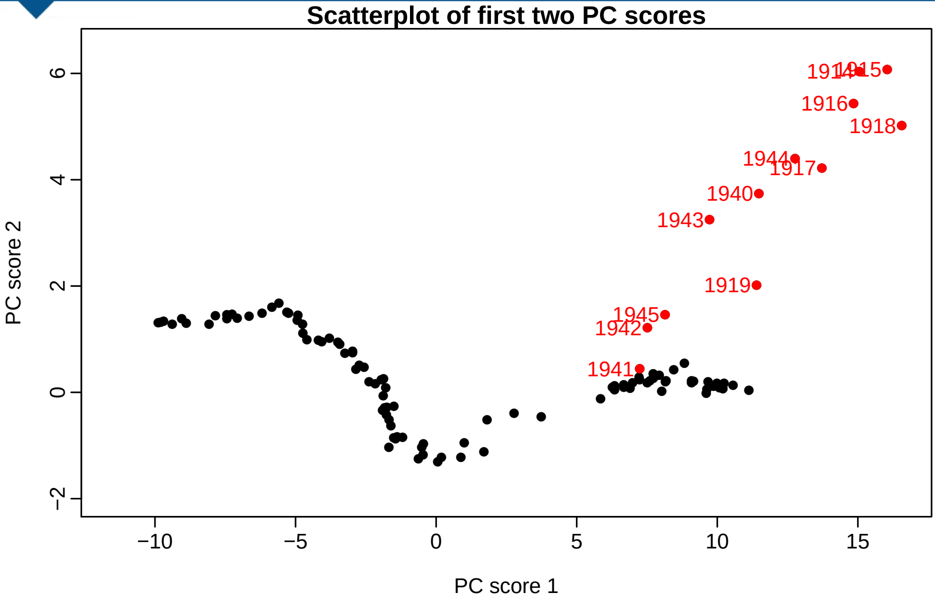
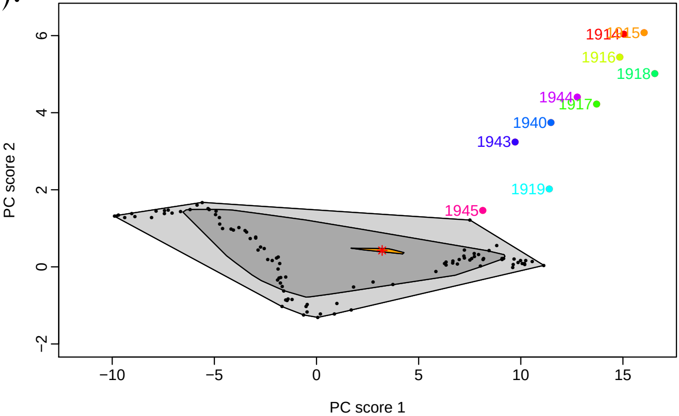


Functional bagplot

- Bivariate bagplot due to Rousseeuw et al. (1999).
- Rank points by halfspace location depth.
- Display median, 50% convex hull and outer convex hull (with 99% coverage if bivariate normal).



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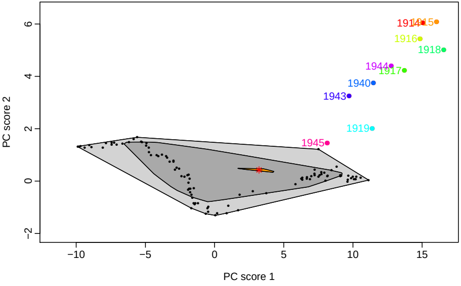
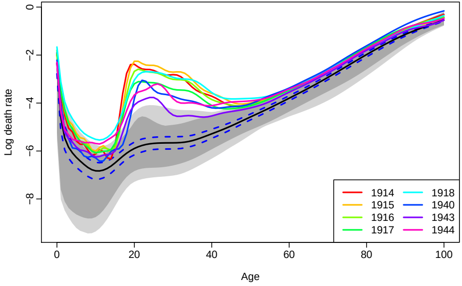
- Boundaries contain all curves inside bags.
- 95% CI for median curve also shown.





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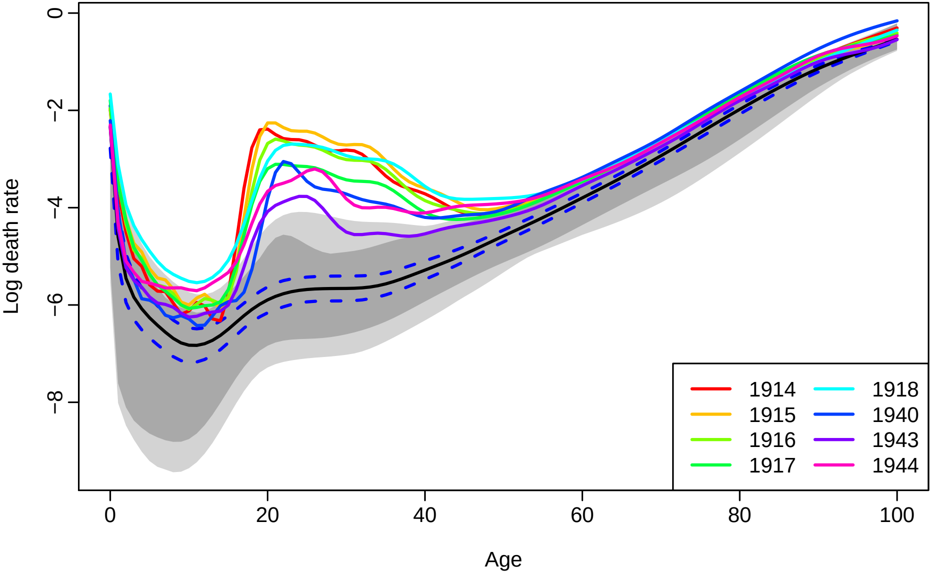
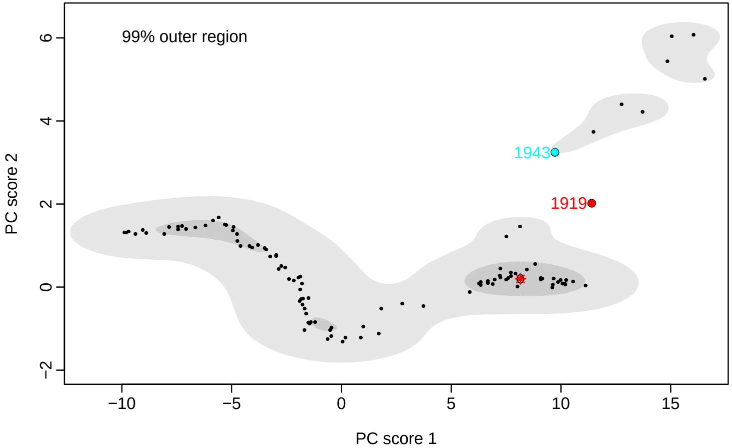


Functional HDR boxplot

- Bivariate HDR boxplot due to Hyndman (1996).
- Rank points by value of kernel density estimate.
- Display mode, 50% and (usually) 99% highest density regions (HDRs) and mode.



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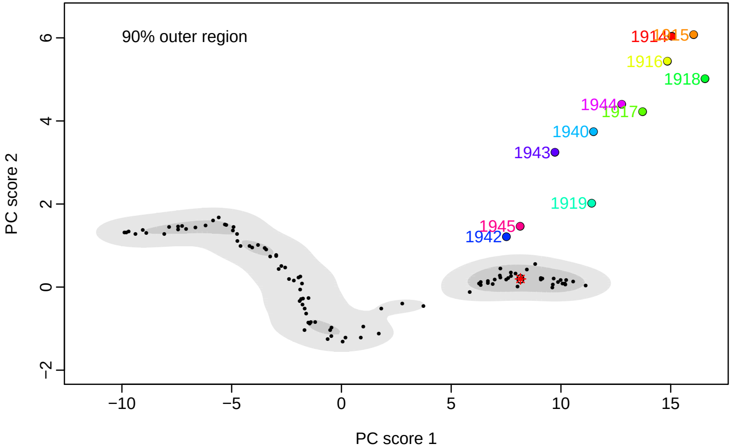

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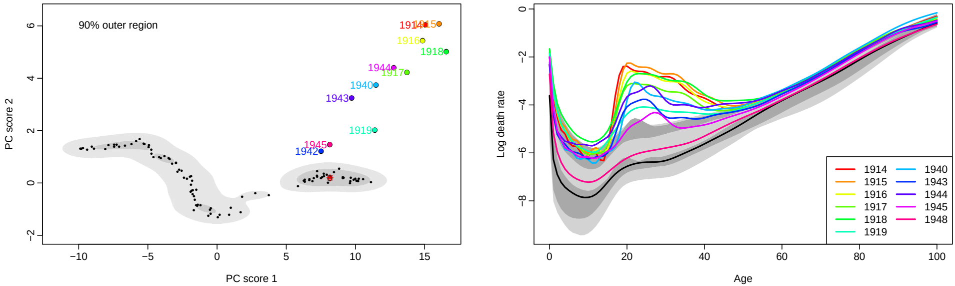

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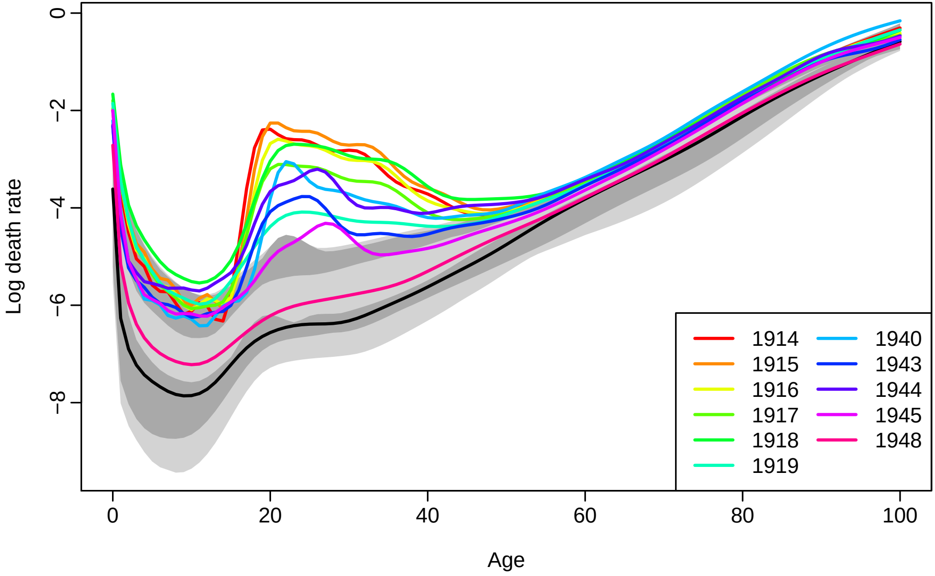

Outline

Functional time series model

$$y_{t,x} = f_t(x) + \sigma_t(x)\varepsilon_{t,x}$$

$$f_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)$$

- The eigenfunctions $\phi_k(x)$ show the main regions of variation.
- The scores $\{\beta_{t,k}\}$ are uncorrelated by construction. So we can forecast each $\beta_{t,k}$ using a univariate time series model.
- Outliers are treated as missing values.
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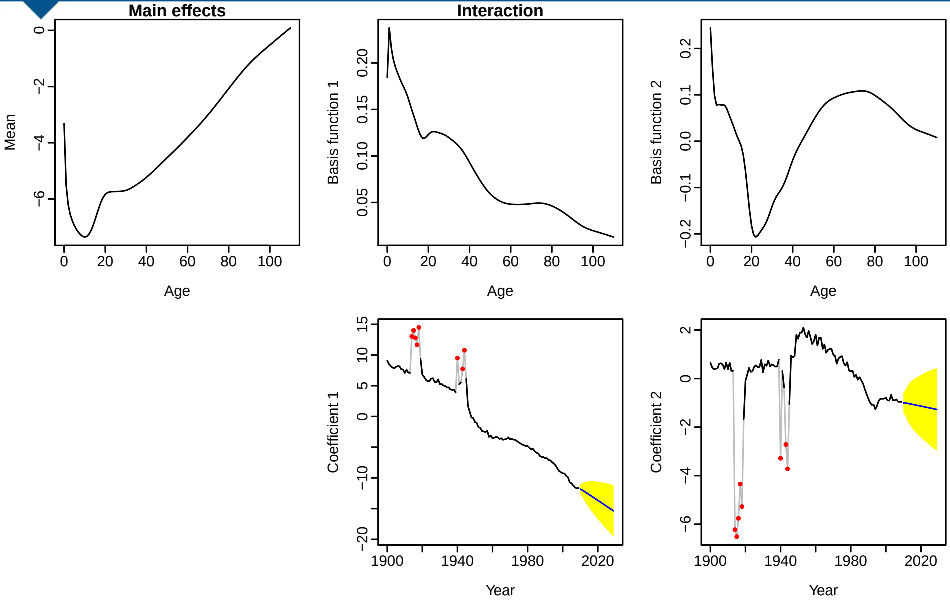
$$\text{E}[y_{n+h,x} \mid \boldsymbol{y}] = \hat{\mu}(x) + \sum_{k=1}^{K} \hat{\beta}_{n+h,k}\,\hat{\phi}_k(x)$$

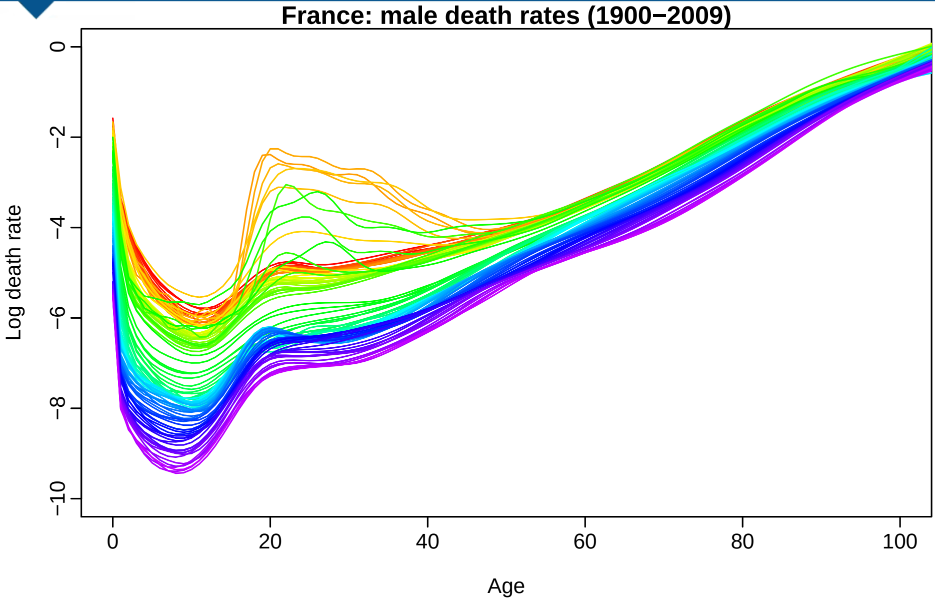
$$\text{Var}[y_{n+h,x} \mid \boldsymbol{y}] = \hat{\sigma}_\mu^2(x) + \sum_{k=1}^{K} v_{n+h,k}\,\hat{\phi}_k^2(x) + \sigma_t^2(x) + v(x)$$

where $v_{n+h,k} = \text{Var}(\beta_{n+h,k} \mid \beta_{1,k}, \dots, \beta_{n,k})$
and $\boldsymbol{y} = [y_{1,x}, \dots, y_{n,x}]$.

Forecasting the PC scores



Forecasts of $f_t(x)$



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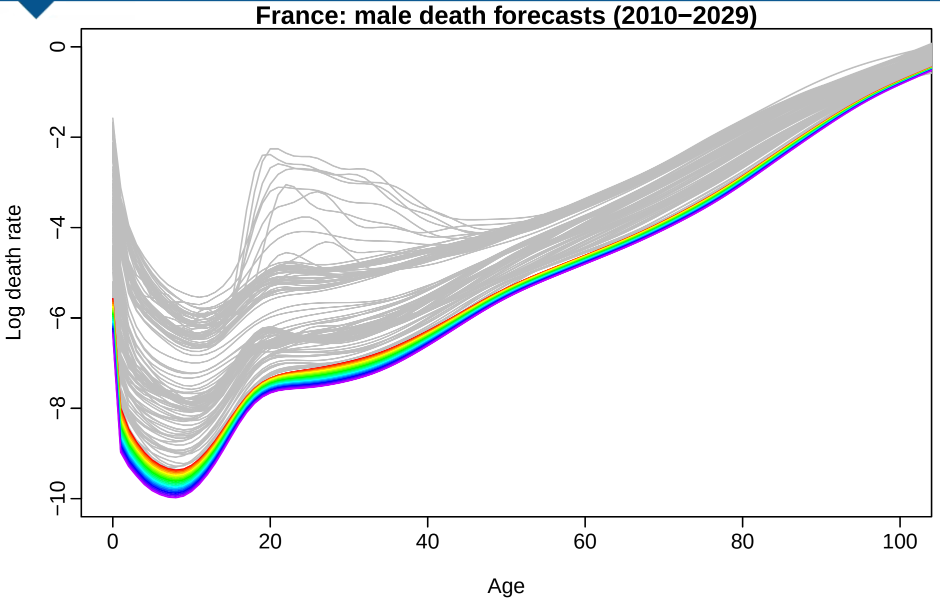

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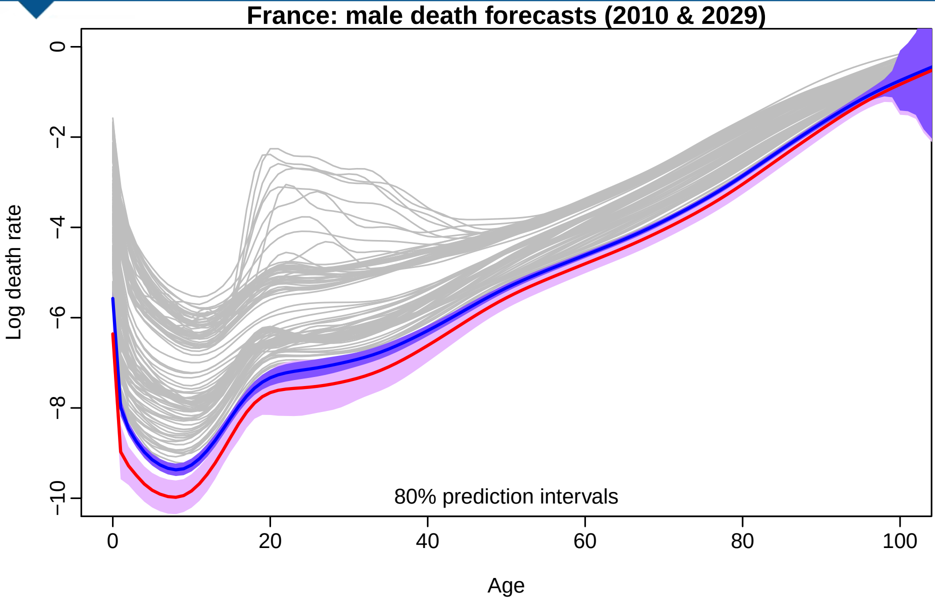
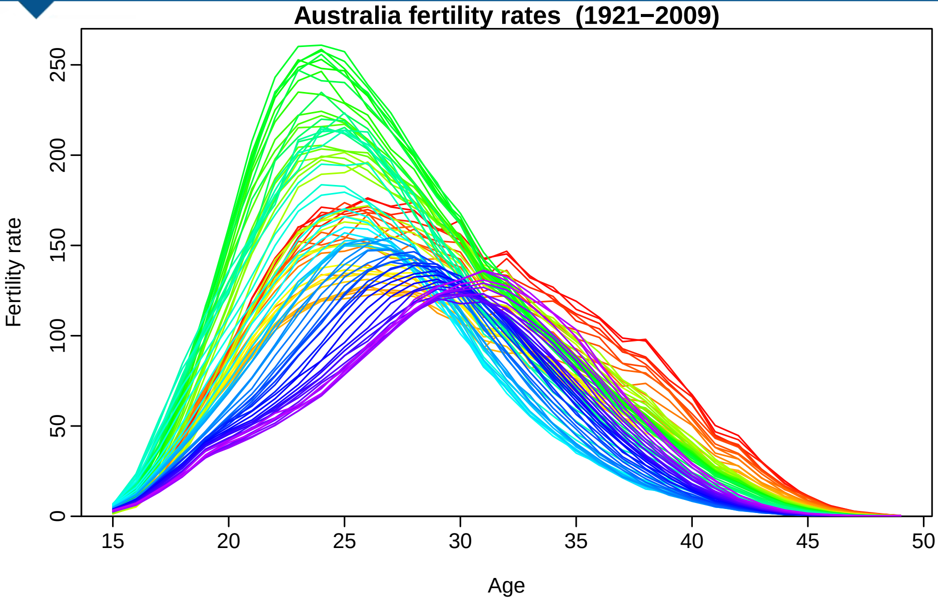
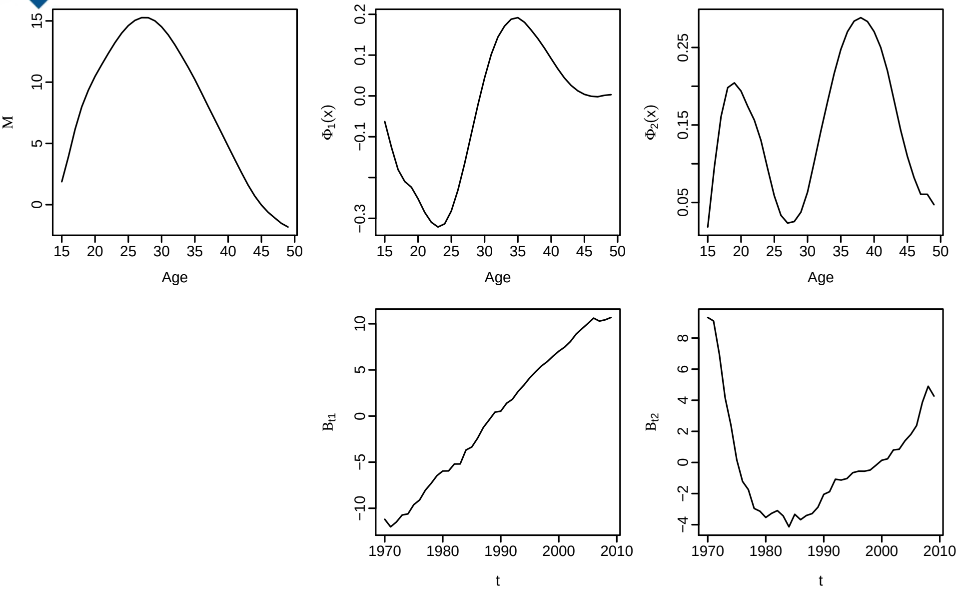


Fertility application



Fertility model



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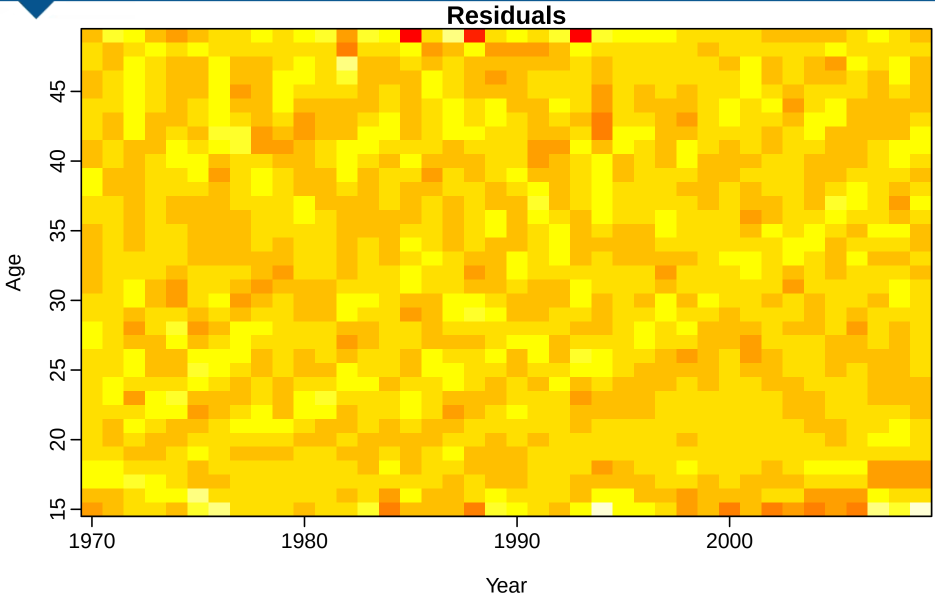

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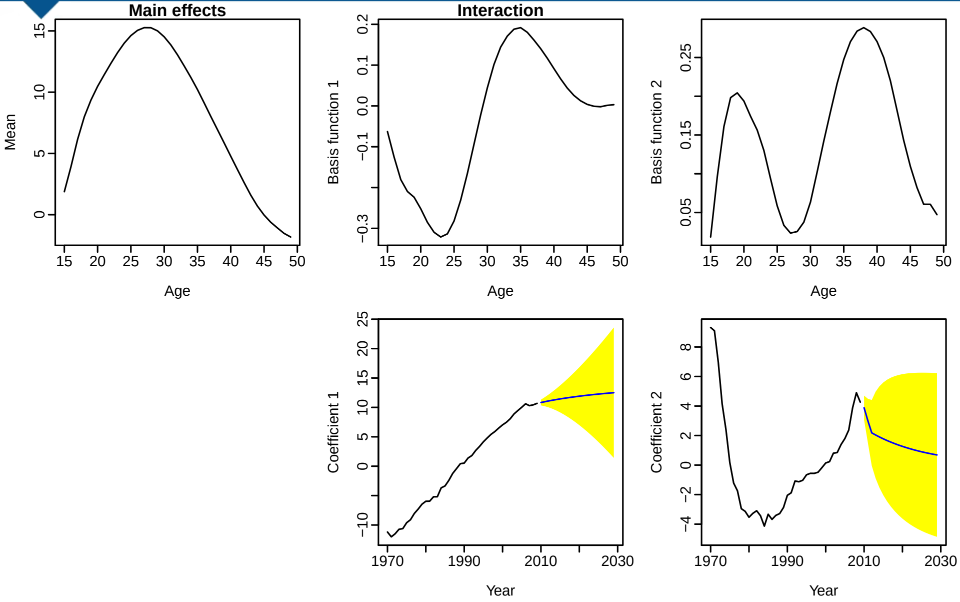

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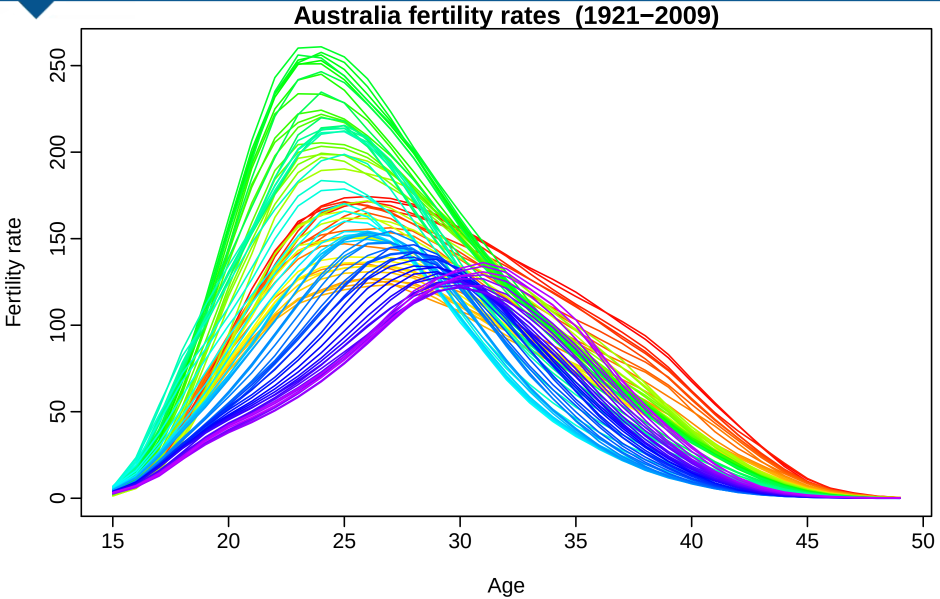

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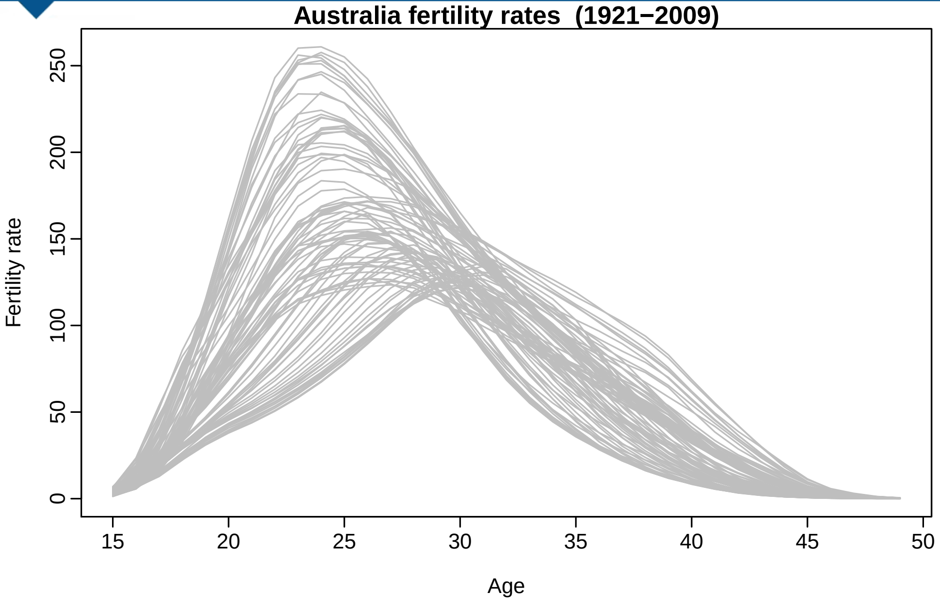

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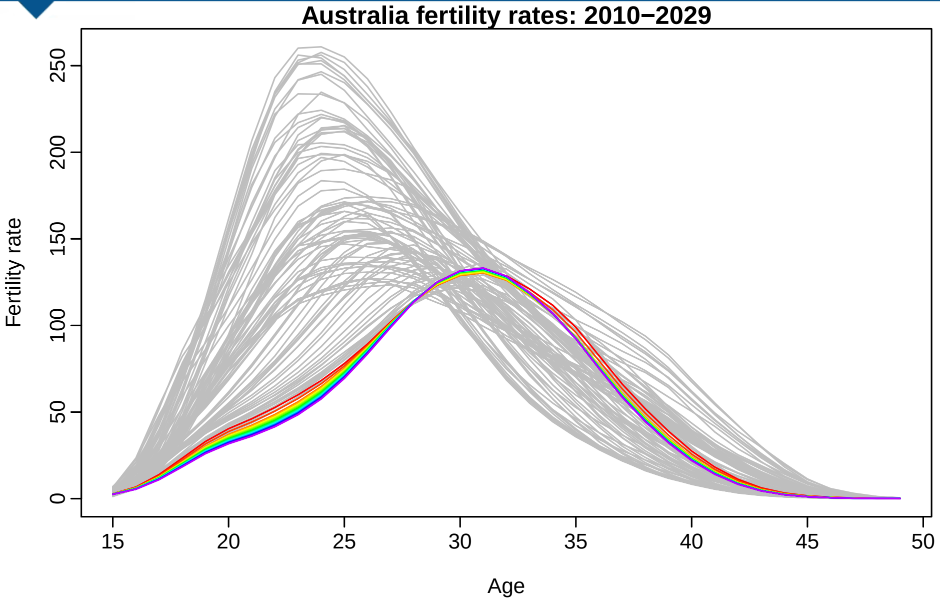

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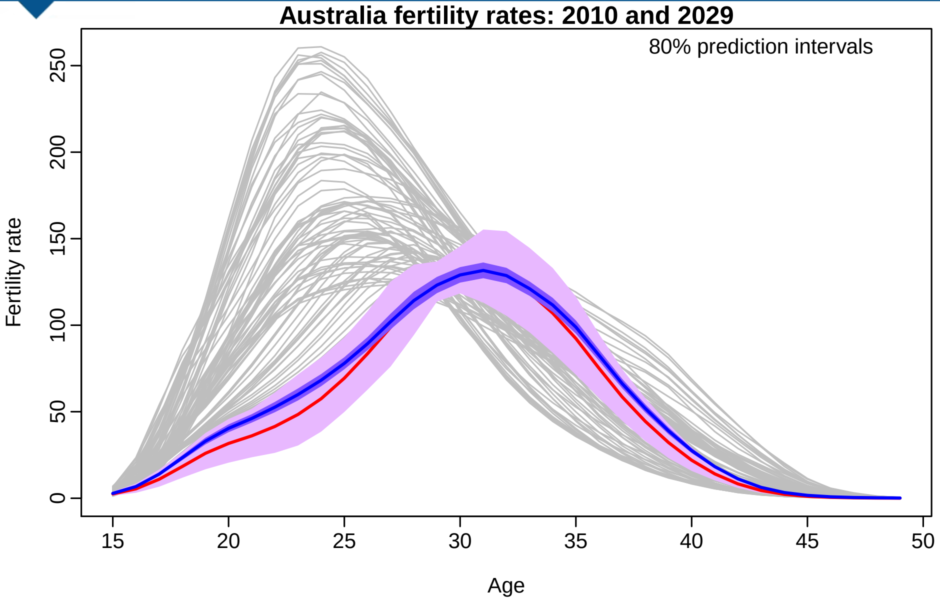

Outline

The problem

Let $f_{t,j}(x)$ be the smoothed mortality rate for age x in group j in year t.

- Groups may be males and females.
- Groups may be states within a country.
- Expected that groups will behave similarly.
- Coherent forecasts do not diverge over time.
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Forecasting the coefficients

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- We use ARIMA models for each coefficient $\{\beta_{1,j,k}, \dots, \beta_{n,j,k}\}$.
- The ARIMA models are non-stationary for the first few coefficients ($k = 1, 2$)
- Non-stationary ARIMA forecasts will diverge. Hence the mortality forecasts are not coherent.

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$$f_t(x) = \mu(x) + \sum_{k=1}^{K} \beta_{t,k}\,\phi_k(x) + e_t(x)$$

- We use ARIMA models for each coefficient $\{\beta_{1,j,k}, \dots, \beta_{n,j,k}\}$.
- The ARIMA models are non-stationary for the first few coefficients ($k = 1, 2$)
- Non-stationary ARIMA forecasts will diverge. Hence the mortality forecasts are not coherent.

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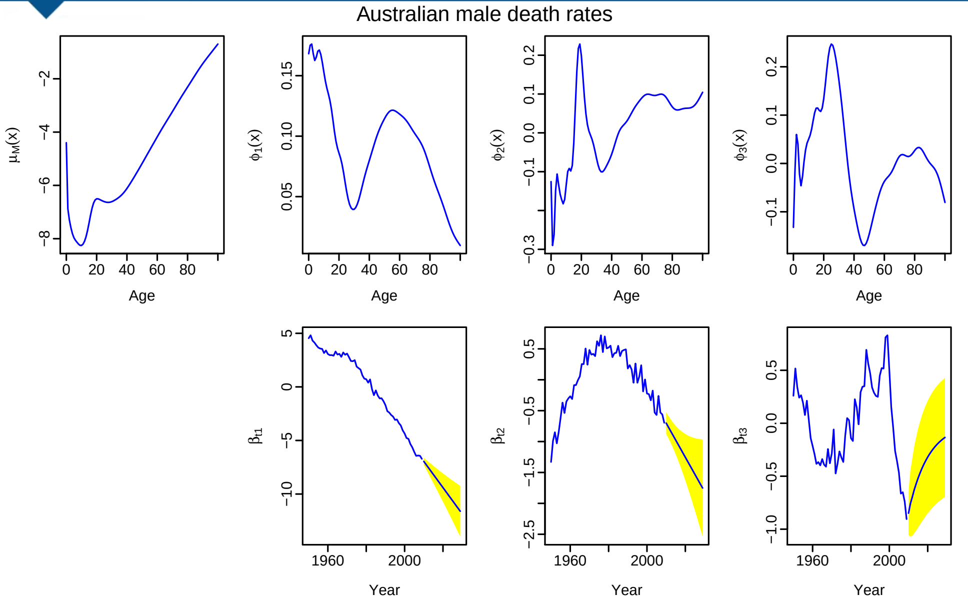
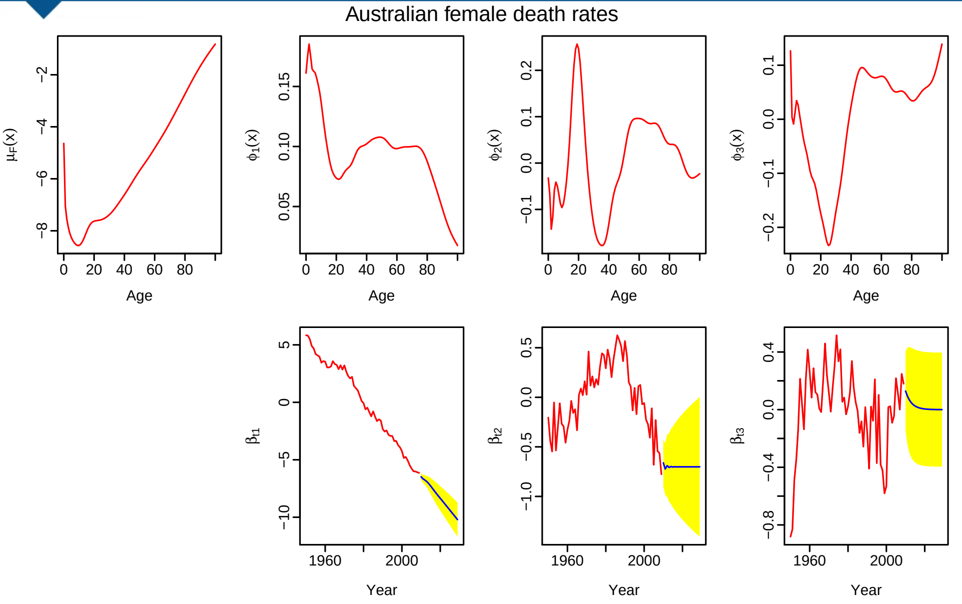
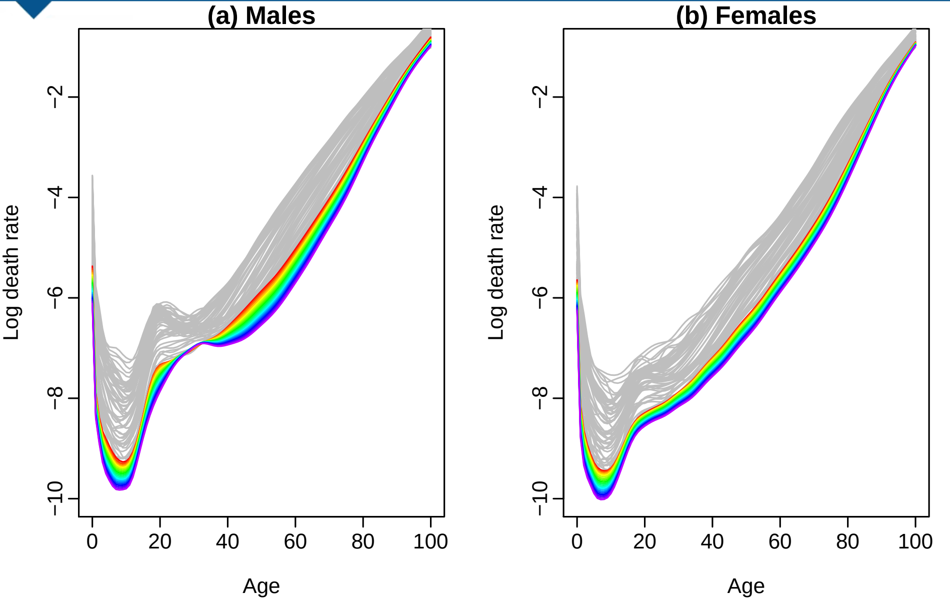
Male fts model



Female fts model



Australian mortality forecasts



Mortality product and ratios

Key idea

Model the geometric mean and the mortality ratio instead of the individual rates for each sex separately.

$$p_t(x) = \sqrt{f_{t,M}(x) f_{t,F}(x)} \quad \text{and} \quad r_t(x) = \sqrt{f_{t,M}(x) \big/ f_{t,F}(x)}.$$

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Mortality rates

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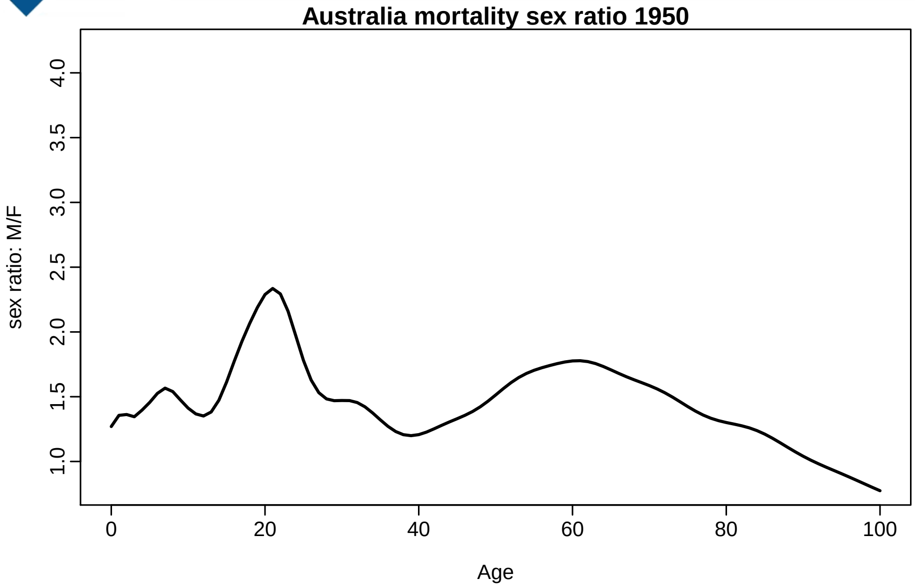

Model product and ratios

$$p_t(x) = \sqrt{f_{t,M}(x) f_{t,F}(x)} \quad \text{and} \quad r_t(x) = \sqrt{f_{t,M}(x) \big/ f_{t,F}(x)}.$$

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- $\{\gamma_{t,\ell}\}$ restricted to be stationary processes: either ARMA(p,q) or ARFIMA(p,d,q).
- No restrictions for $\beta_{t,1},\dots,\beta_{t,K}$.
- Forecasts: $f_{n+h|n,M}(x) = p_{n+h|n}(x) r_{n+h|n}(x)$

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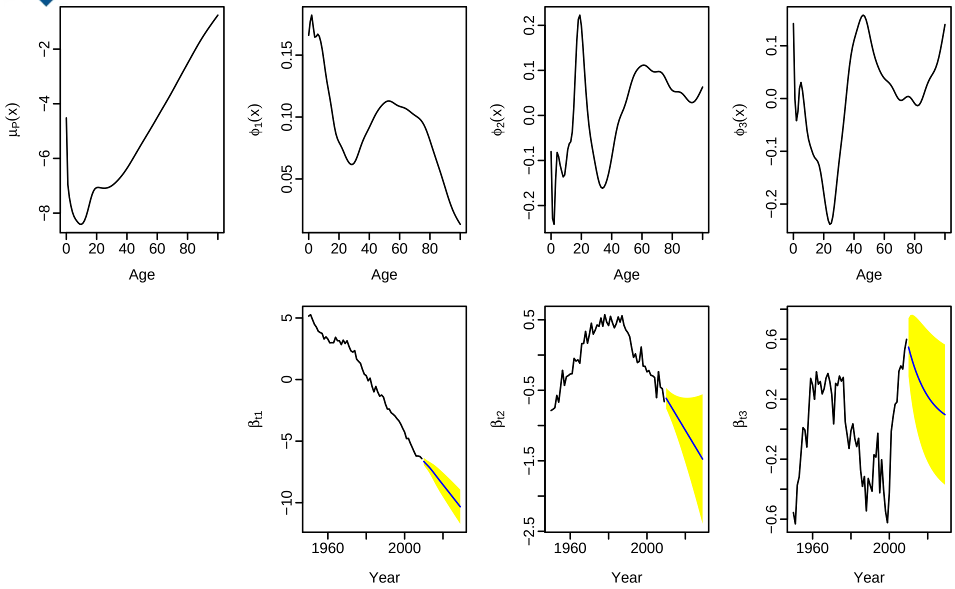
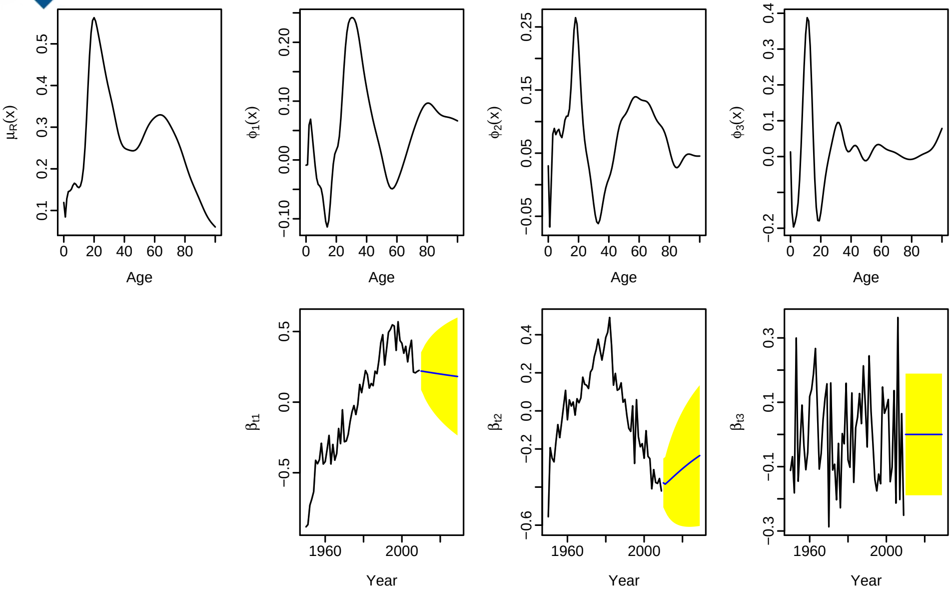
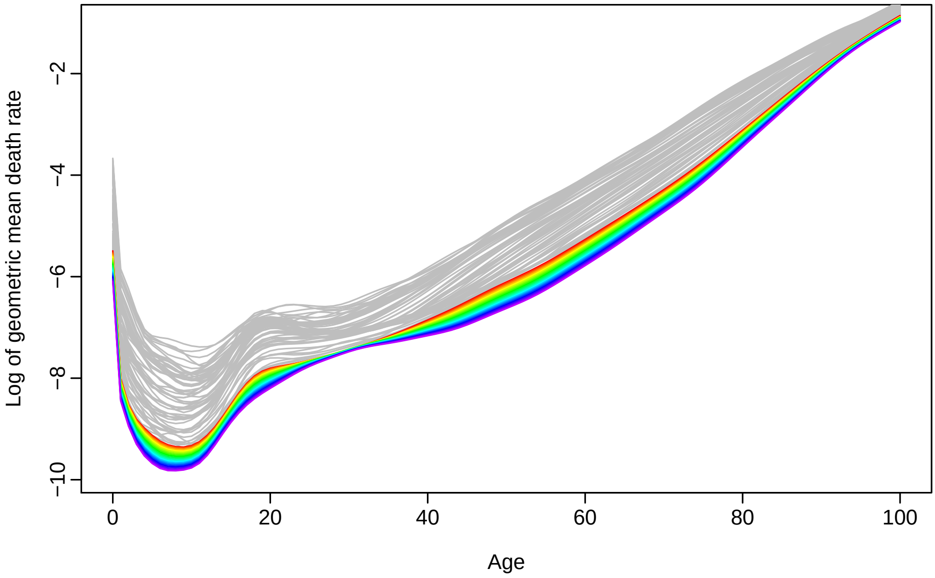


Ratio model



Product forecasts



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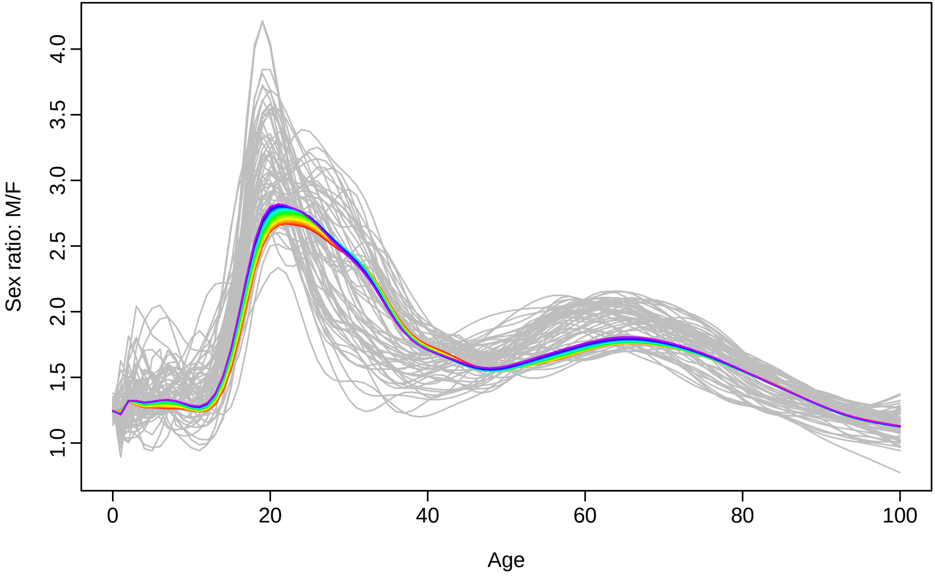
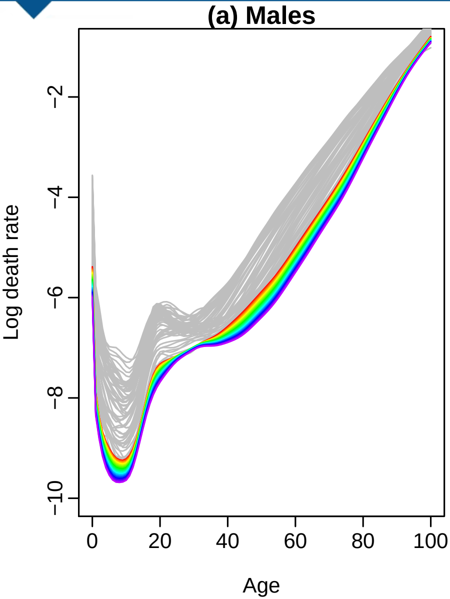


Coherent forecasts



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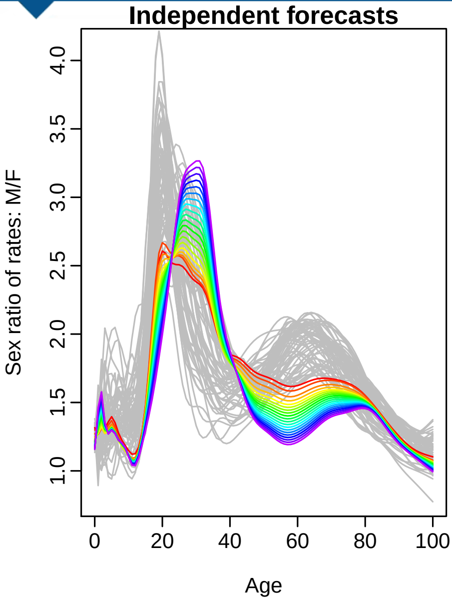
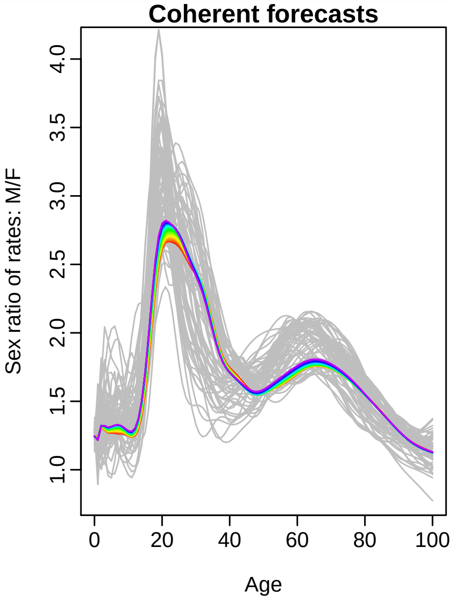
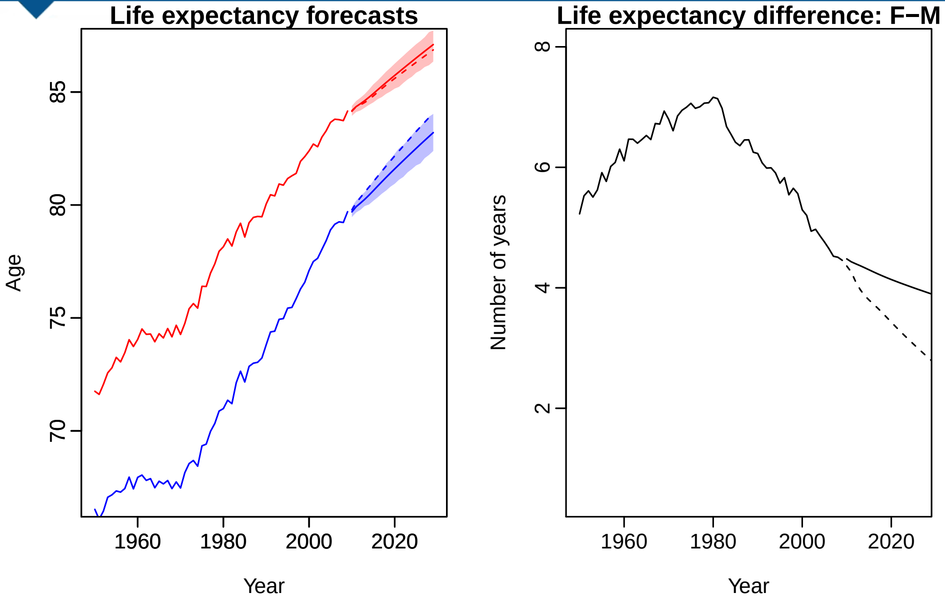




Life expectancy forecasts



Coherent forecasts for J groups

$$p_t(x) = [f_{t,1}(x) f_{t,2}(x) \cdots f_{t,J}(x)]^{1/J}$$

and $$r_{t,j}(x) = f_{t,j}(x) \big/ p_t(x),$$

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Outline

Demographic growth-balance equation

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$$P_{t+1}(x+1) = P_t(x) - D_t(x,x+1) + G_t(x,x+1)$$
$$P_{t+1}(0) = B_t \quad - D_t(B,0) \quad + G_t(B,0)$$

$x = 0,1,2,\ldots.$

$P_t(x) =$ population of age x at 1 January, year t

$B_t =$ births in calendar year t

$D_t(x,x+1) =$ deaths in calendar year t of persons aged x at the beginning of year t

$D_t(B,0) =$ infant deaths in calendar year t

$G_t(x,x+1) =$ net migrants in calendar year t of persons aged x at the beginning of year t

$G_t(B,0) =$ net migrants of infants born in calendar year t

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Key ideas

- Build a **stochastic functional model** for each of mortality, fertility and net migration.
- Treat all observed data as **functional** (i.e., smooth curves of age) rather than discrete values.
- Use the models to **simulate future sample paths** of all components giving the entire age distribution at every year into the future.
- Compute future births, deaths, net migrants. and populations from simulated rates.
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The available data

In most countries, the following data are available:

$P_t(x) =$ **population** of age x at 1 January, year t
$E_t(x) =$ **population** of age x at 30 June, year t
$B_t(x) =$ **births** in calendar year t to females of age x
$D_t(x) =$ **deaths** in calendar year t of persons of age x

From these, we can estimate:

- $m_t(x) = D_t(x)/E_t(x) =$ central death rate in calendar year t;
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Net migration

- We need to *estimate* **migration** data based on difference in population numbers after adjusting for births and deaths.

Demographic growth-balance equation

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Note: "net migration" numbers also include errors associated with all estimates. i.e., a "residual".

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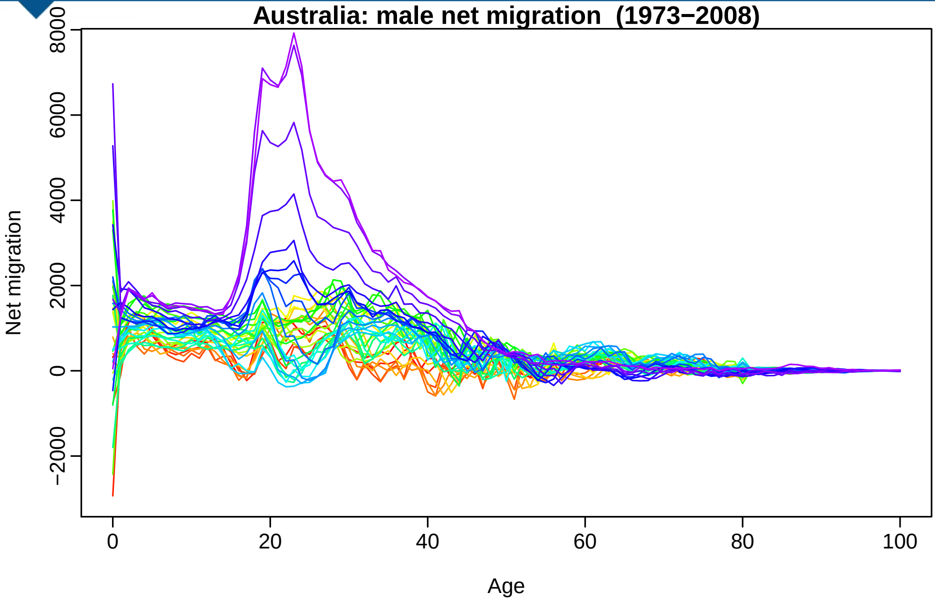
$$G_t(x, x+1) = P_{t+1}(x+1) - P_t(x) + D_t(x, x+1)$$
$$G_t(B, 0) = P_{t+1}(0) \qquad - B_t \qquad + D_t(B, 0)$$

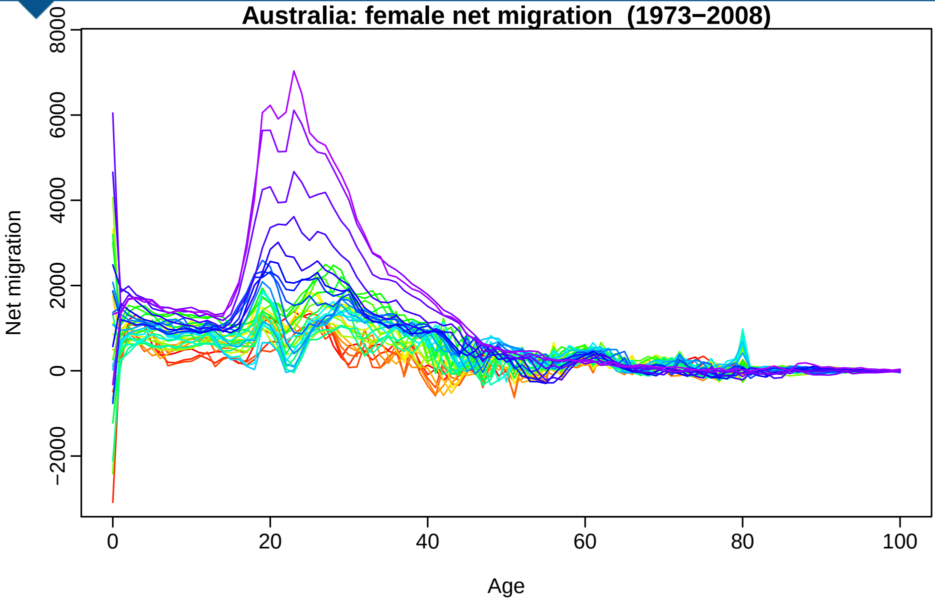
$x = 0, 1, 2, \ldots$

Note: "net migration" numbers also include errors associated with all estimates. i.e., a "residual".

Net migration



Net migration



Stochastic population forecasts

Component models

- Data: age/sex-specific mortality rates, fertility rates and net migration.
- Models: Functional time series models for mortality (M/F), fertility and net migration (M/F) assuming independence between components and coherence between sexes.
- Generate random sample paths of each component conditional on observed data.
- Use simulated rates to generate $B_t(x)$, $D_t^F(x, x+1)$, $D_t^M(x, x+1)$ for $t = n+1, \dots, n+h$, assuming deaths and births are Poisson.

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Simulation

Demographic growth-balance equation used to get population sample paths.

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\begin{aligned}
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$x = 0, 1, 2, \ldots.$

- 10000 sample paths of population $P_t(x)$, deaths $D_t(x)$ and births $B_t(x)$ generated for $t = 2004, \ldots, 2023$ and $x = 0, 1, 2, \ldots,$.
- This allows the computation of the empirical forecast distribution of any demographic quantity that is based on births, deaths and population numbers.

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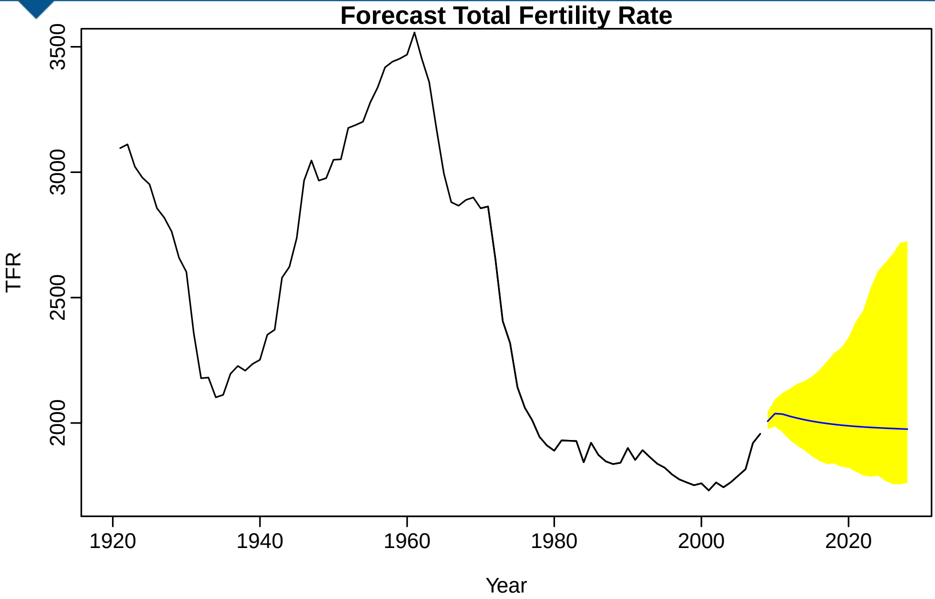
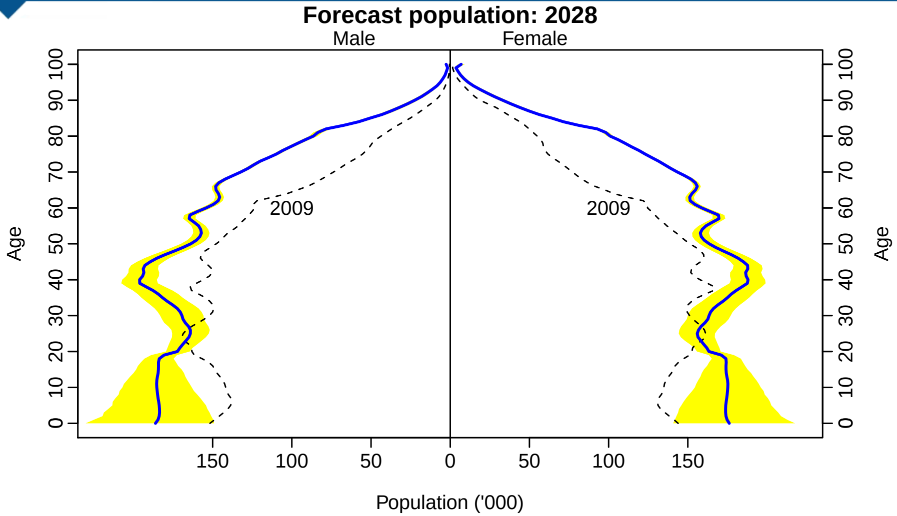
Forecasts of TFR

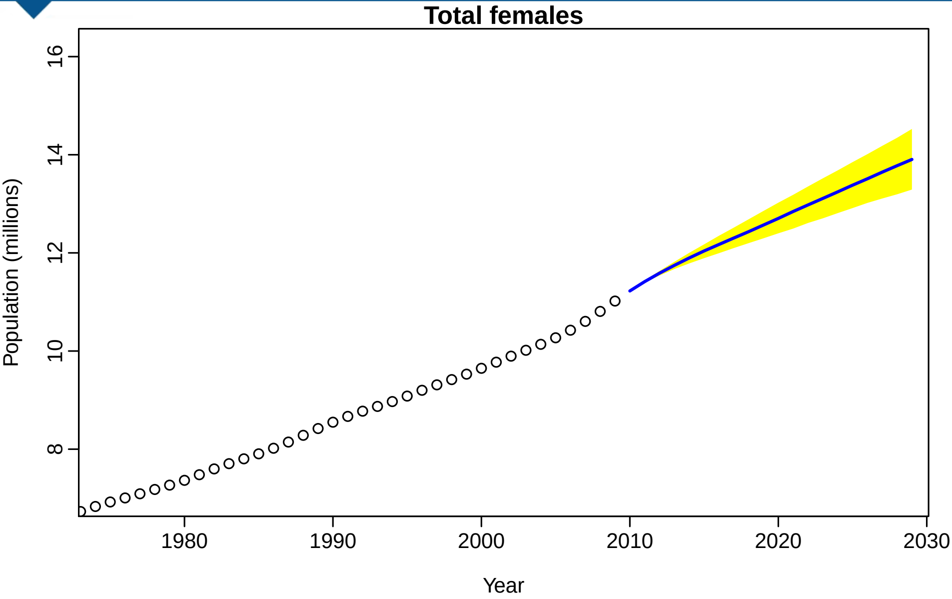


Population forecasts

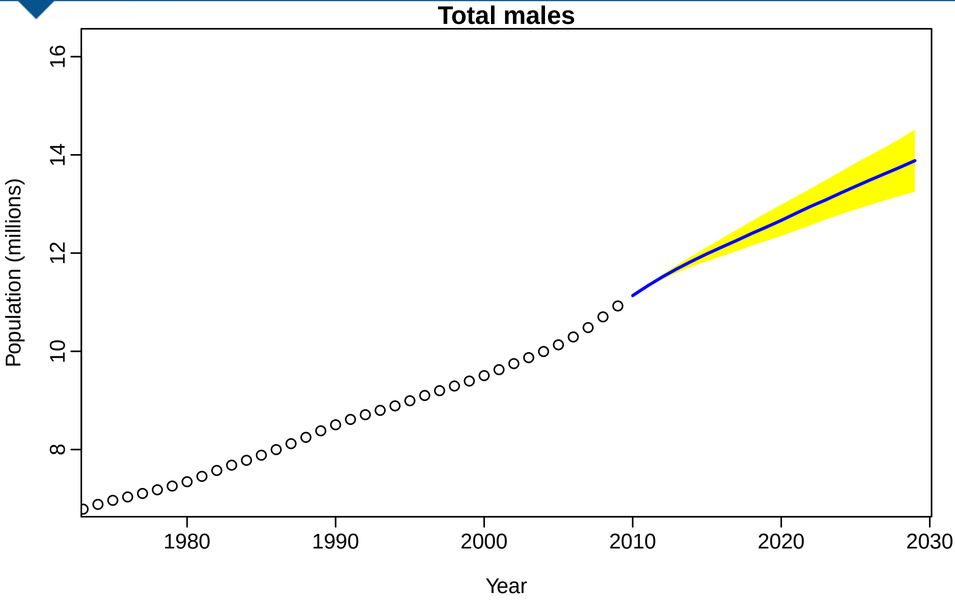


Forecast population pyramid for 2028, along with 80% prediction intervals. Dashed: actual population pyramid for 2009.

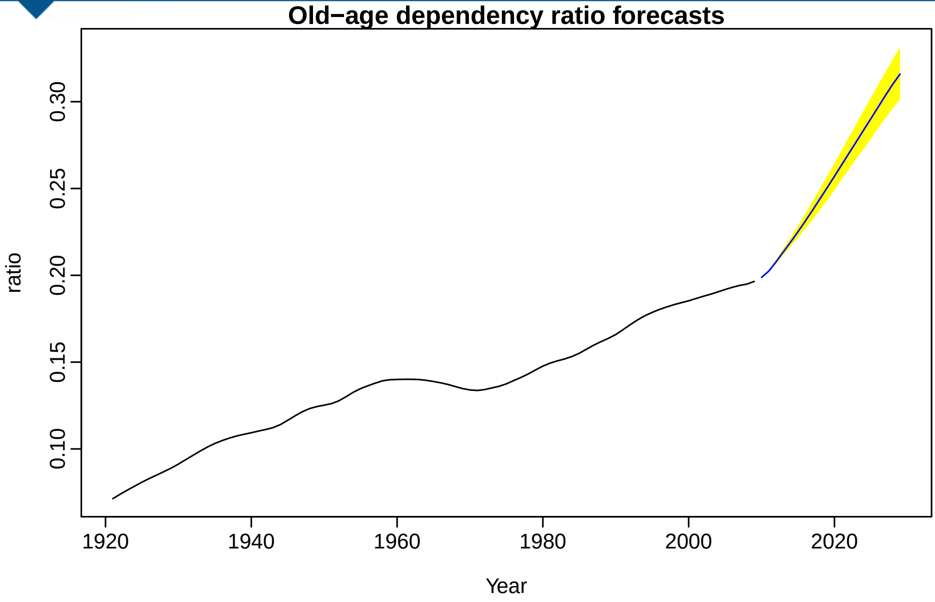
Population forecasts



Population forecasts



Old-age dependency ratio



Advantages of stochastic simulation approach

- Functional data analysis provides a way of forecasting age-specific mortality, fertility and net migration.
- Stochastic age-specific cohort-component simulation provides a way of forecasting many demographic quantities with prediction intervals.
- No need to select combinations of assumed rates.
- True prediction intervals with specified coverage for population and all derived variables (TFR, life expectancy, old-age dependencies, etc.)

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Outline

Selected references

- Hyndman, Shang (2010). "Rainbow plots, bagplots and boxplots for functional data". *Journal of Computational and Graphical Statistics* **19**(1), 29–45
- Hyndman, Ullah (2007). "Robust forecasting of mortality and fertility rates: A functional data approach". *Computational Statistics and Data Analysis* **51**(10), 4942–4956
- Hyndman, Shang (2009). "Forecasting functional time series (with discussion)". *Journal of the Korean Statistical Society* **38**(3), 199–221
- Shang, Booth, Hyndman (2011). "Point and interval forecasts of mortality rates and life expectancy : a comparison of ten principal component methods". *Demographic Research* **25**(5), 173–214
- Hyndman, Booth (2008). "Stochastic population forecasts using functional data models for mortality, fertility and migration". *International Journal of Forecasting* **24**(3), 323–342
- Hyndman, Booth, Yasmeen (2012). "Coherent mortality forecasting: the product-ratio method with functional time series models". *Demography*, to appear
- Hyndman (2012). *demography: Forecasting mortality, fertility, migration and population data.* `cran.r-project.org/package=demography`

Selected references

- Hyndman, Shang (2010). "Rainbow plots, bagplots and boxplots for functional data". *Journal of Computational and Graphical Statistics* **19**(1), 29–45
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➥ Papers and R code:

robjhyndman.com

➥ Email: **Rob.Hyndman@monash.edu**

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