Can Nash inform capital requirements?
Allocating systemic risk measures

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Background: univariate risk measures

For a functional $\rho\colon L^\infty(\mathbb{R}) \to \mathbb{R}$, consider the following (for all $X, Y \in L^\infty(\mathbb{R})$, $m \in \mathbb{R}$, $\lambda \in [0,1]$, $\alpha \geq 0$):

- Monotonicity: $X \geq Y$ a.s. implies $\rho(X) \leq \rho(Y)$.
- Cash-additivity: $\rho(X+m) = \rho(X) - m$.
- Convexity: $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$.
- Positive homogeneity: $\rho(\alpha X) = \alpha\rho(X)$.
- Normalization: $\rho(0) = 0$.
- Lower semicontinuity: ρ is weak* lower semicontinuous.

Monetary risk measure = monotone + cash-additive

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Primal Representation

A monetary risk measure ρ is equivalently characterized by its **acceptance set** $\mathcal{A} := \{X \in L^\infty(\mathbb{R}) \mid \rho(X) \leq 0\}$ via

$$\rho(X) = \inf\{m \in \mathbb{R} \mid X + m \in \mathcal{A}\}.$$

Two ingredients:

- Aggregation Function: $\Lambda : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ increasing function

  $\Lambda(x, m)$ is the impact of a **shock of size** x under **capital allocation** m to society.
- Acceptance Set: $\mathcal{A} \subseteq L^\infty(\mathbb{R})$ of a monetary risk measure ρ

  $\mathcal{A}$ is the set of all **acceptable aggregate values** under a random shock.

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Systemic Risk Measure: For every $X \in L^\infty(\mathbb{R}^N)$, define

$$R(X) := \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}.$$

$R(X)$ is the set of **all capital allocation vectors** yielding acceptable aggregate values under random shock X.

Systemic Risk Measure: $R(X) := \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}$

Typical Examples of Λ:

Single-element aggregation function: $\bar{\Lambda} \colon \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ increasing function

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- Insensitive: (Chen et al. (2013), Kromer et al. (2016))

$$\Lambda^I(x, m) = \bar{\Lambda}(x) + \sum_{i \in [N]} m_i$$

Then: $R^I(X) = \{m \in \mathbb{R}^N \mid \rho(\bar{\Lambda}(X)) \leq \sum_{i \in [N]} m_i\}$

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- Sensitive: (Feinstein et al. (2017), A. & Rudloff (2020))

$$\Lambda^S(x, m) = \bar{\Lambda}(x + m)$$

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- Scenario-dependent: (Biagini et al. (2019)))

$$\Lambda^{SD}(x, m) \coloneqq \sup_{y \in \mathbb{R}^N} \left\{ \bar{\Lambda}(x + y) \mid \sum_{i \in [N]} y_i \leq \sum_{i \in [N]} m_i \right\}$$

Then:
$R^{SD}(X) = \{m \in \mathbb{R}^N \mid \exists Y \in L^\infty(\mathbb{R}^N) \colon \rho(\bar{\Lambda}(X + Y)) \leq 0, \ \sum_{i \in [N]} Y_i \leq \sum_{i \in [N]} m_i\}$
Yields scenario-dependent capital allocations (with deterministic sums)

Capital allocation rules

Systemic Risk Measure: $R(X) := \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}$.

Allocation Problem: How to choose a "good" $m \in R(X)$?

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- Weighted-Sum Scalarization: (Feinstein et al. (2017), A. & Meimanjan (2023))

  Fix a suitable weight vector $w \in \mathbb{R}^N_+ \setminus \{0\}$ and choose

$$m \in \operatorname{argmin} \left\{ \sum_{i \in [N]} w_i m_i \mid \Lambda(X,m) \in \mathcal{A} \right\}.$$

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- Pascoletti-Serafini Scalarization: (Feinstein et al. (2017), A. & Meimanjan (2023))

  Start with some capital level $m^0 \in \mathbb{R}^N$. Fix a direction vector $d \in \mathbb{R}^N_{++}$ and use $m := m^0 + \mu^0 d$, where

  $$\mu^0 := \inf \left\{ \mu \in \mathbb{R}^N \mid \Lambda(X, m^0 + \mu d) \in \mathcal{A} \right\}.$$

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- Duality Approach: (Biagini et al. (2019, 2020, 2021))

  Use $m = \mathbb{E}^{\mathbb{Q}^X}[-Y^X]$, where $\mathbb{Q}^X$ is a dual vector of probability measures and Y^X is the scenario-dependent allocation.

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Today: A more systematic approach?

Attempt #1: Axioms for capital allocation rules

Question: Can we define a "vector-valued risk measure" r such that $r(X) \in R(X)$ for every $X \in L^\infty(\mathbb{R}^N)$?

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Possible Axioms: For a vector-valued functional $r\colon L^\infty(\mathbb{R}^N) \to \mathbb{R}^N$, consider the following
(for all $X, Y \in L^\infty(\mathbb{R}^N)$, $m \in \mathbb{R}^N$, $\lambda \in [0,1]$; $\leq$: componentwise order)

- Monotonicity: $X \geq Y$ a.s. implies $r(X) \leq r(Y)$.
- Marginal domination property: For every $i \in [N]$, there exists a function $f_i\colon \mathbb{R} \to \mathbb{R}$ such that $r_i(m) \leq f_i(m_i)$ for every $m \in \mathbb{R}^N$.
- Cash-(sub)additivity: $r(X+m)(\leq) = r(X) - m$.
- Cash-preserving property: $r(m) = -m$.
- Convexity: $r(\lambda X + (1-\lambda)Y) \leq \lambda r(X) + (1-\lambda) r(Y)$.
- Lower semicontinuity: $r_1, \ldots, r_N$ are weak* lower semicontinuous.

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Vector-valued risk measure = monotone + marginal domination

- Cash-subadditivity / cash-preserving $\Rightarrow$ marginal domination
- The three allocations rules discussed before do *not* satisfy these axioms!

Attempt #1: Axioms for capital allocation rules

Separability of Allocations (A., Feinstein '24 SIFIN)

If $r\colon L^\infty(\mathbb{R}^N) \to \mathbb{R}^N$ is a vector-valued risk measure that is convex and lower semicontinuous, then, r is also separable, i.e.,

$$r(X) = (\bar{r}_1(X_1), \dots, \bar{r}_N(X_N)), \quad X \in L^\infty(\mathbb{R}^N),$$

for some functionals $\bar{r}_1, \dots, \bar{r}_N\colon L^\infty(\mathbb{R}) \to \mathbb{R}$.

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- A negative result! There exists *no* capital allocation rule that is a convex lower semicontinuous vector-valued risk measure.
- Can also be shown in an $L^p(\mathbb{R}^N)$-space setting with $p \geq 1$.
- Extends to set-valued and conditional vector-valued risk measures.

Attempt #2: Nash allocation rules

- Rectangular upper sets:
  $\mathcal{R}(\mathbb{R}^N;\mathbb{R}^N_+) :=$
  $\left\{\bigtimes_{i=1}^N A_i \mid \forall i \in [N]\colon A_i = \mathbb{R} \text{ or } A_i = [x_i, +\infty) \text{ for some } x_i \in \mathbb{R}\right\} \cup \{\emptyset\}$.
- Systemic risk measure: $R(X) = \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}$, where
  - $\Lambda\colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ is an aggregation function, i.e.,
    1. Λ is increasing and concave,
    2. $\operatorname{dom}\Lambda \neq \emptyset$ is closed and Λ is continuous relative to it,
    3. **Rectangular domain:**
       (i) For each $m \in \mathbb{R}^N$, $\operatorname{dom}\Lambda|_m \in \mathcal{R}(\mathbb{R}^N;\mathbb{R}^N_+)$,
       (ii) For each $x \in \mathbb{R}^N$, $\operatorname{dom}\Lambda|^x \in \mathcal{R}(\mathbb{R}^N;\mathbb{R}^N_+)$.
  - $\mathcal{A}$ is a weak* closed acceptance set that is a convex cone,
  i.e., ρ is a coherent risk measure that is weak* lower semicontinuous.

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- For each $X\in L^\infty(\mathbb{R}^N)$, define

$$D(X):=\{m\in\mathbb{R}^N \mid \mathbb{P}\{(X,m)\in\operatorname{dom}\Lambda\}=1\}.$$

- Let $L^\infty_\Lambda(\mathbb{R}^N):=\left\{X\in L^\infty(\mathbb{R}^N) \mid D(X)\neq\emptyset\right\}$.

Well-Definedness (A., Feinstein '25)

For each $X\in L^\infty_\Lambda(\mathbb{R}^N)$ and $m\in D(X)$, it holds $\Lambda(X,m)\in L^\infty(\mathbb{R})$.

Decomposition of aggregation function

- Let $\Lambda_1, \dots, \Lambda_N : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ be aggregation functions. We say that $(\Lambda_i)_{i\in[N]}$ is a decomposition of Λ if:
  - Consistent domain: $\operatorname{dom} \Lambda_i = \operatorname{dom} \Lambda$ for every $i \in [N]$.
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  - Self-feasible: For every $x \in \mathbb{R}^N$ with $D(x) = \operatorname{dom} \Lambda|^x \neq \emptyset$, there exists $\bar{r} \in D(x)$ such that $\Lambda_i(x, (\bar{r}_i, m_{-i})) > 0$ for every $m \in D(x)$ and $i \in [N]$.
  - Self-infeasible: For every $x \in \mathbb{R}^N$ with $D(x) = \operatorname{dom} \Lambda|^x \neq \emptyset$, there exists $\underline{r} \in D(x)$ such that $-\infty \leq \Lambda_i(x, (\underline{r}_i - \varepsilon, m_{-i})) < 0$ for every $m \in D(x)$, $i \in [N]$, and $\varepsilon > 0$.

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- A decomposition $(\Lambda_i)_{i\in[N]}$ is called self-preferential if there exists some constant $L \geq 0$ such that, for every $i \neq j$, $(x, m) \in \operatorname{dom}\Lambda$, and $\delta \geq 0$, it holds

$$\Lambda_i(x, m + L\delta e_i) \geq \Lambda_i(x, m + \delta e_j).$$

- $L((\Lambda_i)_{i\in[N]})$: The self-preferential constant (least $L \geq 0$ satisfying the definition)

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  Take $\lambda_i(x,m) := u_i(x_i + m_i)$. What happens to $\bar{u}$?
- The remaining aggregation function $\Delta := \Lambda - \sum_{i\in[N]} \lambda_i$ requires further decomposition.

Minimal decomposition (A., Feinstein '25)

Under mild conditions, there exists a (minimal) decomposition $(\Delta_i)_{i\in[N]}$ of Δ that minimizes the self-preferential constant $L((\Delta_i)_{i\in[N]})$ among all decompositions of Δ.

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  Ex: Mean-field utility
  Take $\alpha_i \geq 0$ with $\sum_{i\in[N]} \alpha_i = 1$.
  Setting $\Delta_i(x,m) := \alpha_i \bar{u}\left(\frac{1}{N}\sum_{i\in[N]}(x_i + m_i)\right)$ gives a minimal decomposition within the class of "symmetric" decompositions.

Nash allocation rules

- Recall: $R(X) = \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}$.
- $L^\infty_\Lambda(\mathbb{R}^N) := \{X \in L^\infty(\mathbb{R}^N) \mid \exists m \in \mathbb{R}^N : \mathbb{P}\{(X, m) \in \operatorname{dom} \Lambda\} = 1\}$.
- Let $(\Lambda_i)_{i \in [N]}$ be a decomposition of Λ.

Definition: Nash Allocation

A vector-valued functional $r \colon L^\infty_\Lambda(\mathbb{R}^N) \to \mathbb{R}^N$ is called a Nash allocation rule if, for every $X \in L^\infty_\Lambda(\mathbb{R}^N)$ and $i \in [N]$, it holds

$$r_i(X) = \inf\{m_i \in \mathbb{R} \mid \Lambda_i(X, (m_i, r_{-i}(X))) \in \mathcal{A}\}$$

In this case, r is called acceptable if $r(X) \in R(X)$ for every $X \in L^\infty_\Lambda(\mathbb{R}^N)$.

- Recall: $R(X) = \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}$.
- $L^\infty_\Lambda(\mathbb{R}^N) := \{X \in L^\infty(\mathbb{R}^N) \mid \exists m \in \mathbb{R}^N : \mathbb{P}\{(X, m) \in \operatorname{dom} \Lambda\} = 1\}$.
- Let $(\Lambda_i)_{i \in [N]}$ be a decomposition of Λ.

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Acceptable Allocations (A., Feinstein '25)

Every Nash allocation rule is acceptable.

Nash allocation rules: existence and uniqueness

- **Recall:** $R(X) = \{m \in \mathbb{R}^N \mid \Lambda(X, m) \in \mathcal{A}\}$.

Existence and Uniqueness (A., Feinstein '25)

Let $(\Lambda_i)_{i \in [N]}$ be a decomposition of Λ.

1. There exists a Nash allocation rule $r\colon L^\infty_\Lambda(\mathbb{R}^N) \to \mathbb{R}^N$.
2. Suppose that $(\Lambda_i)_{i \in [N]}$ is self-preferential with $L < \frac{1}{N-1}$. Then, there exists a unique Nash allocation rule.

   (Recall: L satisfies $\Lambda_i(x, m + L\delta e_i) \geq \Lambda_i(x, m + \delta e_j)$ for all $i \neq j$, $(x, m) \in \operatorname{dom} \Lambda$, $\delta \geq 0$.)

Examples

$\bar{\Lambda}\colon \mathbb{R}^N \to \mathbb{R} \cup \{-\infty\}$ single-element aggregation function

- Insensitive Systemic Risk Measures:
  - **Recall**: $\Lambda^I(x,m) = \bar{\Lambda}(x) + \sum_{i=1}^N m_i$
  - Consider the decomposition $\Lambda_i^I(x,m) = \bar{\Lambda}_i(x) + m_i$ for monotonic $(\bar{\Lambda}_i)_{i\in[N]}$ dominated by $\bar{\Lambda}$.
  - The unique associated Nash allocation rule is given by

$$r^I(X) := (\rho(\bar{\Lambda}_1(X)), \dots, \rho(\bar{\Lambda}_N(X))).$$

- Aggregate Utility Systemic Risk Measures:
  - Consider $\Lambda^S(x,m) = \sum_{i=1}^N u_i(x_i + m_i)$.
  - Appears in Biagini et al. (2019, 2020).
  - Consider the natural decomposition $\Lambda_i^S(x,m) = u_i(x_i + m_i)$.
  - The unique associated Nash allocation rule is given by

$$r_i(X) = \min\{m_i \in \mathbb{R} \mid u_i(X_i + m_i) \in \mathcal{A}\}.$$

Case study: Eisenberg-Noe allocations

- Network of N banks and one external node (society)
- $x \in \mathbb{R}_+^N$ external assets (e.g., $X(\omega) = x$)
- Bank i has obligations: $\ell_{ij} \geq 0$ to bank j and $\ell_{i0} > 0$ externally
- Relative liabilities: $\pi_{ij} = \ell_{ij} / \sum_{k=0}^{N} \ell_{ik}$
- Payment from i to j is proportional to π_{ij}.
- Total payments:

$$p_i(x) = \left(\sum_{j=0}^{N} \ell_{ij} \right) \wedge \left(x_i + \sum_{j=1}^{N} \pi_{ji} p_j(x) \right).$$

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- Sensitive aggregation function:

$$\Lambda^{\mathsf{EN}}(x, m) := \sum_{i=1}^{N} (\pi_{i0} p_i(x+m) - \gamma \ell_{i0}),$$

  assuming that the regulator requires at least $\gamma \in (0,1)$ fraction of all obligations to society are repaid.
- Decomposition: $\Lambda_i^{\mathsf{EN}}(x, m) := \pi_{i0} p_i(x+m) - \gamma \ell_{i0}$

Eisenberg-Noe allocations

- Take $\Lambda_i^{\mathsf{EN}}(x,m) := \pi_{i0} p_i(x+m) - \gamma \ell_{i0}$ so that $\Lambda^{\mathsf{EN}} = \sum_{i=1}^N \Lambda_i^{\mathsf{EN}}$.

Self-Preferential Constant (A., Feinstein '25)

Assume that $\pi_{i0} > 0$ for every $i \in [N]$. Then, $(\Lambda_i^{\mathsf{EN}})_{i\in[N]}$ is a self-preferential sensitive decomposition with constant L given by

$$L := \max_{\lambda \in \{0,1\}^N} \max_{i \neq j} \frac{[(I_N - \mathrm{diag}(\lambda)\Pi)^{-1}]_{ij}}{[(I_N - \mathrm{diag}(\lambda)\Pi)^{-1}]_{jj}} \leq \max_{i\in[N]} (1 - \pi_{i0}) < 1.$$

- Controlling the entries of Leontief inverse is generally difficult.
- The proof of $\leq$ uses Markov chain theory!
- Condition for uniqueness: $\pi_{i0} > \frac{N-2}{N-1}$ for all banks i is not always guaranteed.

Eisenberg-Noe allocations

- Take $\Lambda_i^{\mathsf{EN}}(x,m) := \pi_{i0} p_i(x+m) - \gamma \ell_{i0}$ so that $\Lambda^{\mathsf{EN}} = \sum_{i=1}^N \Lambda_i^{\mathsf{EN}}$.

Optimization Characterization (A., Feinstein '25)

For each $X \in L^\infty_{\Lambda^{\mathsf{EN}}}(\mathbb{R}^N)$, let

$$r^*(X) \in \operatorname{argmin} \left\{ \sum_{i \in [N]} m_i \mid \Lambda_i^{\mathsf{EN}}(X,m) \in \mathcal{A} \, \forall i \in [N] \right\}.$$

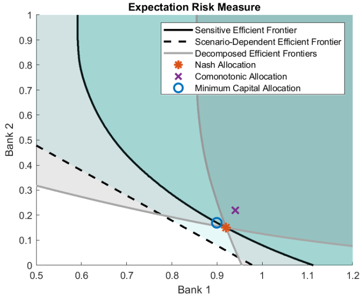
Then, r^* is a Nash allocation rule.

Eisenberg-Noe allocations: a two-bank system

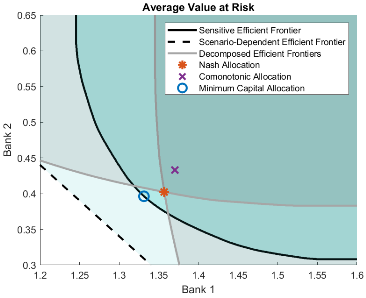
- **External Assets**: $X_i = (\sum_{j=0}^{N} \ell_{ij}) U_i$ for $U = (U_1, \ldots, U_N)$ Uniform with Gaussian copula with correlation $1/2$
- **Obligations** Two banks with

$$\ell = \begin{pmatrix} 0 & 1 & 1 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}.$$

- **Threshold payments** $\gamma = 95\%$



(a) Expectation risk measure



(b) Average Value-at-Risk

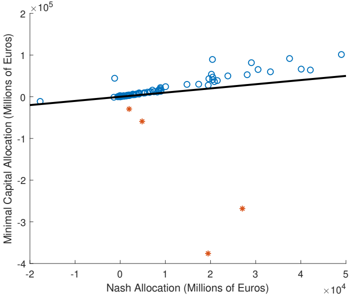
Eisenberg-Noe allocations: European Banking System

- Consider network of 87 banks from 2011 EBA Stress Tests
- Assume a *deterministic* 10% shock to total exposures at each bank
- **Threshold payments** $\gamma = 97.5\%$

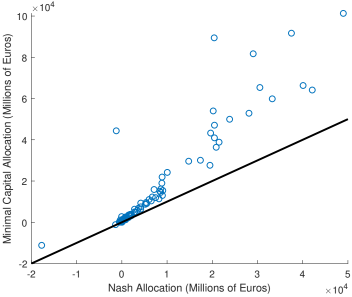
Eisenberg-Noe allocations: European Banking System

- Consider network of 87 banks from 2011 EBA Stress Tests
- Assume a *deterministic* 10% shock to total exposures at each bank
- **Threshold payments** $\gamma = 97.5\%$
- Total Nash Allocation: 728,321.1119 Millions of Euros
- Minimal Total Allocation: 728,316.4122 Millions of Euros

Eisenberg-Noe allocations: European Banking System



(a) All capital allocations.



(b) Only banks with smaller capital requirements under the Nash allocation.

- Black line: Nash = minimal
- Blue circles: banks where Nash $<$ minimal
- Red circles: banks where Nash $>$ minimal

This talk is based on:

- A. & Feinstein, "Short Communication: On the Separability of Vector-Valued Risk Measures", *SIAM Journal on Financial Mathematics*, **15**(4): SC68–SC79, 2024.
- A. & Feinstein, "Can Nash inform capital requirements? Allocating systemic risk measures", arXiv eprint 2504.20413, 2025.

Thank you!