Equilibrium Control Theory for Kihlstrom–Mirman (KM) Preferences in Continuous Time

Sascha Desmettre
(joint work with L. DGA, Y. Havrylenko, M. Steffensen)

Research Seminar of the Institute of Statistics and Mathematics @ WU Vienna

May 7, 2025

MATHEMATICS
JKU LINZ

LSQF LINZ SCHOOL OF QUANTITATIVE FINANCE

Introduction

- What is life?

Introduction

- What is life?
- It can be seen as a process of making **decisions** under **uncertainty**

Introduction

- What is life?
- It can be seen as a process of making **decisions** under **uncertainty**
- Mathematical modelling of decision makers' preferences is challenging

Introduction

- What is life?
- It can be seen as a process of making **decisions** under **uncertainty**
- Mathematical modelling of decision makers' preferences is challenging
- Decision makers (economic agents) may have **different attitudes toward**:

Introduction

- What is life?
- It can be seen as a process of making **decisions** under **uncertainty**
- Mathematical modelling of decision makers' preferences is challenging
- Decision makers (economic agents) may have **different attitudes toward**:
  - **time**, i.e., consume now or later

Introduction

- What is life?
- It can be seen as a process of making **decisions** under **uncertainty**
- Mathematical modelling of decision makers' preferences is challenging
- Decision makers (economic agents) may have **different attitudes toward**:
  - **time**, i.e., consume now or later
  - **risk**, i.e., take more (e.g., financial) risk or less risk

Introduction

- What is life?
- It can be seen as a process of making **decisions** under **uncertainty**
- Mathematical modelling of decision makers' preferences is challenging
- Decision makers (economic agents) may have **different attitudes toward**:
  - **time**, i.e., consume now or later
  - **risk**, i.e., take more (e.g., financial) risk or less risk
  - **temporal resolution of uncertainty**, i.e., is information measurable earlier or later

Examples of models that separate time and risk attitude

- Recursive utility (Kreps and Porteus (1978), Epstein and Zin (1989)):

$$J_{t_n}(c) = W\left(c_{t_n}, \mathcal{M}_{t_n}\left(J_{t_{n+1}}(c)\right)\right)$$

W = intertemporal aggregator
$\mathcal{M}_{t_n}$ = time-t_n certainty equivalent of future continuation value J_{t_n+1}

- Aggregation of CEs (Selden (1978), Jensen and Steffensen (2015)):

$$J_{t_n}(c) = \psi\left(\sum_{k=n}^{N} \delta^{t_k - t_n} \varphi\left(U^{-1}\left(\mathbb{E}_{t_n}\left[U\left(c_{t_k}\right)\right]\right)\right)\right)$$

Ψ: time-global transformation ; t_n : time preferences ; φ: pref. w.r.t. (local) CEs

- Kihlstrom–Mirman model (Kihlstrom and Mirman (1974), Kihlstrom (2009)):

$$J_{t_n}(c) = \mathbb{E}_{t_n}\left[\Phi\left(\sum_{k=n}^{N} H\left(t_k, c_{t_k}, t_n\right)\right)\right]$$

Φ: risk aggregation ; t_n : time preferences ; H: general utility including discounting

How to find "optimal" decisions in those models?

- Optimal control theory is the usual toolbox, which relies on the Bellman principle:

  "A plan for the future deemed optimal at an earlier point in time will remain optimal" - Björk et al. (2021)

  "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." – Bellman (1957)

- For recursive utility this principle applies, i.e., the problem is **time consistent**
- Aggregation of certainty equivalents and KM models are **time inconsistent** as current utility **cannot be separated** from future utility by construction:

  Decision optimal at an earlier point in time may become suboptimal later.

- What is the "optimal" behavior for time-inconsistent (TI) problems?

Time inconsistent decision-making problems

- Strotz (1956) identifies three types of behaviour in TI scenarios:

  1. pre-committment = follow the plan determined at the initial time

Time inconsistent decision-making problems

- Strotz (1956) identifies three types of behaviour in TI scenarios:

  1. pre-committment = follow the plan determined at the initial time
  2. myopic (naive) = recalculate the course of action at each time point

Time inconsistent decision-making problems

- Strotz (1956) identifies three types of behaviour in TI scenarios:

  1. pre-committment = follow the plan determined at the initial time
  2. myopic (naive) = recalculate the course of action at each time point
  3. consistent planning = problem as a non-cooperative game & reach an equilibrium

Time inconsistent decision-making problems

- Strotz (1956) identifies three types of behaviour in TI scenarios:

  1. pre-committment = follow the plan determined at the initial time
  2. myopic (naive) = recalculate the course of action at each time point
  3. consistent planning = problem as a non-cooperative game & reach an equilibrium

  ↷ In all 3 Bellman principle fails due to aggregation of future utilities.

Time inconsistent decision-making problems

- Strotz (1956) identifies three types of behaviour in TI scenarios:

  1. pre-committment = follow the plan determined at the initial time
  2. myopic (naive) = recalculate the course of action at each time point
  3. consistent planning = problem as a non-cooperative game & reach an equilibrium

  ↷ In all 3 Bellman principle fails due to aggregation of future utilities.

- We focus on **consistent planning strategies**, which we call **equilibrium ones**.
- To find equilibrium strategies, we use **equilibrium control theory**.
- Björk et al. (2021) study equilibrium strategies for KM preferences in discrete time
- We **generalize** the results of Björk et al. (2021) **to continuous time**.

Presentation overview

Life, preferences, and decisions

Kihlstrom–Mirman preferences in discrete time

Kihlstrom–Mirman preferences in continuous time

Application to a consumption-investment problem

Summary and concluding remarks

Basic notation

- $\mathcal{T} := [0, T]$ – the decision-making (DM) period, where $T \in (0, \infty)$
- $\{t_0 = 0, t_1, \ldots, t_N = T\}$ – a partition of the DM period
- $X^{\boldsymbol{u}} = \{X^{\boldsymbol{u}}_{t_n}\}_{n=0}^{N}$ – a controlled Markov process (state process)
- $\mathcal{X}$ – the set of values that $X^{\boldsymbol{u}}$ can attain
- $\boldsymbol{u} = \{\boldsymbol{u}_{t_n}(X^{\boldsymbol{u}}_{t_n})\}_{n=0}^{N}$ – a control law (in a feedback form as it depends on $X^{\boldsymbol{u}}_{t_n}$)
- $\boldsymbol{\mathcal{A}}$ – the set of admissible controls

Kihlstrom–Mirman preferences in discrete time

Back to KM CT Back to value function

- The general reward functional of an agent with KM preferences is:

$$J_{t_n}(x, \boldsymbol{u}) = \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X^{\boldsymbol{u}}_{t_k}, \boldsymbol{u}_{t_k}(X^{\boldsymbol{u}}_{t_k}), t_n\Big)\right)\right], \text{ where} \tag{1}$$

$H : \mathcal{T} \times \mathcal{C} \times \mathcal{T} \to \mathbb{R}$ is a discounted utility, $\Phi : \mathbb{R} \to \mathbb{R}$ is nonlinear and increasing

- Example – CRRA-CES specification of (1)

$$J_{t_n}(x, c) = \mathbb{E}_{t_n,x}\left[\frac{1}{1-\alpha}\left(\sum_{k=n}^{N} \delta^{t_k - t_n}\left(c_{t_k}\left(X^{\boldsymbol{u}}_{t_k}\right)\right)^{\rho}\right)^{\frac{1-\alpha}{\rho}}\right], \text{ where}$$

$\alpha \geq 0$ is RRA, $(1-\rho)^{-1}$ is elasticity of intertemporal substitution, $\rho < 1$.

- Attention: This is different from classical recursive utility (CRU), as in CRU the coefficents of RRA and ES are given as the inverse of each other!

CRRA-CES example: RRA as per Kihlstrom and Mirman (1974)

- We now have a different way of defining relative risk aversion (RRA) - one that generalizes the standard (single-argument) Arrow-Pratt measure of risk aversion!
- The *least concave representation* of a consumption bundle c is obtained by the aggregation of all discounted future consumption streams:

$$U^0_{t_n}(c) := \left(\sum_{k=n}^{N} \delta^{t_k - t_n} c^{\rho}_{t_k} \right)^{1/\rho}$$

- For a strictly concave function $v : \mathbb{R} \to \mathbb{R}$, the RRA of $U_{t_n}(c) := v\left(U^0_{t_n}(c)\right)$ is

$$-\frac{U^0_{t_n}(c)\, v''\left(U^0_{t_n}(c)\right)}{v'\left(U^0_{t_n}(c)\right)}$$

$\curvearrowright$ This is now based on a multi-attribute utility, which separates risk and time!

- Choosing $v(x) = x^{1-\alpha}/(1-\alpha)$ as a CRRA function, yields α as RRA coefficient

Equilibrium control - respectively Nash equilibrium

Fix $t_n \in \{0, t_1, \dots, T\}$, $x \in \mathcal{X}$, and a pair of controls $\boldsymbol{u}, \widehat{\boldsymbol{u}} \in \boldsymbol{\mathcal{A}}$. Define $\boldsymbol{u}^{t_n} \in \boldsymbol{\mathcal{A}}$ s.t.:

$$\boldsymbol{u}^{t_n}_{t_k}(x) = \begin{cases} \boldsymbol{u}_{t_n}(x), & \text{for } t_k = t_n, \quad \text{(yourself)} \\ \widehat{\boldsymbol{u}}_{t_k}(x), & \text{for } t_k \in \{t_{n+1}, \dots, T\}. \quad \text{(competitive agents)} \end{cases}$$

Equilibrium control - respectively Nash equilibrium

Fix $t_n \in \{0, t_1, \dots, T\}$, $x \in \mathcal{X}$, and a pair of controls $\boldsymbol{u}, \widehat{\boldsymbol{u}} \in \boldsymbol{\mathcal{A}}$. Define $\boldsymbol{u}^{t_n} \in \boldsymbol{\mathcal{A}}$ s.t.:

$$\boldsymbol{u}^{t_n}_{t_k}(x) = \begin{cases} \boldsymbol{u}_{t_n}(x), & \text{for } t_k = t_n, \quad \text{(yourself)} \\ \widehat{\boldsymbol{u}}_{t_k}(x), & \text{for } t_k \in \{t_{n+1}, \dots, T\}. \quad \text{(competitive agents)} \end{cases}$$

If, for every fixed $t_n \in \{0, t_1, \dots, T\}$, $x \in \mathcal{X}$, it holds that

$$\sup_{\boldsymbol{u}_{t_n}(x) \in \mathcal{A}(t_n, x)} J_{t_n}\left(x, \boldsymbol{u}^{t_n}\right) = J_{t_n}\left(x, \widehat{\boldsymbol{u}}\right),$$

then $\widehat{\boldsymbol{u}}$ is referred to as an **equilibrium control law** (maximizing w.r.t. to others).

$\rightarrow$ **Preview:** In CT we need to maximize a kind of first order derivative as in CT each agent has only impact on the control on a time set of Lebesgue measure zero.

Equilibrium value function and an auxiliary function

The **equilibrium value function** $\widehat{V} = \{\widehat{V}_{t_n}\}_{n=0}^{N}$ is given by:

$$\widehat{V}_{t_n}(x) := J_{t_n}(x, \widehat{\boldsymbol{u}}) \overset{(1)}{=} \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X_{t_k}^{\widehat{\boldsymbol{u}}}, \hat{\boldsymbol{u}}_{t_k}(X_{t_k}^{\widehat{\boldsymbol{u}}}), t_n\Big)\right)\right]$$

Equilibrium value function and an auxiliary function

The **equilibrium value function** $\widehat{V} = \{\widehat{V}_{t_n}\}_{n=0}^{N}$ is given by:

$$\widehat{V}_{t_n}(x) := J_{t_n}(x, \widehat{\boldsymbol{u}}) \stackrel{(1)}{=} \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X_{t_k}^{\widehat{\boldsymbol{u}}}, \hat{\boldsymbol{u}}_{t_k}(X_{t_k}^{\widehat{\boldsymbol{u}}}), t_n\Big)\right)\right]$$

For deriving recursions, we fix the dependence on current time $t_m \in \{0, t_1, \ldots, t_n\}$.

Equilibrium value function and an auxiliary function

The **equilibrium value function** $\widehat{V} = \{\widehat{V}_{t_n}\}_{n=0}^{N}$ is given by:

$$\widehat{V}_{t_n}(x) := J_{t_n}(x, \widehat{\boldsymbol{u}}) \overset{(1)}{=} \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X_{t_k}^{\widehat{\boldsymbol{u}}}, \widehat{\boldsymbol{u}}_{t_k}(X_{t_k}^{\widehat{\boldsymbol{u}}}), t_n\Big)\right)\right]$$

For deriving recursions, we fix the dependence on current time $t_m \in \{0, t_1, \dots, t_n\}$.

Define an auxiliary function $f^{\boldsymbol{u}} = \{f_{t_n}^{\boldsymbol{u}}\}_{n=0}^{N}$ for $t_m \in \{0, t_1, \dots, t_n\}$:

$$f_{t_n}^{\boldsymbol{u}}(x, z, t_m) := \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X_{t_k}^{\boldsymbol{u}}, \boldsymbol{u}_{t_k}(X_{t_k}^{\boldsymbol{u}}), t_m\Big) + z\right)\right],$$

for any $(x, z, t_m) \in \mathcal{X} \times \mathbb{R} \times \{0, t_1, \dots, t_n\}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$. Special cases:

$$f_{t_n}^{\boldsymbol{u}}(x, 0, t_n) = J_{t_n}(x, \boldsymbol{u}), \qquad f_{t_n}^{\widehat{\boldsymbol{u}}}(x, 0, t_n) = \widehat{V}_{t_n}(x)$$

Equilibrium value function and an auxiliary function

The **equilibrium value function** $\widehat{V} = \{\widehat{V}_{t_n}\}_{n=0}^N$ is given by:

$$\widehat{V}_{t_n}(x) := J_{t_n}(x, \widehat{\boldsymbol{u}}) \stackrel{(1)}{=} \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X_{t_k}^{\widehat{\boldsymbol{u}}}, \widehat{\boldsymbol{u}}_{t_k}(X_{t_k}^{\widehat{\boldsymbol{u}}}), t_n\Big)\right)\right]$$

For deriving recursions, we fix the dependence on current time $t_m \in \{0, t_1, \ldots, t_n\}$.

Define an auxiliary function $f^{\boldsymbol{u}} = \{f_{t_n}^{\boldsymbol{u}}\}_{n=0}^N$ for $t_m \in \{0, t_1, \ldots, t_n\}$:

$$f_{t_n}^{\boldsymbol{u}}(x, z, t_m) := \mathbb{E}_{t_n,x}\left[\Phi\left(\sum_{k=n}^{N} H\Big(t_k, X_{t_k}^{\boldsymbol{u}}, \boldsymbol{u}_{t_k}(X_{t_k}^{\boldsymbol{u}}), t_m\Big) + z\right)\right],$$

for any $(x, z, t_m) \in \mathcal{X} \times \mathbb{R} \times \{0, t_1, \ldots, t_n\}$ and $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$. Special cases:

$$f_{t_n}^{\boldsymbol{u}}(x, 0, t_n) = J_{t_n}(x, \boldsymbol{u}), \qquad f_{t_n}^{\widehat{\boldsymbol{u}}}(x, 0, t_n) = \widehat{V}_{t_n}(x)$$

Intuition for z: Tracking variable - keeps track of current utility H of intermediate decisions from t_n to t_{n+1} which cannot be factored out of CE at later points in time.

Extended Bellman system (Proposition 6.3 in Björk et al. (2021))

Using the tower property, $f^{\boldsymbol{u}}_{t_n}$ satisfies the following recursions:

$$
\begin{aligned}
f^{\widehat{\boldsymbol{u}}}_{t_n}(x, z, t_m) &= \mathbb{E}_{t_n, x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\widehat{\boldsymbol{u}}}_{t_{n+1}}, H\left(t_n, x, \widehat{\boldsymbol{u}}_{t_n}(x), t_m\right) + z, t_m\right)\right], \\
f^{\widehat{\boldsymbol{u}}}_{T}(x, z, t_m) &= \Phi\Big(H\big(T, x, \widehat{\boldsymbol{u}}_T(x), t_m\big) + z\Big).
\end{aligned}
\tag{2}
$$

Extended Bellman system (Proposition 6.3 in Björk et al. (2021))

Using the tower property, $f^{\boldsymbol{u}}_{t_n}$ satisfies the following recursions:

$$
\begin{aligned}
f^{\widehat{\boldsymbol{u}}}_{t_n}(x,z,t_m) &= \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\widehat{\boldsymbol{u}}}_{t_{n+1}}, H\left(t_n,x,\widehat{\boldsymbol{u}}_{t_n}(x),t_m\right)+z,t_m\right)\right], \\
f^{\widehat{\boldsymbol{u}}}_{T}(x,z,t_m) &= \Phi\Big(H\big(T,x,\widehat{\boldsymbol{u}}_T(x),t_m\big)+z\Big).
\end{aligned}
\tag{2}
$$

The extended Bellman system is given by (set $t_m = t_n$ and $z = 0$)

$$
\begin{aligned}
\widehat{V}_{t_n}(x) &= \sup_{\boldsymbol{u}_{t_n}(x)\in\mathcal{A}(t_n,x)} \mathbb{E}_{t_n,x}\Big[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H\left(t_n,x,\boldsymbol{u}_{t_n}(x),t_n\right),t_n\right)\Big], \\
\widehat{V}_T(x) &= \sup_{\boldsymbol{u}_T(x)\in\mathcal{A}(T,x)} \Phi\big(H(T,x,\boldsymbol{u}_T(x),T)\big).
\end{aligned}
\tag{3}
$$

Extended Bellman system (Proposition 6.3 in Björk et al. (2021))

Using the tower property, $f^{\boldsymbol{u}}_{t_n}$ satisfies the following recursions:

$$\begin{aligned} f^{\widehat{\boldsymbol{u}}}_{t_n}(x,z,t_m) &= \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\widehat{\boldsymbol{u}}}_{t_{n+1}}, H\left(t_n, x, \widehat{\boldsymbol{u}}_{t_n}(x), t_m\right) + z, t_m\right)\right], \\ f^{\widehat{\boldsymbol{u}}}_{T}(x,z,t_m) &= \Phi\Big(H\big(T, x, \widehat{\boldsymbol{u}}_T(x), t_m\big) + z\Big). \end{aligned} \tag{2}$$

The extended Bellman system is given by (set $t_m = t_n$ and $z = 0$)

$$\begin{aligned} \widehat{V}_{t_n}(x) &= \sup_{\boldsymbol{u}_{t_n}(x)\in\mathcal{A}(t_n,x)} \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H\left(t_n, x, \boldsymbol{u}_{t_n}(x), t_n\right), t_n\right)\right], \\ \widehat{V}_{T}(x) &= \sup_{\boldsymbol{u}_T(x)\in\mathcal{A}(T,x)} \Phi\big(H(T, x, \boldsymbol{u}_T(x), T)\big). \end{aligned} \tag{3}$$

Classic Bellman equation – for $\Phi(y) = y$ and $H(\dots) = \overline{H}(X^{\boldsymbol{u}}_{t_k}, \boldsymbol{u}_{t_k}(X^{\boldsymbol{u}}_{t_k}))$, i.e., no future dependence:

$$\widehat{V}_{t_n}(x) = \sup_{\boldsymbol{u}_{t_n}(x)\in\mathcal{A}(t_n,x)} \left\{\overline{H}\left(x, \boldsymbol{u}_{t_n}(x)\right) + \mathbb{E}_{t_n,x}\left[\widehat{V}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}\right)\right]\right\}.$$

Presentation overview

Life, preferences, and decisions

Kihlstrom–Mirman preferences in discrete time

Kihlstrom–Mirman preferences in continuous time

Application to a consumption-investment problem

Summary and concluding remarks

Kihlstrom–Mirman preferences in continuous time

- $(\Omega, \mathcal{F}, \mathbb{P})$ – probability space
- $\boldsymbol{\mathcal{F}} := (\mathcal{F}_t)_{t\in\mathcal{T}}$ – standard filtration generated by 1-dim. Wiener proc. $W := (W_t)_{t\in\mathcal{T}}$
- $X^{\boldsymbol{u}} := (X^{\boldsymbol{u}}_t)_{t\in\mathcal{T}}$ – controlled state process taking values in $\mathcal{X} \subseteq \mathbb{R}$, with SDE:

$$dX^{\boldsymbol{u}}_t = \mu(t, X^{\boldsymbol{u}}_t, \boldsymbol{u}(t, X^{\boldsymbol{u}}_t))dt + \sigma(t, X^{\boldsymbol{u}}_t, \boldsymbol{u}(t, X^{\boldsymbol{u}}_t))dW_t, \quad X^{\boldsymbol{u}}_0 = x_0 \in \mathcal{X}, \tag{4}$$

  where:
  - $\mu : \mathcal{T} \times \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}$ is a continuous mapping representing the controlled drift
  - $\sigma : \mathcal{T} \times \mathcal{X} \times \mathbb{R}^d \to \mathbb{R}$ is a continuous mapping representing the controlled volatility
  - $\boldsymbol{u} : \mathcal{T} \times \mathcal{X} \to \mathbb{R}^d$ is a control law chosen by the agent, for some dimensionality $d \in \mathbb{N}$
- Whenever there is no confusion, we will write for brevity $\boldsymbol{u}(t) := \boldsymbol{u}(t, X^{\boldsymbol{u}}_t)$

Kihlstrom–Mirman reward functional - cf. De Gennaro Aquino et al. (2024)

- The agent's reward functional is given by

$$J(t, x, \boldsymbol{u}) = \mathbb{E}_{t,x}\left[\Phi\left(\int_t^T H(s, X_s^{\boldsymbol{u}}, \boldsymbol{u}(s, X_s^{\boldsymbol{u}}), t)\, ds + G\left(X_T^{\boldsymbol{u}}, t\right)\right)\right], \qquad (5)$$

  where

  - $G : \mathcal{X} \times \mathcal{T} \to \mathbb{R}$ represents the discounted utility of terminal state
  - $H : \mathcal{T} \times \mathcal{X} \times \mathbb{R}^d \times \mathcal{T} \to \mathbb{R}$ is the discounted utility of intermediate decisions, continuous
  - $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing function

- Recall the DT KM-functional (1).

Admissible controls

- A control $\boldsymbol{u}$ is admissible if, for all $(t,x) \in \mathcal{T} \times \mathcal{X}$, the following conditions hold:

  (i) $\boldsymbol{u}(t,x) \in \mathcal{A}(t,x)$, where $\mathcal{A} : \mathcal{T} \times \mathcal{X} \to 2^{\mathbb{R}^k}$ is a continuous set-valued function representing the admissible values attained by $\boldsymbol{u}(t,x)$.

  (ii) The SDE (4) has a unique strong solution $X^{\boldsymbol{u}}$.

  (iii) $J(t,x,\boldsymbol{u})$ is well defined and finite.

- The set of admissible controls is denoted by $\boldsymbol{\mathcal{A}}$.

Equilibrium control

- Consider a point $(t, x) \in [0, T) \times \mathcal{X}$, a pair of controls $\widehat{\boldsymbol{u}}, \boldsymbol{u} \in \boldsymbol{\mathcal{A}}$, and a real number $h \in (0, T - t]$. Define a new control $\boldsymbol{u}_h \in \boldsymbol{\mathcal{A}}$ by setting

$$\boldsymbol{u}_h(s, y) = \begin{cases} \boldsymbol{u}(s, y), & \text{for } t \leq s < t + h, \quad y \in \mathcal{X}, \\ \widehat{\boldsymbol{u}}(s, y), & \text{for } t + h \leq s < T, \quad y \in \mathcal{X}. \end{cases}$$

- If for any $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$ and $(t, x) \in [0, T) \times \mathcal{X}$ the following inequality holds

$$\liminf_{h \downarrow 0} \frac{J(t, x, \widehat{\boldsymbol{u}}) - J(t, x, \boldsymbol{u}_h)}{h} \geq 0,$$

  then $\widehat{\boldsymbol{u}}$ is an **(intrapersonal) equilibrium control**.
- If we do not divide by h, in most models $\lim_{h \to 0} J(t, x, \widehat{\boldsymbol{u}}) - J(t, x, \boldsymbol{u}_h) = 0$
- Also called **intrapersonal equilibrium** since it can be viewed as a game between different future manifestations of one's own preferences.

Equilibrium value function and an auxiliary function

Back to verification Back to challenges

When $\widehat{\boldsymbol{u}}$ exists, the corresponding **equilibrium value function** $\widehat{V}$ is defined as

$$\widehat{V}(t,x) := J(t,x,\widehat{\boldsymbol{u}}) \tag{6}$$

For any $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$, $(t,x,z,\tau) \in \mathcal{T} \times \mathcal{X} \times \mathbb{R} \times [0,t]$, the **auxiliary function** $f^{\boldsymbol{u}}$ is defined by

$$f^{\boldsymbol{u}}(t,x,z,\tau) = \mathbb{E}_{t,x}\left[\Phi\left(\int_t^T H(s,X_s^{\boldsymbol{u}},\boldsymbol{u}(s),\tau)\,ds + G\left(X_T^{\boldsymbol{u}},\tau\right) + z\right)\right] \tag{7}$$

Analogue to DT, $\tau \in [0,t]$ fixes the dependence on current time (t_m in DT).

$$f^{\boldsymbol{u}}(t,x,0,t) = J(t,x,\boldsymbol{u}), \qquad f^{\widehat{\boldsymbol{u}}}(t,x,0,t) = J(t,x,\widehat{\boldsymbol{u}}) = \widehat{V}(t,x)$$

Remark: This is the exact CT analogue to the auxiliary function $f^{\boldsymbol{u}} = \{f^{\boldsymbol{u}}_{t_n}\}_{n=0}^N$.

Auxiliary stochastic processes and differential operator

- For an arbitrary fixed $\tau \in [0, t]$, introduce an auxiliary $Z^{\boldsymbol{u}} = (Z^{\boldsymbol{u}}_s)_{s \in [t,T]}$ with

$$dZ^{\boldsymbol{u}}_s = H(s, X^{\boldsymbol{u}}_s, \boldsymbol{u}(s), \tau)ds, \qquad Z^{\boldsymbol{u}}_t = z$$

  This stochastic process is the analogue to z and keeps track of the current utility from t to $t + h$ at later points in time .

- Let $\xi : (t, x, z, \tau) \in \mathcal{T} \times \mathcal{X} \times \mathbb{R} \times \mathcal{T} \mapsto \mathbb{R}$ such that $\xi \in \mathfrak{C}^{1,2,1,1}(\mathcal{T} \times \mathcal{X} \times \mathbb{R} \times \mathcal{T})$.

  Then for any $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$, the controlled infinitesimal operator $\mathcal{D}^{\boldsymbol{u}}$ applied to ξ is def. as

$$\begin{aligned} \mathcal{D}^{\boldsymbol{u}}\xi(t, x, z, \tau) = {} & \partial_t \xi(t, x, z, \tau) + \mu(t, x, \boldsymbol{u}(t, x))\partial_x \xi(t, x, z, \tau) \\ & + \frac{1}{2}\left(\sigma(t, x, \boldsymbol{u}(t, x))\right)^2 \partial_{xx}\xi(t, x, z, \tau) \\ & + H(t, x, \boldsymbol{u}(t, x), \tau)\partial_z \xi(t, x, z, \tau) + \partial_\tau \xi(t, x, z, \tau), \end{aligned}$$

  where $\partial_y \, \xi(y, \cdot)$ $(\partial_{yy} \, \xi(y, \cdot))$ denotes the partial derivatives in y.

Function space $\mathcal{L}^2(X^{\boldsymbol{u}})$

Consider an arbitrary $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$. ξ is said to belong to $\mathcal{L}^2(X^{\boldsymbol{u}})$ if:

1. $\xi \in \mathfrak{C}^{1,2,1,1}(\mathcal{T} \times \mathcal{X} \times \mathbb{R} \times \mathcal{T})$
2. for any $(t, x, z, \tau) \in [0, T) \times \mathcal{X} \times \mathbb{R} \times [0, t]$, there exists $\bar{h} \in (0, T-t)$ s. t.

$$\mathbb{E}_{t,x,z}\left[\sup_{0\le h\le \bar{h}}\left|\int_t^{t+h}\frac{1}{h}\mathcal{D}^{\boldsymbol{u}}\xi(s, X_s^{\boldsymbol{u}}, Z_s^{\boldsymbol{u}}, \tau)ds\right| + \int_t^{t+\bar{h}}\left(\partial_x\xi(s, X_s^{\boldsymbol{u}}, Z_s^{\boldsymbol{u}}, \tau)\sigma\left(s, X_s^{\boldsymbol{u}}, \boldsymbol{u}(s, X_s^{\boldsymbol{u}})\right)\right)^2 ds\right] < \infty,$$

where $\mathbb{E}_{t,x,z}[\cdot]$ denotes the conditional expectation given $X_t^{\boldsymbol{u}} = x$ and $Z_t^{\boldsymbol{u}} = z$

Characterization of the auxiliary function

- Let $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$. Then $f^{\boldsymbol{u}}$ satisfies the **recursion** (analogue to discrete time)

$$f^{\boldsymbol{u}}(t,x,z,\tau) = \mathbb{E}_{t,x}\left[f^{\boldsymbol{u}}\left(t+h, X^{\boldsymbol{u}}_{t+h}, \int_t^{t+h} H(s, X^{\boldsymbol{u}}_s, \boldsymbol{u}(s), \tau)\, ds + z, \tau \right)\right],$$

$$f^{\boldsymbol{u}}(T,x,z,\tau) = \Phi\left(G(x,\tau)+z\right)$$

for any $(t,x,z,\tau) \in [0,T) \times \mathcal{X} \times \mathbb{R} \times [0,t]$, and $h \in [0, T-t]$

- If $f^{\boldsymbol{u}|\tau} \in \mathcal{L}^2(X^{\boldsymbol{u}})$, then $f^{\boldsymbol{u}|\tau}$ solves

$$\mathcal{D}^{\boldsymbol{u}} f^{\boldsymbol{u}|\tau}(t,x,z) = 0,$$

where $f^{\boldsymbol{u}|\tau}(t,x,z)$ indicates that $\tau \in [0,t]$ is arbitrary but fixed.

The extended Hamilton-Jacobi-Bellman (HJB) system

- Starting with the discrete-time system (3), we can heuristically derive an extended HJB system by taking the proper limit of (3):

$$0 = \sup_{u \in \mathcal{A}(t,x)} \left\{ \mathcal{D}^u V(t,x) + \partial_z f(t,x,0,t) H(t,x,u,t) - \partial_\tau f(t,x,0,t) \right\}, \tag{S1}$$

$$0 = \mathcal{D}^{\widehat{\boldsymbol{u}}} f^{|\tau}(t,x,z), \tag{S2}$$

$$V(T,x) = \Phi\left(G\left(x,\ T\right)\right), \tag{S3}$$

$$f(T,x,z,\tau) = \Phi\left(G\left(x,\tau\right) + z\right), \tag{S4}$$

for $(t,x,z,\tau) \in [0,\ T) \times \mathcal{X} \times \mathbb{R} \times [0,t]$.

- The novelty lies in the term $\partial_z f(t,x,0,t) H(t,x,u,t)$.

- The time-inconsistency induced by the non-linearity of Φ is encoded in the z-derivative of f at $z = 0$, which becomes an adjustment factor of H.[1]

[1] When Φ is the identity function, i.e. $\Phi(y) = y$, we have $\partial_z f(t,x,0,t) = 1$.

Heuristic proof of the convergence from DT to CT

- Recall the discrete-time Bellman equation as in (3) with $u := \boldsymbol{u}_{t_n}(x)$:

$$\widehat{V}_{t_n}(x) = \sup_{u \in \mathcal{A}(t_n,x)} \mathbb{E}_{t_n,x}\left[f_{t_{n+1}}^{\widehat{\boldsymbol{u}}}\left(X_{t_{n+1}}^{\boldsymbol{u}^{t_n}}, H\left(t_n, x, u, t_n\right), t_n \right) \right]$$

Heuristic proof of the convergence from DT to CT

- Recall the discrete-time Bellman equation as in (3) with $u := \boldsymbol{u}_{t_n}(x)$:

$$\widehat{V}_{t_n}(x) = \sup_{u \in \mathcal{A}(t_n,x)} \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H\left(t_n, x, u, t_n\right), t_n \right) \right]$$

- Using $\mathbb{E}_{t_n,x}\left[\widehat{V}_{t_{n+1}}(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}) \right] = \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, 0, t_{n+1} \right) \right]$, we obtain

$$\widehat{V}_{t_n}(x) = \sup_{u \in \mathcal{A}(t_n,x)} \left\{ \mathbb{E}_{t_n,x}\left[\widehat{V}_{t_{n+1}}(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}) \right] + \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H(t_n, x, u, t_n), t_n \right) \right] \right.$$
$$\left. - \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, 0, t_{n+1} \right) \right] \right\}.$$

Heuristic proof of the convergence from DT to CT

- Recall the discrete-time Bellman equation as in (3) with $u := \boldsymbol{u}_{t_n}(x)$:

$$\widehat{V}_{t_n}(x) = \sup_{u \in \mathcal{A}(t_n,x)} \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H\left(t_n, x, u, t_n\right), t_n \right) \right]$$

- Using $\mathbb{E}_{t_n,x}\left[\widehat{V}_{t_{n+1}}(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}) \right] = \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, 0, t_{n+1} \right) \right]$, we obtain

$$\begin{aligned}\widehat{V}_{t_n}(x) = \sup_{u \in \mathcal{A}(t_n,x)} \Bigg\{ & \mathbb{E}_{t_n,x}\left[\widehat{V}_{t_{n+1}}(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}) \right] + \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H(t_n, x, u, t_n), t_n \right) \right] \\ & - \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, 0, t_{n+1} \right) \right] \Bigg\}.\end{aligned}$$

- Thus, for any $\boldsymbol{u}^{t_n}$, the following inequality holds:

$$\begin{aligned}0 \geq\ & \mathbb{E}_{t_n,x}\left[\widehat{V}_{t_{n+1}}(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}) \right] - \widehat{V}_{t_n}(x) \\ & + \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, H(t_n, x, \boldsymbol{u}^{t_n}_{t_n}(x), t_n), t_n \right) \right] - \mathbb{E}_{t_n,x}\left[f^{\widehat{\boldsymbol{u}}}_{t_{n+1}}\left(X^{\boldsymbol{u}^{t_n}}_{t_{n+1}}, 0, t_{n+1} \right) \right].\end{aligned}$$

Heuristic proof of the convergence from DT to CT

- We obtain for a general control $\boldsymbol{u}_h \in \boldsymbol{\mathcal{A}}$ by writing V instead of $\widehat{V}$, $\boldsymbol{u}_h$ instead of $\boldsymbol{u}^{t_n}$, and replacing t_n by t as well as t_{n+1} by $t+h$, with $h = t_{n+1} - t_n$:

$$
\begin{aligned}
0 \geq\ & \mathbb{E}_{t,x}\left[V(t+h, X^{\boldsymbol{u}_h}_{t+h})\right] - V(t,x) \\
& + \mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, \int_t^{t+h} H(s, X^{\boldsymbol{u}_h}_s, \boldsymbol{u}_h(s), t)\, ds, t\right)\right] - \mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, 0, t+h\right)\right]
\end{aligned}
$$

Heuristic proof of the convergence from DT to CT

- We obtain for a general control $\boldsymbol{u}_h \in \boldsymbol{\mathcal{A}}$ by writing V instead of $\widehat{V}$, $\boldsymbol{u}_h$ instead of $\boldsymbol{u}^{t_n}$, and replacing t_n by t as well as t_{n+1} by $t+h$, with $h = t_{n+1} - t_n$:

$$
\begin{aligned}
0 \geq\ & \mathbb{E}_{t,x}\left[V(t+h, X^{\boldsymbol{u}_h}_{t+h})\right] - V(t,x) \\
& + \mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, \int_t^{t+h} H(s, X^{\boldsymbol{u}_h}_s, \boldsymbol{u}_h(s), t)\, ds, t\right)\right] - \mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, 0, t+h\right)\right]
\end{aligned}
$$

- Subtracting and adding the term $\mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, 0, t\right)\right]$, dividing the above inequality by $h > 0$, and taking the limit as $h \downarrow 0$, we anticipate to obtain

$$
0 \geq \mathcal{D}^u V(t,x) + \partial_z f\left(t,x,0,t\right) H(t,x,u,t) - \partial_\tau f(t,x,0,t).
$$

Heuristic proof of the convergence from DT to CT

- We obtain for a general control $\boldsymbol{u}_h \in \boldsymbol{\mathcal{A}}$ by writing V instead of $\widehat{V}$, $\boldsymbol{u}_h$ instead of $\boldsymbol{u}^{t_n}$, and replacing t_n by t as well as t_{n+1} by $t+h$, with $h = t_{n+1} - t_n$:

$$0 \geq \mathbb{E}_{t,x}\left[V(t+h, X^{\boldsymbol{u}_h}_{t+h})\right] - V(t,x)$$

$$+ \mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, \int_t^{t+h} H(s, X^{\boldsymbol{u}_h}_s, \boldsymbol{u}_h(s), t)\, ds, t\right)\right] - \mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, 0, t+h\right)\right]$$

- Subtracting and adding the term $\mathbb{E}_{t,x}\left[f\left(t+h, X^{\boldsymbol{u}_h}_{t+h}, 0, t\right)\right]$, dividing the above inequality by $h > 0$, and taking the limit as $h \downarrow 0$, we anticipate to obtain

$$0 \geq \mathcal{D}^u V(t,x) + \partial_z f\left(t,x,0,t\right) H(t,x,u,t) - \partial_\tau f(t,x,0,t).$$

- Taking the supremum gives us the desired extended HJB equation.

Formal proof: Verification theorem

Assume that the following conditions are satisfied:

(C1) An admissible equilibrium control $\widehat{\boldsymbol{u}}$ exists and realizes the sup in $(S1)$.

(C2) $V(t,x)$ and $f(t,x,z,\tau)$ solve the extended HJB system $(S1)$-$(S4)$.

(C3) $V, f \in \mathcal{L}^2(X^{\boldsymbol{u}})$ for any $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$.

(C4) For any $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$, there exists $\overline{h} > 0$ such that

$$\sup_{h\in(0,\overline{h}),\, \eta:\Omega\to[t,t+h]} \mathbb{E}_{t,x,0}\left[\left|\partial_z f(t+h, X^{\boldsymbol{u}}_{t+h}, Z^{\boldsymbol{u}|t}_{\eta}, t) H(\eta, X^{\boldsymbol{u}}_{\eta}, \boldsymbol{u}(\eta, X^{\boldsymbol{u}}_{\eta}), t)\right|\right] < \infty.$$

(C5) For any $\boldsymbol{u} \in \boldsymbol{\mathcal{A}}$, there exists $\overline{h} > 0$ such that

$$\sup_{h\in(0,\overline{h}),\, \iota:\Omega\to[t,t+h]} \mathbb{E}_{t,x}\left[\left|\partial_\tau f(t+h, X^{\boldsymbol{u}}_{t+h}, 0, t)\right| + \left|f_{\tau\tau}(t+h, X^{\boldsymbol{u}}_{t+h}, 0, \iota) h\right|\right] < \infty.$$

Verification theorem

Then we have:

(R1) $f(t, x, z, \tau) = f^{\widehat{\boldsymbol{u}}}(t, x, z, \tau)$ and has the probabilistic representation (7).

(R2) $V(t, x) = J(t, x, \widehat{\boldsymbol{u}})$ for $\widehat{\boldsymbol{u}}$ realizing the sup in $(S1)$.

(R3) $\widehat{\boldsymbol{u}}$ is an equilibrium control.

(R4) $\widehat{V}(t, x) = V(t, x)$ is the equilibrium value function and has the probabilistic representation (6).

Presentation overview

Life, preferences, and decisions

Kihlstrom–Mirman preferences in discrete time

Kihlstrom–Mirman preferences in continuous time

Application to a consumption-investment problem

Summary and concluding remarks

Financial market, controls and controlled process

- Two assets – a risk-free asset B and a risky asset S with price dynamics

$$dB_t = B_t r dt, \quad B_0 = 1,$$
$$dS_t = S_t (r + \lambda) dt + S_t \sigma dW_t, \quad S_0 = s_0 \in \mathbb{R}^+,$$

- $\boldsymbol{u}(t,x) := (\pi(t,x), c(t,x))$, where x is the state variable (wealth) and:
  - $\pi(t,x)$ – the fraction of wealth invested in S
  - $c(t,x)$ – the consumption rate

- The controlled process (state, wealth) $\left(X_t^{\pi,c}\right)_{t\in\mathcal{T}}$ solves

$$dX_t^{\pi,c} = \left(X_t^{\pi,c}\left(r + \lambda\pi(t, X_t^{\pi,c})\right) - c(t, X_t^{\pi,c})\right)dt + X_t^{\pi,c}\pi(t, X_t^{\pi,c})\sigma dW_t,$$
$$X_0^{\pi,c} = x_0 \in \mathbb{R}^+.$$

Reward functional

Back to KM vs. ET

- In the general form (5) of the KM reward functional:

$$J(t,x,\boldsymbol{u}) = \mathbb{E}_{t,x}\left[\Phi\left(\int_t^T H(s,X_s^{\boldsymbol{u}},\boldsymbol{u}(s,X_s^{\boldsymbol{u}}),t)\,ds + G\left(X_T^{\boldsymbol{u}},t\right)\right)\right]$$

Reward functional

Back to KM vs. ET

- In the general form (5) of the KM reward functional:

$$J(t,x,\boldsymbol{u}) = \mathbb{E}_{t,x}\left[\Phi\left(\int_t^T H(s, X_s^{\boldsymbol{u}}, \boldsymbol{u}(s, X_s^{\boldsymbol{u}}), t)\, ds + G\left(X_T^{\boldsymbol{u}}, t\right)\right)\right]$$

set $\Phi(x) = \dfrac{1}{1-\alpha} x^{\frac{1-\alpha}{\rho}}$, $H(s,x,(\pi,c),t) = e^{-\delta(s-t)}c^{\rho}$, $G(x,t) = e^{-\delta(T-t)}x^{\rho}$.

- CRRA-CES reward functional in continuous time:

$$J(t,x,(\pi,c)) = \mathbb{E}_{t,x}\left[\frac{1}{1-\alpha}\left(\int_t^T e^{-\delta(s-t)}\left(c(s)\right)^{\rho}\, ds + e^{-\delta(T-t)}\left(X_T^{\pi,c}\right)^{\rho}\right)^{\frac{1-\alpha}{\rho}}\right] \tag{8}$$

Extended HJB system for CRRA-CES decision maker

The extended HJB system is given by

$$
\begin{aligned}
0 &= \sup_{(\pi,c)\in\mathcal{A}(t,x)} \Big\{ \partial_t V(t,x) + \partial_x V(t,x)(x(r+\pi\lambda)-c) + \frac{1}{2}\partial_{xx}V(t,x)\sigma^2\pi^2x^2 \\
&\qquad + \partial_z f(t,x,0,t)c^\rho - \partial_\tau f(t,x,0,t) \Big\}, \\
0 &= \mathcal{D}^{\widehat{\pi},\widehat{c}} f^{|\tau}(t,x,z), \\
&\quad V(T,x) = \frac{1}{1-\alpha}x^{1-\alpha}, \\
&\quad f(T,x,z,\tau) = \frac{1}{1-\alpha}\left(e^{-\delta(T-\tau)}x^\rho + z\right)^{\frac{1-\alpha}{\rho}}.
\end{aligned}
\tag{9}
$$

for all $(t,x,z,\tau) \in [0,T) \times \mathcal{X} \times \mathbb{R} \times [0,t]$.

Challenges and probabilistic representation

- (9) appears challenging, mainly due to f (function of 4 variables).

- We obtain an equivalent system using the probabilistic representation of V and f:

$$V(t,x) = \mathbb{E}_{t,x}\left[\frac{1}{1-\alpha}\left(\int_t^T e^{-\delta(s-t)}\left(\widehat{c}(s)\right)^{\rho}\,ds + e^{-\delta(T-t)}\left(X_T^{\widehat{\pi},\widehat{c}}\right)^{\rho}\right)^{\frac{1-\alpha}{\rho}}\right],$$

$$f(t,x,z,\tau) = \mathbb{E}_{t,x}\left[\frac{1}{1-\alpha}\left(\int_t^T e^{-\delta(s-\tau)}\left(\widehat{c}(s)\right)^{\rho}\,ds + e^{-\delta(T-\tau)}\left(X_T^{\widehat{\pi},\widehat{c}}\right)^{\rho} + z\right)^{\frac{1-\alpha}{\rho}}\right].$$

- Recall: $V(t,x) = J(t,x,\hat{\boldsymbol{u}})$ and $f(t,x,z,\tau)$ is given via (7).

Alternative characterization of an equilibrium control

Using the specific expressions for f_z and f_τ, the extended HJB system (9) can be written in the form

$$
\begin{aligned}
0 &= \sup_{(\pi,c)\in\mathcal{A}(t,x)} \Big\{ \partial_t V(t,x) + \partial_x V(t,x)(x(r+\pi\lambda)-c) + \frac{1}{2}\partial_{xx}V(t,x)\sigma^2\pi^2x^2 \\
&\qquad + \frac{1}{\rho}\widetilde{V}^{(1)}(t,x)c^\rho - \delta\frac{1-\alpha}{\rho}V(t,x)\Big\}, \\
0 &= \widetilde{V}^{(k)}_t(t,x) + \partial_x \widetilde{V}^{(k)}(t,x)(x(r+\widehat{\pi}\lambda)-\widehat{c}) + \frac{1}{2}\partial_{xx}\widetilde{V}^{(k)}(t,x)\sigma^2\widehat{\pi}^2x^2 \\
&\quad + \left(\frac{1-\alpha}{\rho}-k\right)\widetilde{V}^{(k+1)}(t,x)\widehat{c}^\rho - \delta\left(\frac{1-\alpha}{\rho}-k\right)\widetilde{V}^{(k)}(t,x), \\
V(T,x) &= \frac{1}{1-\alpha}x^{1-\alpha}, \\
\widetilde{V}^{(k)}(T,x) &= x^{1-\alpha-k\rho} \quad \text{for} \quad k\in\{1,2,3,\dots\}
\end{aligned}
\tag{10}
$$

This is an **infinte system of PDEs** as $\widetilde{V}^{(1)}_t$ depends recursively on all following $\widetilde{V}^{(k)}_t$!

An equilibrium investment-consumption strategy

By employing FOCs and separation of x and t we obtain:

$$\widehat{\pi}(t,x) = \frac{\lambda}{\alpha\sigma^2} \text{ and } \widehat{c}(t,x) = x\left(\frac{A(t)}{A^{(1)}(t)}\right)^{\frac{1}{\rho-1}},$$

where $A(t)$ and $A^{(1)}(t)$ are computed from the following system of ODEs:

$$0 = \partial_t A(t) + (1-\alpha)A(t)\left(r + \frac{\lambda^2}{2\alpha\sigma^2} - \frac{\delta}{\rho}\right) - (1-\alpha)\left(1 - \frac{1}{\rho}\right)(A(t))^{\frac{\rho}{\rho-1}}(A^{(1)}(t))^{-\frac{1}{\rho-1}},$$

$$\begin{aligned} 0 = {} & \partial_t A^{(k)}(t) + (1-\alpha-k\rho)A^{(k)}(t)\left(r + \frac{\lambda^2}{2\alpha\sigma^2} - \frac{\delta}{\rho} - k\rho\frac{\lambda^2}{2\alpha^2\sigma^2}\right) \\ & - (1-\alpha-k\rho)A^{(k)}(t)\left(\frac{A(t)}{A^{(1)}(t)}\right)^{\frac{1}{\rho-1}} + \left(\frac{1-\alpha}{\rho} - k\right)A^{(k+1)}(t)\left(\frac{A(t)}{A^{(1)}(t)}\right)^{\frac{\rho}{\rho-1}}, \\ & A(T) = 1, A^{(k)}(T) = 1, \quad k \in \{1,2,3,\dots\} \end{aligned}$$

Note: If $\frac{1-\alpha}{\rho} = k$, i.e. an integer, then the system is truncated at that integer!

Link to Epstein-Zin (EZ) utility

Back to KM vs. EZ

- Consider CRRA-CES continuous-time EZ utility:

$$J(t) = \mathbb{E}_{t,x}\left[\int_t^T m\big(c(s), J(s)\big)\,ds + \frac{1}{1-\alpha}\left(X_T^{\pi,c}\right)^{1-\alpha}\right],$$
$$\text{with} \quad m(c,J) := \frac{1-\alpha}{\rho}\delta J\left(c^{\rho}\left(\frac{1}{(1-\alpha)J}\right)^{\frac{\rho}{1-\alpha}} - 1\right), \tag{11}$$

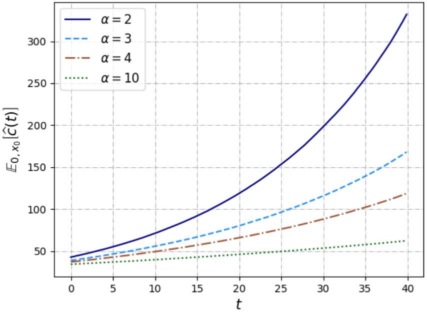
  where α = RRA, ρ = ES (= 1/RRA), and δ = time preference.

- The optimal decisions for an agent with (11) are known in closed form:

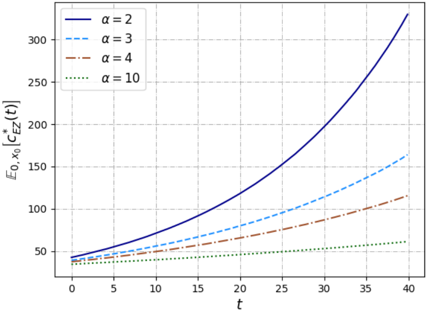
$$\pi^*_{EZ}(t,x) = \frac{\lambda}{\alpha\sigma^2}, \quad c^*_{EZ}(t,x) = \frac{x}{a(t)},$$
$$\text{where} \quad a(t) = \frac{1}{\nu} + \left(1 - \frac{1}{\nu}\right)e^{\nu(t-T)}, \quad \nu = \frac{\delta}{1-\rho} + \left(1 - \frac{1}{1-\rho}\right)\left(r + \frac{\lambda^2}{2\alpha\sigma^2}\right)$$

- **Note:** Yields time-dependent consumption as in KM model.

Average consumption (figures from De Gennaro Aquino et al. (2024))

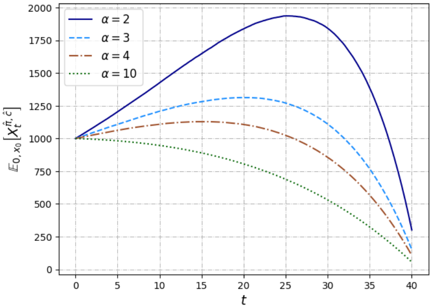


(a) Average consumption for KM preferences

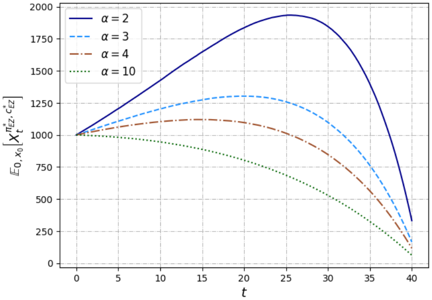


(b) Average consumption for EZ preferences

Average wealth (figures from De Gennaro Aquino et al. (2024))



(a) Average wealth for KM preferences



(b) Average wealth for EZ preferences

Comparison of KM and EZ preferences

- The actual values of consumption and wealth are very similar but not the same.
- Both models align on a fundamental level of constant EIS and RRA.
- For deterministic $\bar{c} := (\bar{c}_t)_{t \in \mathcal{T}}$, EZ and KM models yield the same decisions, as:
- In a first step, we obtain from (8) for deterministic $\bar{c}$:

$$U_t^{\bar{c}} := \frac{1}{1-\alpha} \left(\int_t^T e^{-\delta(s-t)} (\bar{c}_s)^\rho ds + e^{-\delta(T-t)} (X_T^{\bar{c}})^\rho \right)^{\frac{1-\alpha}{\rho}}$$

- In a second step, differentiating w.r.t. t and integration by parts we obtain

$$U_t^{\bar{c}} = \int_t^T \left(\frac{1}{\rho} (\bar{c}(s))^\rho \left((1-\alpha) U_s^{\bar{c}} \right)^{1-\frac{\rho}{1-\alpha}} - \delta \frac{1-\alpha}{\rho} U_s^{\bar{c}} \right) ds + \frac{1}{1-\alpha} (X_T^{\bar{c}})^{1-\alpha}.$$

- This is very similar to EZ preferences as given by (11).
- Consistent with observations made by Kihlstrom himself - cf. Kihlstrom (2015).

Presentation overview

Life, preferences, and decisions

Kihlstrom–Mirman preferences in discrete time

Kihlstrom–Mirman preferences in continuous time

Application to a consumption-investment problem

Summary and concluding remarks

Summary

- Kihlstrom-Mirman preferences :
  - separate risk aversion and the elasticity of intertemporal substitution
  - are time-inconsistent, i.e., do not admit the Bellman principle of optimality
- We derive and verify equilibrium decisions in continuous-time environments.
- Future research directions:
  - links between general KM preferences and EZ preferences on the fundamental level
  - forward-backward looking KM preferences:
    $\mathbb{E}_{t,x}\left[\Phi\left(\int_{s\geq 0} H(s,\boldsymbol{u}(s),X_s^{\boldsymbol{u}},t)ds\right)\right], \quad t\geq 0$
    $\curvearrowright$ This yields a path-dependent notion of TI.
  - implications for concrete decision-making problems in economics, finance, and insurance
  - Continuous notion(s) of temporal resolution of uncertainty.

Which lottery do you prefer? Illustrations taken from Strzalecki (2013).

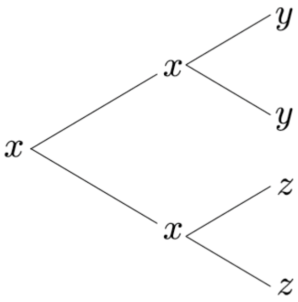


FIGURE 1.—Uncertainty resolves early.

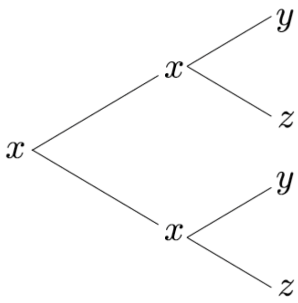


FIGURE 2.—Uncertainty resolves late.

- Say, you prefer info at time 2 rather than time 3 (early resolution of uncertainty).
- For $T > 3$, one would need to establish an ordering for every pair of dates.
- So it is unclear what happens if, say, one prefers info at 2 over 3, but 4 over 3.

References I

Björk, T., M. Khapko, and A. Murgoci (2021). *Time-Inconsistent Control Theory with Finance Applications*. Springer Cham.

De Gennaro Aquino, L., S. Desmettre, Y. Havrylenko, and M. Steffensen (2024). Equilibrium control theory for Kihlstrom-Mirman preferences in continuous time.

Epstein, L. G. and S. E. Zin (1989). Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica 57*(4), 937–969.

Jensen, N. R. and M. Steffensen (2015). Personal Finance and Life Insurance Under Separation of Risk Aversion and Elasticity of Substitution. *Insurance: Mathematics and Economics 62*, 28–41.

Kihlstrom, R. (2009). Risk Aversion and the Elasticity of Substitution in General Dynamic Portfolio Theory: Consistent Planning by Forward Looking, Expected Utility Maximizing Investors. *Journal of Mathematical Economics 45*(9-10), 634–663.

References II

Kihlstrom, R. E. (2015). Consumption Based Asset Pricing in a General Model of Expected Utility Maximizing Investors. *Preprint available at https://www.tse-fr.eu/sites/default/files/TSE/documents/sem2015/macro/kihlstrom.pdf*.

Kihlstrom, R. E. and L. J. Mirman (1974). Risk Aversion With Many Commodities. *Journal of Economic Theory 8*(3), 361–388.

Kreps, D. M. and E. L. Porteus (1978). Temporal Resolution of Uncertainty and Dynamic Choice Theory. *Econometrica 46*(1), 185–200.

Selden, L. (1978). A New Representation of Preferences Over "Certain x Uncertain" Consumption Pairs: The "Ordinal Certainty Equivalent" Hypothesis. *Econometrica 46*(5), 1045–1060.

Strotz, R. H. (1956). Myopia and Inconsistency in Dynamic Utility Maximization. *Review of Economic Studies 23*, 165–180.

Strzalecki, T. (2013). Temporal Resolution of Uncertainty and Recursive Models of Ambiguity Aversion. *Econometrica 81*(3), 1039–1074.

Thank you!

SCAN ME

Figure: Our paper as PDF: https://arxiv.org/pdf/2407.16525

Time-additivity of the RU functional and CRRA-CES specification

- Time-additivity is obtained if both the CE and aggregator are linear, i.e.,

$$\mathcal{M}_{t_n}\left(J_{t_{n+1}}(c)\right) = \mathbb{E}_{t_n}\left[J_{t_{n+1}}(c)\right] \text{ and}$$

$$W\left(c_{t_n}, \mathcal{M}_{t_n}\left(J_{t_{n+1}}(c)\right)\right) = H(c_{t_n}) + \delta^{t_{n+1}-t_n}\mathcal{M}_{t_n}\left(J_{t_{n+1}}(c)\right).$$

- For a CES aggregator and CRRA certainty equivalent, (3) assumes the popular specification

$$J_{t_n}(c) = \left(c_{t_n}^{\rho} + \delta^{t_{n+1}-t_n}\left(\mathbb{E}_{t_n}\left[\left(J_{t_{n+1}}(c)\right)^{1-\alpha}\right]\right)^{\frac{\rho}{1-\alpha}}\right)^{\frac{1}{\rho}}.$$