A BSDE-based optimal reinsurance in a model with jump clusters

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Outline

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Exploring the background

- The optimal reinsurance-investment problem is a fundamental research issue in actuarial science. Acquiring reinsurance serves as a crucial safeguard for insurers against adverse claims experiences.
- Considerable literature exists on this topic, under different criteria (e.g., minimizing ruin probability or maximizing expected utility). See for instance, among others, [Schmidli 2007], [Liang et al. IME 2014], [Zhang et al. IME 2009], [Zhu et al. IME 2015], [Brachetta et al. IME (2019)].
- Most of the literature relies on the classical Cramér-Lundberg model or its diffusion approximation.

- Classical models assume constant claims arrival intensity.
- This assumption is often far from realistic.

  Example: claims associated with natural catastrophes are in general affected by environmental stochastic factors;

- As about stochastic intensity models in non-life insurance:
  - Stochastic factor models: (Liang & Bayraktar, IME 2014), (Brachetta & Ceci, IME 2019);
  - Cox process with shot noise intensity: (Dassios & Jang, Finance Stoch., 2003), (Schmidt, Risks 2014);
  - Contagion models: (Cao, Landriault, & Li, IME 2020), (Brachetta, Callegaro, Ceci & Sgarra, Finance Stoch., 2023).

Optimal reinsurance with jump clusters

- Jump clustering effect: in catastrophic situations the jumps in the claims arrival process can exhibit clustering feature. We combine Cox with shot-noise intensity and Hawkes processes (with exponential kernel) and we get a shot-noise self-exciting counting process. This modeling framework is inspired by the concept initially proposed in Dassios and Zhao AAP 2011.

- in Brachetta, Callegaro, Ceci & Sgarra, Finance Stoch. 2023 the optimal reinsurance problem is analyzed under partial information via a BSDE-approach.
- in Ceci-Cretarola https://arxiv.org/abs/2404.11482 the problem is discussed under full information with two methodologies: HJB-approach and BSDE-approach.

Outline

The dynamic contagion claim model

On $(\Omega, \mathcal{F}, \mathbf{P}; \mathbb{F})$ with $T > 0$ the maturity of a reinsurance contract, introduce the cumulative claim process $C = \{C_t,\ t \in [0, T]\}$:

$$C_t = \sum_{n=1}^{N_t^{(1)}} \underbrace{Z_n^{(1)}}_{claims\ size} = \sum_{n \geq 1} Z_n^{(1)} 1_{\{T_n^{(1)} \leq t\}}$$

where the claims arrival process $N_t^{(1)} = \sum_{n \geq 1} 1_{\{T_n^{(1)} \leq t\}}$ is a point process with stochastic intensity:

$$\lambda_t = \beta + (\lambda_0 - \beta) e^{-\alpha t} + \sum_{j=1}^{N_t^{(1)}} e^{-\alpha(t - T_j^{(1)})} \underbrace{\ell(Z_j^{(1)})}_{Int-exc.jump} + \sum_{j=1}^{N_t^{(2)}} e^{-\alpha(t - T_j^{(2)})} \underbrace{Z_j^{(2)}}_{Ext-exc.jump}$$

Assumption

$N^{(2)}$ Poisson process with intensity $\rho > 0$; $\{Z_n^{(1)}\}_{n \geq 1}$ ($\{Z_n^{(2)}\}_{n \geq 1}$) i.i.d. $\mathbb{R}^+$-valued rvs with distribution function $F^{(1)}$ ($F^{(2)}$). $N^{(2)}$, $\{Z_n^{(1)}\}_{n \geq 1}$ and $\{Z_n^{(2)}\}_{n \geq 1}$ are independent.

The integer-valued random measures

- We introduce the random measures

$$m^{(i)}(\mathrm{d}t, \mathrm{d}z) = \sum_{n \geq 1} \delta_{(T_n^{(i)}, Z_n^{(i)})}(\mathrm{d}t, \mathrm{d}z) \mathbb{1}_{\{T_n^{(i)} < +\infty\}}, \quad i = 1, 2$$

- The predictable projections measures (the so-called compensator measures) of $m^{(1)}(\mathrm{d}t, \mathrm{d}z)$ and $m^{(2)}(\mathrm{d}t, \mathrm{d}z)$ are

$$\nu^{(1)}(\mathrm{d}t, \mathrm{d}z) = \lambda_{t^-} F^{(1)}(\mathrm{d}z)\mathrm{d}t, \quad \nu^{(2)}(\mathrm{d}t, \mathrm{d}z) = \rho F^{(2)}(\mathrm{d}z)\mathrm{d}t.$$

  In particular, λ_{t^-} is the intensity of $N_t^{(1)}$ hence $\mathbb{E}[N_t^{(1)}] = \mathbb{E}[\int_0^t \lambda_s \mathrm{d}s]$.

- The compensated random measures:

$$\widetilde{m}^{(i)}(\mathrm{d}t, \mathrm{d}z) := m^{(i)}(\mathrm{d}t, \mathrm{d}z) - \nu^{(1)}(\mathrm{d}t, \mathrm{d}z), \quad i = 1, 2$$

For any $\mathbb{F}$-predictable nonnegative random field $\{H(t,z),\ t \in [0,T],\ z \in [0,+\infty)\}$, $i = 1,2$, $t \in [0,T]$

$$\mathbb{E}\left[\int_0^t \int_0^{+\infty} H(s,z) m^{(i)}(\mathrm{d}s,\mathrm{d}z)\right] = \mathbb{E}\left[\int_0^t \int_0^{+\infty} H(s,z) \nu^{(i)}(\mathrm{d}s,\mathrm{d}z)\right].$$

Moreover, under the condition

$$\mathbb{E}\left[\int_0^T \int_0^{+\infty} |H(s,z)| \nu^{(i)}(\mathrm{d}s,\mathrm{d}z)\right] < +\infty,$$

the process

$$\left\{\int_0^t \int_0^{+\infty} H(s,z) \underbrace{\left(m^{(i)}(\mathrm{d}s,\mathrm{d}z) - \nu^{(i)}(\mathrm{d}s,\mathrm{d}z)\right)}_{\widetilde{m}^{(i)}(\mathrm{d}t,\mathrm{d}z)},\ t \in [0,T]\right\},$$

is an $(\mathbb{F},\mathbf{P})$-martingale.

The reinsurance contract

The insurer selects a reinsurance strategy $\{u_t,\ t \in [0,T]\}$, so that the aggregate losses covered by the insurer are

$$C_t^u = \sum_{j=1}^{N_t^{(1)}} \Phi(Z_j^{(1)}, u_{T_j^{(1)}}) = \int_0^t \int_0^{+\infty} \Phi(z, u_s) m^{(1)}(\mathrm{d}s, \mathrm{d}z), \quad t \in [0,T],$$

(the remaining $C_t - C_t^u$ will be undertaken by the reinsurer). We assume:

- The retention function $\Phi(z,u)$ is continuous in $u \in U$;
- There exists at least two points u_N and $u_M \in U$ such that

$$0 \leq \Phi(z, u_M) \leq \Phi(z,u) \leq \Phi(z, u_N) = z, \quad \forall u \in U$$

(u_M=maximal reinsurance, u_N=null reinsurance).

Different type of contracts

a) Proportional reinsurance: the insurer transfers a percentage $1 - u$ of any future loss to the reinsurer, so $U = [0, 1]$ and $\Phi(z, u) = uz$.

b) Excess-of-loss: the reinsurer covers all the losses exceeding a threshold u, hence $U = [0, +\infty]$ and $\Phi(z, u) = u \wedge z$.

c) Limited stop-loss reinsurance: the reinsurer covers the losses exceeding a threshold u_1, up to a maximum level $u_2 > u_1$, so that the maximum loss is limited to $(u_2 - u_1)$ on the reinsurer's side. In this case: $\Phi(z, u) = z - (z - u_1)^+ + (z - u_2)^+$, so that $U = \{(u_1, u_2) : u_1 \geq 0, u_2 \in [u_1, +\infty]\}$ and $u = (u_1, u_2)$. Here $u_M = (u_{M,1}, u_{M,2}) = (0, +\infty)$ and u_N can be any point on the line $u_1 = u_2$.

d) Limited stop-loss with fixed reinsurance coverage: $u_2 = u_1 + \beta$, $\beta > 0$. Here $U = [0, +\infty]$, $u_N = +\infty$ and $u_M = 0$ corresponds to the maximum reinsurance coverage β.

The surplus and the wealth

Under $\{u_t,\ t \in [0,T]\}$, the surplus process R^u of the primary insurer follows:

$$dR_t^u = \left(c_t - q_t^u\right) dt - dC_t^u, \quad R_0^u = R_0 \in \mathbb{R}^+$$

with $\mathbb{F}$-predictable processes

- c_t insurance premium rate;
- q_t^u the reinsurance premium rate.

The insurance company invests its surplus in a risk-free asset with interest rate $r > 0$, so that the wealth is $X_0^u = R_0 \in \mathbb{R}^+$

$$dX_t^u = dR_t^u + rX_t^u\, dt = (c_t - q_t^u)\, dt - \int_0^{+\infty} \Phi(z, u_t)\, m^{(1)}(dt, dz) + rX_t^u\, dt$$

Problem formulation

The insurance company aims at solving (with $\eta > 0$ the insurer's risk aversion)

$$\sup_{u \in \mathcal{U}} \mathbb{E}\left[1 - e^{-\eta X_T^u}\right] = 1 - \inf_{u \in \mathcal{U}} \mathbb{E}\left[e^{-\eta X_T^u}\right]$$

Definition (Admissible strategies)

$\mathcal{U}$: all the U-valued, $\mathbb{F}$-predictable processes s.t. $\mathbb{E}\left[e^{-\eta X_T^u}\right] < +\infty$.

Assumption

For every $a > 0$:

$$\mathbb{E}\left[e^{a\ell(Z^{(1)})}\right] < \infty, \ \mathbb{E}\left[e^{aZ^{(1)}}\right] < \infty \ \mathbb{E}\left[e^{aZ^{(2)}}\right] < \infty, \ \mathbb{E}\left[e^{a \int_0^T q_t^{u_M} \, dt}\right] < \infty.$$

Under these assumptions, every U-valued $\mathbb{F}$-predictable process is admissible and $\mathbb{E}\left[e^{aC_T}\right] < \infty$, $\mathbb{E}\left[e^{a \int_0^T \lambda_t \, dt}\right] < \infty$, for every $a > 0$.

Premium principles

- The expected cumulative losses

$$\mathbb{E}[C_t] = \mathbb{E}\left[\int_0^t \int_0^{+\infty} z m^{(1)}(\mathrm{d}s, \mathrm{d}z)\right] = \mathbb{E}\left[\int_0^t \lambda_s \mathrm{d}s\right] \mathbb{E}[Z^{(1)}]$$

- According to the *expected value principle (EVP)*, the insurance premium c is given by

$$c_t = (1+\theta_I)\lambda_{t^-} \int_0^{+\infty} z F^{(1)}(\mathrm{d}z) = (1+\theta_I)\lambda_{t^-} \mathbb{E}[Z^{(1)}]$$

where $\theta_I > 0$ denotes the safety loading applied by the insurer. This implies that *the net profit condition* holds

$$\mathbb{E}[\int_0^t c_s \mathrm{d}s] = (1+\theta_I)\mathbb{E}[C_t] > \mathbb{E}[C_t]$$

Premium principles

- Under the *expected value principle (EVP)* the reinsurance premium q^u is given by

$$q_t^u = (1+\theta_R)\lambda_{t^-} \int_0^{+\infty} (z - \Phi(z, u_t))\, F^{(1)}(\mathrm{d}z)$$

where $\theta_R > 0$ denotes the safety loading applied by reinsurer.

- This implies that for any $u \in \mathcal{U}$

$$\mathbb{E}\left[\int_0^t q_s^u \mathrm{d}s\right] = (1+\theta_R)\mathbb{E}[\underbrace{C_t - C_t^u}_{\text{losses covered by reinsurer}}].$$

Premium principles

- Under the *variance premium principle (VPP)*, the insurance and reinsurance premiums are given by

$$c_t = \lambda_{t^-} \left\{ \int_0^{+\infty} z F^{(1)}(\mathrm{d}z) + \eta_I \int_0^{+\infty} z^2 F^{(1)}(\mathrm{d}z) \right\}$$

$$q_t^u = \lambda_{t^-} \left\{ \int_0^{+\infty} (z - \Phi(z, u_t))\, F^{(1)}(\mathrm{d}z) + \eta_R \int_0^{+\infty} (z - \Phi(z, u_t))^2\, F^{(1)}(\mathrm{d}z) \right\},$$

  respectively, where $\eta_I > 0$ and $\eta_R > 0$ are the variance loadings applied by insurer and reinsurer, respectively.

- Thus for any $u \in \mathcal{U}$ and $t \in [0, T]$ the *the net profit condition* holds

$$\mathbb{E}\left[\int_0^t c_s \mathrm{d}s \right] > \mathbb{E}[C_t]$$

$$\mathbb{E}\left[\int_0^t q_s^u \mathrm{d}s \right] > \mathbb{E}[C_t - C_t^u].$$

Outline

HJB-approach

- Assuming $c_t = c(t, \lambda_t)$ and $q_t^u = q(t, u_t, \lambda_t)$, for each $t \in [0, T]$;
- (X^u, λ) is a Markov process (for any constant or Markovian control);
- Value function:

$$v(t, x, \lambda) = \inf_{u \in \mathcal{U}} \mathbb{E}_{t,x,\lambda}\left[e^{-\eta X_T^u}\right], \quad (t, x, \lambda) \in [0, T) \times \mathbb{R} \times (0, +\infty),$$

where the notation $\mathbb{E}_{t,x,\lambda}[\cdot]$ stands for the expectation with initial data $(t, x, \lambda) \in [0, T] \times \mathbb{R} \times (0, +\infty)$.

- If $v(t, x, \lambda)$ is sufficiently regular it solves the Hamilton-Jacobi-Bellman equation:

$$\inf_{u \in U} \mathcal{L}^{X,\lambda,u} v(t, x, \lambda) = 0, \quad v(T, x, \lambda) = e^{-\eta x}$$

where $\mathcal{L}^{X,\lambda,u}$ denotes the Markov generator of the pair (X^u, λ) associated to a constant control $u \in U$.

- We can prove that $v(t,x,\lambda) = e^{-\eta x e^{r(T-t)}} \varphi(t,\lambda)$ with

$$\varphi(t,\lambda) = \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,\lambda}\left[e^{-\eta \int_t^T e^{r(T-s)}(c_s - q_s^u)\,\mathrm{d}s + \eta \int_t^T \int_0^{+\infty} e^{r(T-s)} \Phi(z,u_s)\, m^{(1)}(\mathrm{d}s,\mathrm{d}z)} \right]$$

- The reduced HJB equation:

$$\frac{\partial \varphi}{\partial t}(t,\lambda) + \alpha(\beta - \lambda)\frac{\partial \varphi}{\partial \lambda}(t,\lambda) + \int_0^{+\infty} [\varphi(t,\lambda+z) - \varphi(t,\lambda)]\, \rho F^{(2)}(\mathrm{d}z)$$
$$- \eta e^{r(T-t)} \varphi(t,\lambda) c(t,\lambda) + \inf_{u \in U} \Psi^u(t,\lambda) = 0, \tag{1}$$

with final condition $\varphi(T,\lambda) = 1$, where the function Ψ^u is given by

$$\Psi^u(t,\lambda) = \eta e^{r(T-t)} \varphi(t,\lambda) q(t,\lambda,u)$$
$$+ \int_0^{+\infty} \left[e^{\eta \Phi(z,u) e^{r(T-t)}} \varphi(t,\lambda + \ell(z)) - \varphi(t,\lambda) \right] \lambda F^{(1)}(\mathrm{d}z).$$

Theorem (Verification Theorem)

Let $\widetilde{\varphi} \in C^1((0,T) \times (0,+\infty)) \cap C([0,T] \times (0,+\infty))$ *be a classical solution of the HJB equation* (1).
Let $\widetilde{v}(t,x,\lambda) = e^{-\eta x e^{r(T-t)}} \widetilde{\varphi}(t,\lambda)$ *and assume that for any* $u \in \mathcal{U}$ *the family* $\{\widetilde{v}(\tau, X^u_\tau, \lambda_\tau);\ \tau \text{ stopping time},\ \tau \leq T\}$ *is uniformly integrable.*
Let $u^*(t,\lambda)$ *be a minimizer of* $\Psi^u(t,\lambda)$.
Then $\widetilde{v}(t,x,\lambda) = v(t,x,\lambda)$ *is the value function. Furthermore,* $u^*_t = u^*(t,\lambda_{t-}) \in \mathcal{U}$ *is an optimal (Markovian) strategy.*

Proposition (The optimal strategy)

Under the assumptions of the Verification Theorem. Suppose moreover that $\Phi(z,u)$ *is differentiable in* u *for almost every* z *and* $\Psi^u(t,\lambda)$ *is strictly concave in* $u \in [u_M, u_N]$. *Then, the optimal reinsurance strategy* $u_t^* = \{u^*(t,\lambda_{t^-}), t \in [0,T]\}$ *is given by*

$$u^*(t,\lambda_{t^-}) = \begin{cases} u_M & (t,\lambda_{t^-}) \in A_0 \\ u_N & (t,\lambda_{t^-}) \in A_1, \\ \bar{u}(t,\lambda_{t^-}) & \textit{otherwise} \end{cases}$$

where $A_0 = \{(t,\lambda) : h(t,\lambda,u_M) \le 0\}$, $A_1 = \{(t,\lambda) : h(t,\lambda,u_N) \ge 0\}$, *with*

$$h(t,\lambda,u) = -\varphi(t,\lambda)\frac{\partial q(u,\lambda)}{\partial u} - \int_0^\infty \varphi(t,\lambda + l(z)) e^{\eta e^{r(T-t)}\Phi(z,u)} \frac{\partial \Phi(z,u)}{\partial u} \lambda F^{(1)}(dz)$$

and $\bar{u}(t,\lambda) \in (u_M, u_N)$ *solves the following equation:*

$$-\varphi(t,\lambda)\frac{\partial q(\lambda,u)}{\partial u} = \int_0^\infty \varphi(t,\lambda + l(z)) e^{\eta e^{r(T-t)}\Phi(z,u)} \frac{\partial \Phi(z,u)}{\partial u} \lambda F^{(1)}(dz).$$

Some problems...

- Regularity of the value function;
- The verification approach requires to prove existence and uniqueness of the solution to Eq.(1) (partial integro-differential equation with an embedded optimization).

Outline

The BSDE-approach

We define, for $\mathcal{U}(t,u) = \left\{ \bar{u} \in \mathcal{U} : \bar{u}_s = u_s \text{ a.s., } s \leq t \leq T \right\}$, the Snell envelope (see, N. El Karuoi (1981))

$$W_t^u = \operatorname*{ess\,inf}_{\bar{u} \in \mathcal{U}(t,u)} \mathbb{E}\left[e^{-\eta X_T^{\bar{u}}} \Big| \mathcal{F}_t \right],$$

so that if $\widehat{X}_t^u := e^{-rt} X_t^u$ is the discounted wealth, then

$$W_t^u = e^{-\eta \widehat{X}_t^u e^{rT}} \varphi(t, \lambda_t),$$

for every $u \in \mathcal{U}$. Where

$$\varphi(t,\lambda) = \inf_{u \in \mathcal{U}_t} \mathbb{E}_{t,\lambda}\left[e^{-\eta \int_t^T e^{r(T-s)} (c(s,\lambda_s) - q(s,\lambda_s,u_s))\, \mathrm{d}s + \eta \int_t^T \int_0^{+\infty} e^{r(T-s)} \Phi(z,u_s)\, m^{(1)}(\mathrm{d}s,\mathrm{d}z)} \right]$$

In particular, choosing $u = u_N$ (null reinsurance) the value process $\varphi(t, \lambda_t)$ and the Snell envelope associated to null reinsurance ${W_t}^N$ satisfy

$$\varphi(t, \lambda_t) = e^{\eta {\widehat{X}_t}^{u_N} e^{rT}} {W_t}^N.$$

Idea

To develop a BSDEs characterization for $\{W_t^N,\ t \in [0,T]\}$ (the Snell envelope associated to null reinsurance) to get a complete description of $\{\varphi(t,\lambda_t),\ t \in [0,T]\}$ and of the optimal control, without needing the regularity of $\varphi(t,\lambda)$.

We define three classes of stochastic processes

- $\mathcal{S}^2$ denotes the space of càdlàg $\mathbb{F}$-adapted processes Y such that:

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]}|Y_t|\right)^2\right] < +\infty.$$

- $\mathcal{L}^2$ denotes the space of càdlàg $\mathbb{F}$-adapted processes Y such that:

$$\mathbb{E}\left[\int_0^T |Y_t|^2 \mathrm{d}t\right] < +\infty.$$

- $\widehat{\mathcal{L}}^{(1)}$ ($\widehat{\mathcal{L}}^{(2)}$) denotes the space of $[0,+\infty)$-indexed $\mathbb{F}$-predictable random fields $\Theta = \{\Theta_t(z), t \in [0,T], z \in [0,+\infty)\}$ such that:

$$\mathbb{E}\left[\int_0^T \int_0^{+\infty} \Theta_t^2(z)\lambda_{t^-} F^{(1)}(\mathrm{d}z)\,\mathrm{d}t\right] < +\infty$$

$$\left(\mathbb{E}\left[\int_0^T \int_0^{+\infty} \Theta_t^2(z)\rho F^{(2)}(\mathrm{d}z)\,\mathrm{d}t\right] < +\infty \quad \text{respectively}\right).$$

The two-dimensional random measure

Let $Z = (C^{(1)}, C^{(2)})$, $C_t^{(1)} = C_t = \sum_{n=1}^{N_t^{(1)}} Z_n^{(1)}$, $\quad C_t^{(2)} = \sum_{n=1}^{N_t^{(2)}} Z_n^{(2)}$ and $m(\mathrm{d}t, \mathrm{d}z_1, \mathrm{d}z_2)$ the associated integer-valued measure. Since $C^{(1)}$ and $C^{(2)}$ have not common jump times, then

$$m(\mathrm{d}t, \mathrm{d}z_1, \mathrm{d}z_2) = m^{(1)}(\mathrm{d}t, \mathrm{d}z_1)\delta_0(\mathrm{d}z_2) + m^{(2)}(\mathrm{d}t, \mathrm{d}z_2)\delta_0(\mathrm{d}z_1)$$

and the $\mathbb{F}$-dual predictable projection is given by

$$\nu(\mathrm{d}t, \mathrm{d}z_1, \mathrm{d}z_2) = \lambda_{t-} F^{(1)}(\mathrm{d}z_1)\delta_0(\mathrm{d}z_2) + \rho F^{(2)}(\mathrm{d}z_2)\delta_0(\mathrm{d}z_1).$$

Proposition (Martingale representation theorem)

Let $\mathbb{F} = \mathbb{F}^{m^{(1)}} \vee \mathbb{F}^{m^{(2)}}$. *Any square-integrable* $(\mathbb{F}, \mathbf{P})$*-martingale* $M = \{M_t, t \in [0, T]\}$ *has the following representation*

$$M_t = M_0 + \int_0^t \int_0^{+\infty} \Gamma_s^{(1)}(z)\widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_0^{+\infty} \Gamma_s^{(2)}(z)\widetilde{m}^{(2)}(\mathrm{d}s, \mathrm{d}z), \quad (2)$$

$\Gamma_s^{(i)}(z) \in \widehat{\mathcal{L}}^{(i)}$, $i = 1, 2$.

Theorem (Main result)

i) $(W^N, \Theta^{(1)}, \Theta^{(2)}) \in \mathcal{S}^2 \times \widehat{\mathcal{L}}^{(1)} \times \widehat{\mathcal{L}}^{(2)}$ *is the unique solution to BSDE*

$$\begin{aligned} W_t^N = \xi - \int_t^T \int_0^{+\infty} \Theta_s^{(1)}(z)\widetilde{m}^{(1)}(\mathrm{d}s, \mathrm{d}z) - \int_t^T \int_0^{+\infty} \Theta_s^{(2)}(z)\widetilde{m}^{(2)}(\mathrm{d}s, \mathrm{d}z) \\ - \int_t^T \operatorname*{ess\,sup}_{u \in \mathcal{U}} f(s, W_{s-}^N, \Theta_s^{(1)}(\cdot), u_s)\,\mathrm{d}s, \end{aligned} \tag{3}$$

with terminal condition $\xi = e^{-\eta X_T^N}$, *where*

$$\begin{aligned} f(t, W_{t-}^N, \Theta_t^{(1)}(\cdot), u_t) = -W_{t-}^N \eta e^{r(T-t)} q_t^u \\ + \int_0^{+\infty} [W_{t-}^N + \Theta_t^{(1)}(z)]\left[1 - e^{-\eta e^{r(T-t)}(z - \Phi(z, u_t))}\right] \lambda_{t-} F^{(1)}(\mathrm{d}z). \end{aligned} \tag{4}$$

ii) *Any process* $u^* \in \mathcal{U}$ *which maximizes* $f(t, W_t^N, \Theta_t^{(1)}(\cdot), u_t)$ *furnishes an optimal reinsurance strategy.*

Idea of the proof

- Existence of a solution of BSDE (3):
  - The generator of the BSDE satisfies a stochastic Lipschitz condition;
  - We apply [Theorem 3.5 in Papapantoleon et al. EJP 2018]

- Verification Result:
  Let Y be a solution to BSDE (3) and $u^* \in \mathcal{U}$ which attains the ess-sup. Then $Y_t = W_t^N$, $\mathbf{P}$-a.s. and u^* is an optimal control.

Existence of solution

- The BSDE (3) can be written via $m(\mathrm{d}t, \mathrm{d}z_1, \mathrm{d}z_2)$:

$$Y_t = \xi - \int_t^T \int_0^\infty \int_0^\infty \Theta_s^Y(z_1, z_2)\widetilde{m}(\mathrm{d}s, \mathrm{d}z_1, \mathrm{d}z_2) - \int_t^T F(s, Y_s, \Theta_s^Y(\cdot,\cdot))\mathrm{d}s \tag{5}$$

  where

$$F(t, Y_t, \Theta_t^Y(\cdot,\cdot), u_t) = \operatorname*{ess\,sup}_{u \in \mathcal{U}} f(t, Y_t, \Theta_t^Y(\cdot, 0), u_t) \tag{6}$$

  and $f(t, Y_t, \Theta_t^Y(\cdot, 0), u_t)$ is given in (4).
- The generator of the BSDE satisfies a stochastic Lipschitz condition:

$$\left|F(t,\omega,y,\theta(\cdot,\cdot)) - F(t,\omega,y',\theta'(\cdot,\cdot))\right|^2 \leq \gamma_t(\omega)|y-y'|^2 + \bar{\gamma}_t(\omega)||\theta(\cdot,\cdot)-\theta'(\cdot,\cdot)||_t^2$$

  where $\gamma_t = 3\eta^2 e^{2r(T-t)}(q_t^{u_M})^2 + 3\lambda_{t-}^2$, $\bar{\gamma}_t = 3\lambda_{t-}$.
- Thanks to $\mathbb{E}[e^{a\int_0^T \lambda_s ds}] < \infty$ for any $a > 0$ we can apply [Theorem 3.5 in Papapantoleon et al. EJP 2018].

Verification Result

Lemma

Let D be an $\mathbb{F}$-adapted process such that
(1) $D_T = 1$;
(2) $\{D_t e^{-\eta \widehat{X}_t^u e^{rT}},\ t \in [0,T]\}$ *is a sub-martingale for any* $u \in \mathcal{U}$ *and a martingale for some* $u^* \in \mathcal{U}$.
Then, $D_t = \varphi(t, \lambda_t)$ $\mathbf{P}$*-a.s. and* u^* *is an optimal control.*

Theorem (Verification Result)

Let $(Y, \Theta^{Y,(1)}, \Theta^{Y,(2)}) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^{(1)} \times \widehat{\mathcal{L}}^{(2)}$ *be a solution to BSDE* (3) *and* $u^* \in \mathcal{U}$ *satisfies the ess-sup. Then* $Y_t = W_t^N$, $\mathbf{P}$*-a.s. (that is* $\varphi(t, \lambda_t) = e^{\eta \widehat{X}_t^{u_N} e^{rT}} Y_t$*) and* u^* *is an optimal control.*

Proof.

Let $D_t := Y_t e^{\eta \widehat{X}_t^N e^{rT}}$. It verifies (1) and (2) and we apply the Lemma. □

Bellman Principle

Proposition

- *The Snell-envelope* $\{W_t^u\}_{t\in[0,T]}$ *is a sub-martingale for any* $u \in \mathcal{U}$*;*
- $\{W_t^{u^*}\}_{t\in[0,T]}$ *is a martingale if and only if any* $u^* \in \mathcal{U}$ *is an optimal control.*

Proof.

Let $(Y, \Theta^{Y,(1)}, \Theta^{Y,(2)}) \in \mathcal{L}^2 \times \widehat{\mathcal{L}}^{(1)} \times \widehat{\mathcal{L}}^{(2)}$ be a solution to BSDE (3). Then $Y_t = W_t^N$ and we get the Bellman Principle by the equality $W_t^u = W_t^N e^{\eta \widehat{X}_t^N e^{rT}} e^{-\eta \widehat{X}_t^u e^{rT}} = D_t e^{-\eta \widehat{X}_t^u e^{rT}}$. □

Proposition (Jump sizes of W^N)

The following representations hold

$$\Theta_t^{(1)}(z) = e^{-\eta \widehat{X}_{t-}^N e^{rT}} \left[e^{\eta z e^{r(T-t)}} \varphi(t, \lambda_{t-} + l(z)) - \varphi(t, \lambda_{t-}) \right], \ F^{(1)}(\mathrm{d}z)\mathrm{d}t\mathrm{d}\mathbf{P} - a.e.,$$

$$\Theta_t^{(2)}(z) = e^{-\eta \widehat{X}_{t-}^N e^{rT}} \left[\varphi(t, \lambda_{t-} + z) - \varphi(t, \lambda_{t-}) \right], \ F^{(2)}(\mathrm{d}z)\mathrm{d}t\mathrm{d}\mathbf{P} - a.e.,$$

for every $t \in [0, T]$.

Outline

Proposition

Suppose that $\Phi(z,u)$ *is differentiable in* $u \in [u_M, u_N]$ *for a.e.* $z \in (0,+\infty)$ *and* f *is strictly concave in* $u \in [u_M, u_N]$*. Then,*$u_t^* = \{u^*(t,\lambda_{t^-}), t \in [0,T]\}$ *is given by*

$$u^*(t,\lambda_{t^-}) = \begin{cases} u_M & (t,\lambda_{t^-}) \in A_0 \\ \bar{u}(t,\lambda_{t^-}) & \textit{otherwise} \\ u_N & (t,\lambda_{t^-}) \in A_1, \end{cases}$$

where

$$A_0 = \{(t,\lambda) \in [0,T] \times (0,+\infty) : h(t,\lambda,u_M) \leq 0\}$$
$$A_1 = \{(t,\lambda) \in [0,T] \times (0,+\infty) : h(t,\lambda,u_N) \geq 0\},$$

$$h(t,\lambda,u) = -\varphi(t,\lambda)\frac{\partial q(\lambda,u)}{\partial u} - \int_0^\infty \varphi(t,\lambda+l(z))e^{\eta e^{r(T-t)}\Phi(z,u)}\frac{\partial \Phi(z,u)}{\partial u}\lambda F^{(1)}(dz)$$

and $\bar{u}(t,\lambda) \in (u_M, u_N)$ *solves the following equation:*

$$-\varphi(t,\lambda)\frac{\partial q(\lambda,u)}{\partial u} = \int_0^\infty \varphi(t,\lambda+l(z))e^{\eta e^{r(T-t)}\Phi(z,u)}\frac{\partial \Phi(z,u)}{\partial u}\lambda F^{(1)}(dz).$$

Proportional reinsurance $\Phi(z, u) = zu$

- Expected Value Principle: $q_t^u = (1+\theta_R)\mathbb{E}[Z^{(1)}]\lambda_{t^-}(1-u_t)$
- The optimal retention level u^* is obtained "explicitly" and

$$u_t^* = u^*(t, \lambda_{t-}) = \begin{cases} 0 & \text{if } \theta_R \leq \theta^F(t, \lambda_{t-}) \\ 1 & \text{if } \theta_R \geq \theta^N(t, \lambda_{t-}) \\ \bar{u}(t, \lambda_{t^-}) & \text{otherwise,} \end{cases}$$

The stochastic thresholds ($\theta^F(t, \lambda_{t-}) < \theta^N(t, \lambda_{t-})$) are:

$$\theta^F(t, \lambda) = \frac{1}{\mathbb{E}[Z^{(1)}]} \int_0^\infty \frac{\varphi(t, \lambda + l(z))}{\varphi(t, \lambda)} zF^{(1)}(dz) - 1,$$

$$\theta^N(t, \lambda) = \frac{1}{\mathbb{E}[Z^{(1)}]} \int_0^\infty \frac{\varphi(t, \lambda + l(z))}{\varphi(t, \lambda)} e^{\eta e^{r(T-t)} z} zF^{(1)}(\mathrm{d}z) - 1$$

and $\bar{u}(t, \lambda) \in (0, 1)$ solves the following equation, w.r.t. u:

$$(1+\theta_R)\mathbb{E}[Z^{(1)}] = \int_0^{+\infty} \frac{\varphi(t, \lambda + l(z))}{\varphi(t, \lambda)} z e^{\eta e^{r(T-t)} zu} F^{(1)}(\mathrm{d}z).$$

Limited Stop-Loss Reinsurance with fixed maximum reinsurance coverage $\beta > 0$

- According to the Expected Value Principle

$$q_t^u = (1+\theta_R)\lambda_{t^-} \int_{u_t}^{u_t+\beta} (1 - F^{(1)}(z))\mathrm{d}z.$$

- The optimal control u^* is given by

$$u_t^* = u^*(t,\lambda_{t-}) = \begin{cases} 0 & \text{if } \theta_R \leq \theta^L(t,\lambda_{t-}) \\ \bar{u}(t,\lambda_{t-}) & \text{if } \theta_R > \theta^L(t,\lambda_{t-}) \end{cases}$$

where

$$\theta^L(t,\lambda) = \frac{1}{F^{(1)}(\beta)} \int_0^\beta \frac{\varphi(t,\lambda + l(z))}{\varphi(t,\lambda)} F^{(1)}(\mathrm{d}z) - 1.$$

and $\bar{u}(t,\lambda) \in (0,+\infty)$ solves the following equation w.r.t. u:

$$(1+\theta_R)\big(F^{(1)}(u+\beta) - F^{(1)}(u)\big) = e^{\eta e^{r(T-t)} u} \int_u^{u+\beta} \frac{\varphi(t,\lambda + l(z))}{\varphi(t,\lambda)} F^{(1)}(\mathrm{d}z).$$

Cox with shot noise intensity model, $\ell(z) = 0$

Under the Expected Value Principle

- proportional reinsurance $\theta^F = 0$ (i.e. full reinsurance is never optimal) the optimal reinsurance is deterministic:

$$u^{*,cox}(t) = \begin{cases} 1 & \text{if } \theta_R \geq \theta_t^N(t) \\ \bar{u}^{cox}(t) & \text{if } \theta_R < \theta_t^N(t), \end{cases} \tag{7}$$

  where $\theta^N(t) = \frac{1}{\mathbb{E}[Z^{(1)}]} \int_0^\infty e^{\eta e^{r(T-t)} z} z F^{(1)}(dz) - 1$ and $\bar{u}^{cox}(t) \in (0,1)$ is the solution to $(1+\theta_R)\mathbb{E}[Z^{(1)}] = \int_0^{+\infty} z e^{\eta e^{r(T-t)} z u} F^{(1)}(dz)$.

- Limited Excess-of-Loss with fixed reinsurance coverage and Excess-of-Loss, $\theta^L = 0$ (i.e. maximal reinsurance is never optimal) and the optimal reinsurance is deterministic:

$$u^{*,cox}(t) = \frac{\log(1+\theta_R)}{\eta} e^{-r(T-t)},$$

Comparison results

Under EVP, proportional reinsurance or limited excess of loss reinsurance, assuming $\varphi(t, \lambda)$ increasing in λ:

Whenever there is the self-exciting component $\ell(z) \neq 0$, the insurance company transfers more risk to the reinsurance company:

$$u_t^* \leq u^{*,cox}(t), \quad t \in [0, T].$$

Monotonicity of the value function

- Preliminary result:

$$\varphi(t,\lambda) = \inf_{u\in\mathcal{U}_t} \mathbb{E}^{\mathbb{Q}}\left[H(t,T,u.)e^{\lambda\int_t^T e^{-\alpha(s-t)}\{\int_0^{+\infty} B(s,z,u.)F^{(1)}(\mathrm{d}z) - a(s,u_s)\}\mathrm{d}s}\right]$$

where $\mathbb{Q}$ is a probability measure equivalent to $\mathbf{P}$ such that under $\mathbb{Q}$, $m^{(i)}(\mathrm{d}t,\mathrm{d}z)$, $i=1,2$, are Poisson random measures;
$a(t,u_t) := 1 + \eta e^{r(T-t)}(c(t) - d(t,u_t))$,
$A(t,u.) := \int_t^T a(s,u_s)e^{-\alpha(s-t)}\mathrm{d}s$, $B(t,z,u.) := e^{\eta e^{r(T-t)}\Phi(z,u_t) - A(t,u.)\ell(z)}$,
$H(t,T,u.)$ is a strictly positive r.v.

- Under the assumption, for any $u\in\mathcal{U}$ and $t>0$:

$$\int_0^{+\infty} B(t,z,u.)F^{(1)}(\mathrm{d}z) - a(t,u_t) \geq 0, \quad \mathbf{P}-\text{a.s.}$$

Then, $\varphi(t,\lambda)$ is an increasing function of $\lambda\in(0,+\infty)$.

Thanks for your attention!

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