# Towards multi-objective stochastic control under model uncertainty

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based on joint work with Igor Cialenco

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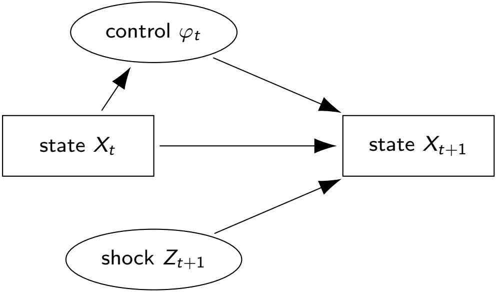
# Introduction: Single-objective stochastic control

# Stochastic control system




E.g. investor's portfolio

- state variable $X$
  ... portfolio value
- control variable $\varphi$
  ... portfolio position
- stochastic factor $Z$
  ... asset returns

- Objective: e.g. minimize $\mathbb{E}[\ell(X_T)]$, minimize $\rho(\ell(X_T))$, etc.
- Applications in finance, engineering, energy management, epidemiology, etc.

# Dynamic programming (Richard Bellman, 1950s)

- Control problem with a **value function** $\mathcal{V}_t(X_t)$

$$\mathcal{V}_t(X_t) := \inf \left\{ \mathbb{E}_t \left[ \ell(X_T) \right] \mid (\varphi_t, \ldots, \varphi_{T-1}) \text{ feasible} \right\}$$

- ... is time consistent and the **Bellman equations** are satisfied,

$$\begin{aligned} \mathcal{V}_t(X_t) &= \inf_{\varphi_t \in \mathcal{A}_t(X_t)} \mathbb{E}_t \left[ \mathcal{V}_{t+1}(X_{t+1}) \right], \\ \mathcal{V}_T(X_T) &:= \ell(X_T). \end{aligned}$$

  - problem can be solved recursively
  - lies at the core of reinforcement learning

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  - problem can be solved recursively
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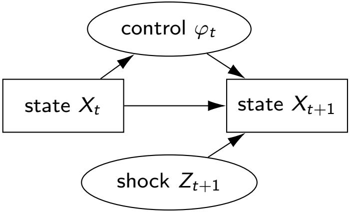
- What if your problem is multi-objective $\ell : \mathbb{X} \to \mathbb{R}^d$?
  ...**set-valued Bellman's principle**
- What if the distribution of the stochastic factor is unknown?
  ...**model uncertainty**

# Multi-objective stochastic control without model uncertainty

# Stochastic control system

- (for now) known distribution of factor process $(Z_t)_{t=1,\dots,T}$
- controlled dynamics & admissibility

$$\varphi_t \in A_t(X_t),$$
$$X_{t+1} = F_t(X_t, \varphi_t, Z_{t+1}), \quad t = 0, \dots, T-1$$




- technical assumptions
  - probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration generated by $(Z_t)_{t=1,\dots,T}$
  - measurable $F_t : \mathbb{X} \times \mathbb{A} \times \mathbb{Z} \to \mathbb{X}$, $\exists$ measurable selector of $A_t : \mathbb{X} \rightrightarrows \mathbb{A}$

# Stochastic control system

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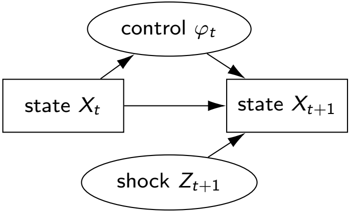



- multi-loss $\ell : \mathbb{X} \to \mathbb{R}^d$
- aim: $\underset{(\varphi_t,\dots,\varphi_{T-1}) \in \mathcal{A}^t(X_t)}{\text{minimize}} \quad \mathbb{E}_t[\ell(X_T)]$

- technical assumptions
  - probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration generated by $(Z_t)_{t=1,\dots,T}$
  - measurable $F_t : \mathbb{X} \times \mathbb{A} \times \mathbb{Z} \to \mathbb{X}$, $\exists$ measurable selector of $A_t : \mathbb{X} \rightrightarrows \mathbb{A}$
  - bounded, measurable $\ell : \mathbb{X} \to \mathbb{R}^d$

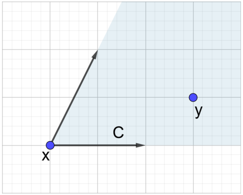
# Vector and multi-objective optimization

Modeling choice: How to compare values of $\ell : \mathbb{X} \to \mathbb{R}^d$?

- preorder $\preceq$ on $R^d$ $\iff$ ordering cone $C \subseteq \mathbb{R}^d$

$$x \preceq y \quad \Leftrightarrow \quad y \in x + C$$

  - preorder: non-trivial ($\{0\} \subsetneq C \subsetneq \mathbb{R}^d$), convex cone $C$
  - partial order: pointed ($-C \cap C = \{0\}$) ordering cone $C$
  - standard assum.: solid ($\operatorname{int} C \neq \emptyset$), closed, ordering cone




- preorder $\preceq^t$ on $\mathscr{L}_t(\mathbb{R}^d) := L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$

$$X \preceq^t Y \quad \Leftrightarrow \quad Y \in X + \mathscr{L}_t(C)$$

- talk: focus on component-wise order $\leq$ generated by $C = \mathbb{R}^d_+$
[Cialenco, K. '26]: also general preorders and partial orders

# Vector and multi-objective optimization

- **multi-objective optimization problem** with $f : \mathbb{X} \to \mathscr{L}_t(\mathbb{R}^d)$

$$\text{minimize} \quad f(x) \quad \text{with respect to} \quad \leq \quad \text{subject to} \quad x \in \mathbb{X}$$

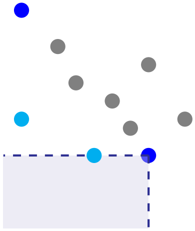
  - notions of optimality? value function?

- $\bar{x} \in \mathbb{X}$ is minimizer iff

$$\left(\{f(\bar{x})\} - \mathscr{L}_t(\mathbb{R}^d_+) \setminus \{0\}\right) \cap f[\mathbb{X}] = \emptyset$$

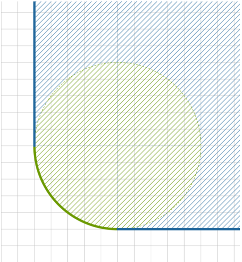
- $\bar{x} \in \mathbb{X}$ is **weak minimizer** iff

$$\left(\{f(\bar{x})\} - \operatorname{int} \mathscr{L}_t(\mathbb{R}^d_+)\right) \cap f[\mathbb{X}] = \emptyset$$



# Value function

- Single-objective: $\mathcal{V}_t(X_t) := \inf \{\mathbb{E}_t\ [\ell(X_T)]\ |\ \varphi \text{ feasible}\}$
- Multi-objective: appropriate notion of infimum?
  - Vector-valued candidate: several drawbacks
  - Set-valued candidates: image of the feasible set $f[\mathbb{X}]$ and **upper image**



$$\mathcal{V} = \mathrm{cl}\left(f[\mathbb{X}] + \mathscr{L}_t(\mathbb{R}^d_+)\right)$$

- *conlinear space* of closed, upper sets is a *complete lattice*
- set optimization: $\mathcal{V}$ is an *infimum* of the VOP [Löhne '11]; [Hamel et.al. '14, '15]

- Value function of a multi-objective problem is *set-valued*

# Multi-objective control and dynamic programming

$$\begin{aligned} &\text{minimize } \ \mathbb{E}_t[\ell(X_T)] \ \text{ with respect to } \ \leq \\ &\text{subject to } \ \varphi_s \in A_s(X_s),\ X_{s+1} = F_s(X_s, \varphi_s, Z_{s+1}), \quad s = t, \dots, T-1 \end{aligned}$$

- value function given a state $X_t \in \mathscr{L}_t(\mathbb{X})$,

$$\mathcal{V}_t(X_t) = \operatorname{cl}\Big(\Big\{ \mathbb{E}_t[\ell(X_T)] \mid \varphi_s \in A_s(X_s),\ X_{s+1} = F_s(X_s, \varphi_s, Z_{s+1}), \\ s = t, \dots, T-1 \Big\} + \mathscr{L}_t(\mathbb{R}^d_+)\Big)$$

# Multi-objective control and dynamic programming

$$\text{minimize} \;\; \mathbb{E}_t[\ell(X_T)] \;\; \text{with respect to} \;\; \leq$$
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**Set-valued Bellman's principle [K., Rudloff '21; Cialenco, K. '25]**

The value function has a recursive form $\mathcal{V}_T(X_T) = \ell(X_T) + \mathscr{L}_T(\mathbb{R}^d_+)$ and

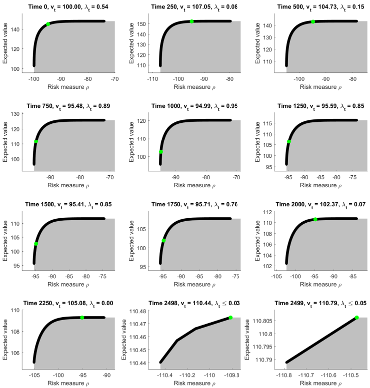
$$\mathcal{V}_t(X_t) = \mathrm{cl}\Big\{ \mathbb{E}_t[Y] \mid \varphi_s \in A_s(X_s),\; Y \in \mathcal{V}_{t+1}\big(F_t(X_t, \varphi_t, Z_{t+1})\big) \Big\}.$$

- set optimization: recursion interpreted as an *infimum*
- recursion $\leftrightarrow$ *one-step* multi-objective problem

# Case study of mean-risk portfolio selection [K., Rudloff '21]

minimize $\begin{pmatrix} -\mathbb{E}_t(X_T) \\ \rho_t(X_T) \end{pmatrix}$ with respect to $\leq$

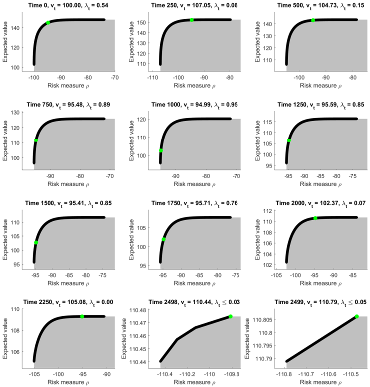
- scalarized mean-risk problem is **time inconsistent**
  [Bäuerle and Mundt '09]; [Cui et.al.'12]



# Case study of mean-risk portfolio selection [K., Rudloff '21]

$$\text{minimize} \quad \begin{pmatrix} -\mathbb{E}_t(X_T) \\ \rho_t(X_T) \end{pmatrix} \quad \text{with respect to} \ \leq$$

- scalarized mean-risk problem is **time inconsistent** [Bäuerle and Mundt '09]; [Cui et.al.'12]
- bi-objective mean-risk problem is **time consistent**
  - principle of optimality w.r.t. weak minimizers
  - **time varying state dependent risk aversion**




- set-valued $\mathcal{V}_t(X_t) \approx$ efficient frontier
  - set-valued Bellman's principle holds
  - recursive computation of the frontier

# Set-valued Bellman's principle

- Multivariate risk measurement
  - set-valued risk measures, superhedging, systemic risk [Feinstein, Rudloff '13, '15, '17]; [Löhne, Rudloff '14]; [Ararat, Feinstein '21]
- Control problems
  - set-valued Bellman's principle, set-valued HJB, time consistency [K., Rudloff '21]; [Hamel, Visetti '20]; [K., Rudloff, Cialenco '22]; [Iseri and Zhang '23]; [Visetti '23]; [Cialenco, K. '26]
- Game theory
  - recursion for sets of Nash equilibria, mean field games [Feinstein, Rudloff, Zhang '22]; [Iseri and Zhang '24]

# Towards model uncertainty

# Motivation

- Consider the mean-risk problem
  - $\mathbb{E}[X_T]$ and $\rho(X_T)$ computed w.r.t. the distribution of asset returns $Z$
  - distribution of asset returns are assumed to be *known*
- In reality, distribution of the stochastic factor $Z$ is often *unknown*
  - mean-risk: an investor does not know the true return distribution
- Model uncertainty: consider several models for the stochastic factor $Z$
  - e.g. parametrized family of distributions $\mathbb{Q}(\boldsymbol{\Theta}) = \{\mathbb{Q}^\theta \ : \ \theta \in \boldsymbol{\Theta}\}$
  - mean-risk: incorporate investor's uncertainty about return distribution

# Approaches to model uncertainty: single-objective

- Loss function $\ell : \mathbb{X} \to \mathbb{R}$ and parameter set $\boldsymbol{\Theta}$
- If true $\theta^* \in \boldsymbol{\Theta}$ were known: stochastic control problem

$$\min_{\varphi^t \in \mathcal{A}^t(X_t)} \quad \mathbb{E}_t^{\theta^*}[\ell(X_T^{\varphi})]$$

# Approaches to model uncertainty: single-objective

- Loss function $\ell : \mathbb{X} \to \mathbb{R}$ and parameter set $\boldsymbol{\Theta}$
- Robust approach
  - optimize the worst case scenario: $\min\limits_{\varphi^t \in \mathcal{A}^t(X_t)} \sup\limits_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_t^\theta[\ell(X_T^\varphi)]$
- Adaptive approach
  - point estimate $\theta^t$ of $\theta^*$ at time $t$: $\min\limits_{\varphi^t \in \mathcal{A}^t(X_t)} \mathbb{E}_t^{\theta^t}[\ell(X_T^\varphi)]$
- Bayesian approach
  - prior distribution $\xi$ on $\boldsymbol{\Theta}$: $\min\limits_{\varphi^t \in \mathcal{A}^t(X_t)} \int_{\boldsymbol{\Theta}} \mathbb{E}_t^\theta[\ell(X_T^\varphi)]\xi(d\theta)$
- Adaptive robust approach
  - confidence region $\boldsymbol{\Theta}^t \subseteq \boldsymbol{\Theta}$ at time $t$: $\min\limits_{\varphi^t \in \mathcal{A}^t(X_t)} \sup\limits_{\theta \in \boldsymbol{\Theta}^t} \mathbb{E}_t^\theta[\ell(X_T^\varphi)]$

see e.g. [Nutz '14], [Rieder '75], [Bielecki et.al. '19], [Shapiro et.al. '23]

# Robust multi-objective stochastic control [Cialenco, K. '26]

# Robust multi-objective optimization: static case

- Objective $f : \mathbb{X} \times \boldsymbol{\Theta} \to \mathbb{R}^d$
  - variable $x \in \mathbb{X}$ and uncertainty parameter $\theta \in \boldsymbol{\Theta}$

- Robustified objective $f^{RC}(x) = \begin{pmatrix} \sup_{\theta \in \boldsymbol{\Theta}} f_1(x, \theta) \\ \vdots \\ \sup_{\theta \in \boldsymbol{\Theta}} f_d(x, \theta) \end{pmatrix}$
  - vector-valued (*ideal point*) notion of $\sup_{\theta \in \boldsymbol{\Theta}} f(x, \theta) := f^{RC}(x)$
  - multi-objective robust counterpart and robust efficient frontier see e.g. [Kuroiwa, Lee '12]; [Fliege, Werner '13]

- Set-valued objective $f^{\boldsymbol{\Theta}}(x) = \left\{ f(x, \theta) \ : \ \theta \in \boldsymbol{\Theta} \right\}$
  - set-valued notion of $\sup_{\theta \in \boldsymbol{\Theta}} f(x, \theta) := f^{\boldsymbol{\Theta}}(x)$
  - set-valued robust counterpart and robust efficiency see e.g. [Ehrgott, Ide, Schöbel '14]; [Ide et.al. '14]

- [Cialenco, K. '26]: preorder $\preceq^t$ on $\mathscr{L}_t(\mathbb{R}^d)$ with ordering cone $\mathscr{L}_t(C)$

Vector $\underset{\theta\in\mathbf{\Theta}}{\text{v-sup}}\, X^\theta \in \mathscr{L}_t(\mathbb{R}^d)$ is a supremum of $\{X^\theta\}_{\theta\in\mathbf{\Theta}} \subseteq \mathscr{L}_t(\mathbb{R}^d)$ w.r.t. $\preceq^t$ iff

(i) $\forall\theta \in \mathbf{\Theta}: \; X^\theta \preceq^t \underset{\theta\in\mathbf{\Theta}}{\text{v-sup}}\, X^\theta$

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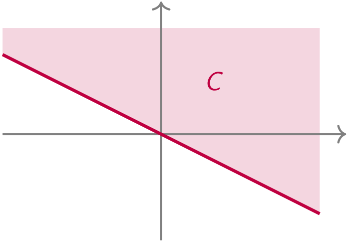
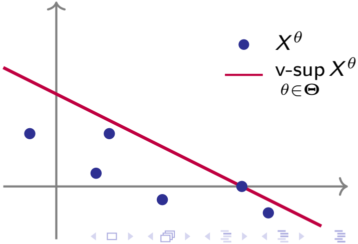
$$\bigcap_{\theta\in\mathbf{\Theta}} (X^\theta + \mathscr{L}_t(C)) = \underset{\theta\in\mathbf{\Theta}}{\text{v-sup}}\, X^\theta + \mathscr{L}_t(C)$$

Challenge 1: Non-uniqueness

$$C = \{x \in \mathbb{R}^d \mid w^\top x \geq 0\}$$



# Ideal point supremum

- [Cialenco, K. '26]: preorder $\preceq^t$ on $\mathscr{L}_t(\mathbb{R}^d)$ with ordering cone $\mathscr{L}_t(C)$

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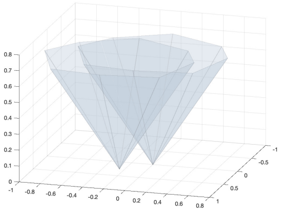
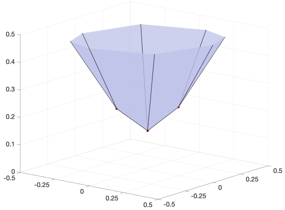
Challenge 2: Non-existence

$$C = \text{cone}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0.75\\0.75\\1\end{pmatrix}, \begin{pmatrix}-0.75\\0.75\\1\end{pmatrix}, \begin{pmatrix}-0.75\\-0.75\\1\end{pmatrix}, \begin{pmatrix}0.75\\-0.75\\1\end{pmatrix}\right\}$$





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$$\bigcap_{\theta\in\boldsymbol{\Theta}} (X^\theta + \mathscr{L}_t(C)) = \operatorname*{v-sup}_{\theta\in\boldsymbol{\Theta}} X^\theta + \mathscr{L}_t(C)$$

  - sufficient condition for existence of ideal point supremum
  - uniqueness for partial orders

- This talk: component-wise order $\leq$ on $\mathscr{L}_t(\mathbb{R}^d)$ with ordering cone $\mathscr{L}_t(\mathbb{R}^d_+)$
- Bounded set of vectors $\{X^\theta\}_{\theta\in\boldsymbol{\Theta}} \subseteq \mathscr{L}_t(\mathbb{R}^d)$

$$\operatorname*{v-sup}_{\theta\in\boldsymbol{\Theta}} X^\theta := \begin{pmatrix} \operatorname*{ess\,sup}_{\theta\in\boldsymbol{\Theta}} X_1^\theta \\ \vdots \\ \operatorname*{ess\,sup}_{\theta\in\boldsymbol{\Theta}} X_d^\theta \end{pmatrix}$$

# Robust multi-objective control

- Compact parameter set $\Theta$, bounded set $\{\mathbb{E}_t^\theta[\ell(X_T)]\}_{\theta\in\Theta}$

$$\begin{aligned} &\text{minimize} \quad \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[\ell(X_T)] \quad \text{with respect to} \quad \leq \\ &\text{subject to} \quad \varphi_s \in A_s(X_s),\ X_{s+1} = F_s(X_s, \varphi_s, Z_{s+1}), \quad s = t, \dots, T-1 \end{aligned}$$

- vector-valued v-sup $\rightarrow$ multi-objective stochastic control problem

- value function given a state $X_t \in \mathscr{L}_t(\mathbb{X})$

$$\mathcal{V}_t(X_t) = \text{cl}\Big(\Big\{ \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[\ell(X_T)] \mid \varphi_s \in A_s(X_s),\ X_{s+1} = F_s(X_s, \varphi_s, Z_{s+1}), \\ s = t, \dots, T-1 \Big\} + \mathscr{L}_t(\mathbb{R}^d_+)\Big)$$

- question: dynamic programming for the robust problem?
  - set-valued Bellman's principle? recursive problem? time consistency?

# Recursive problem and dynamic programming

- In backward fashion, construct $\mathcal{W}_T(X_T) := \mathcal{V}_T(X_T)$ and (one-step) problem

$$\text{minimize} \quad \underset{\theta\in\boldsymbol{\Theta}}{\text{v-sup}}\,\mathbb{E}_t^{\theta}[Y] \quad \text{with respect to} \quad \leq$$
$$\text{subject to} \quad \varphi_t \in A_t(X_t),\ \ Y \in \mathcal{W}_{t+1}\big(F_t(X_t,\varphi_t,Z_{t+1})\big)$$

with value function

$$\mathcal{W}_t(X_t) = \mathrm{cl}\Big\{\underset{\theta\in\boldsymbol{\Theta}}{\text{v-sup}}\,\mathbb{E}_t^{\theta}[Y] \mid \varphi_t \in A_t(X_t),\ \ Y \in \mathcal{W}_{t+1}\big(F_t(X_t,\varphi_t,Z_{t+1})\big)\Big\}$$

- Set-valued Bellman's principle?

$$\mathcal{V}_t(X_t) \overset{?}{=} \mathcal{W}_t(X_t)$$

# Weak set-valued Bellman's principle under uncertainty

- **Weak set-valued Bellman's principle** (inclusion) without further assumptions

**Theorem [Cialenco, K. '26]**

For all times $t = 0, \dots, T-1$ and all states $X_t \in \mathscr{L}_t(\mathbb{X})$ it holds

$$\mathcal{V}_t(X_t) \preccurlyeq^t \mathcal{W}_t(X_t) \quad \text{i.e.} \quad \mathcal{W}_t(X_t) \subseteq \mathcal{V}_t(X_t) + \mathscr{L}_t(\mathbb{R}^d_+)$$

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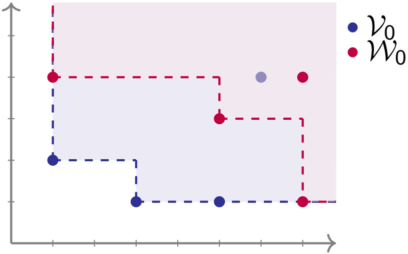


- Example: $\mathcal{V}_t(X_t) \neq \mathcal{W}_t(X_t)$
  - binomial tree, $T = 2$, $|\mathcal{A}| = 4$
  - $|\boldsymbol{\Theta}| = 4$ with indep. $Z_1, Z_2$
- Sufficient condition(s) for $\mathcal{V}_t(X_t) = \mathcal{W}_t(X_t)$?
  - $|\boldsymbol{\Theta}| = 1$, but...

# Strong set-valued Bellman's principle under uncertainty

Family of models $\boldsymbol{\Theta}$ is *m-rectangular* if for all time $t = 0, \dots, T-1$ and all random variables $Y \in \mathscr{L}_T(\mathbb{R})$ it holds

$$\sup_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_t^{\theta} \left[ \sup_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_{t+1}^{\theta}[Y] \right] = \sup_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_t^{\theta}[Y].$$

- see e.g. [Epstein, Schneider '03]; [Iyengar '05]; [Shapiro '16]
- e.g. [Shapiro '16] provides a construction of $m$-rectangular families
- $m$-rectangularity implies *$\leq$-rectangularity*, i.e. for vector $Y \in \mathscr{L}_T(\mathbb{R}^d)$

$$\underset{\theta \in \boldsymbol{\Theta}}{\text{v-sup}}\, \mathbb{E}_t^{\theta} \left[ \underset{\theta \in \boldsymbol{\Theta}}{\text{v-sup}}\, \mathbb{E}_{t+1}^{\theta}[Y] \right] = \underset{\theta \in \boldsymbol{\Theta}}{\text{v-sup}}\, \mathbb{E}_t^{\theta}[Y]$$

- preorder $\preceq$: verification and construction open

# Strong set-valued Bellman's principle under uncertainty

Family of models $\boldsymbol{\Theta}$ is *$\leq$-rectangular* if for all time $t = 0, \dots, T-1$ and all random vectors $Y \in \mathscr{L}_T(\mathbb{R}^d)$ it holds

$$\operatorname*{v-sup}_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_t^\theta \left[ \operatorname*{v-sup}_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_{t+1}^\theta [Y] \right] = \operatorname*{v-sup}_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_t^\theta [Y].$$

**Theorem [Cialenco, K. '26]**

Assume that $\boldsymbol{\Theta}$ is $\leq$-rectangular. Then, for all times $t = 0, \dots, T-1$ and all states $X_t \in \mathscr{L}_t(\mathbb{X})$ it holds $\mathcal{V}_t(X_t) = \mathcal{W}_t(X_t)$, i.e.

$$\mathcal{V}_t(X_t) = \operatorname{cl} \left\{ \operatorname*{v-sup}_{\theta \in \boldsymbol{\Theta}} \mathbb{E}_t^\theta [Y] \mid \varphi_t \in A_t(X_t),\ Y \in \mathcal{V}_{t+1}\big(F_t(X_t, \varphi_t, Z_{t+1})\big) \right\}.$$

# Time consistency under uncertainty

- Time consistency – principle of optimality
- Multi-objective control without uncertainty
  - optimality notions: minimizers and weak minimizers
  - time consistency w.r.t. weak minimizers holds
  - time consistency w.r.t. minimizers requires additional assumptions

**Theorem [Cialenco, K. '26]**

Assume that $\boldsymbol{\Theta}$ is $\leq$-rectangular. The family of robust multi-objective stochastic control problems is *time consistent w.r.t. weak minimizers*.

$$(\varphi_t, \dots, \varphi_{T-1}) \in \mathcal{A}^t(X_t) \text{ is a } \textit{weak minimizer} \text{ at time } t \text{ and state } X_t \in \mathscr{L}_t(\mathbb{X})$$
$$\Downarrow$$
$$(\varphi_{t+1}, \dots, \varphi_{T-1}) \in \mathcal{A}^{t+1}\big(F_t(X_t, \varphi_t, Z_{t+1})\big) \text{ is a } \textit{weak minimizer}$$
$$\text{at time } t+1 \text{ and state } F_t(X_t, \varphi_t, Z_{t+1})$$

# Summary

## multi-objective stochastic control

- value function of a *multi-objective* problem is *set-valued*
- DPP = set-valued Bellman's principle
- time consistency w.r.t. *weak minimizers*

## robust multi-objective stochastic control

- *ideal point* supremum approach
- *weak* set-valued Bellman's principle without assumptions
- strong set-valued Bellman's principle under *rectangularity*

**Thank you for your attention!**

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