Direction-Free Approximation Algorithms for Bounded Convex Vector Optimization Problems

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This work was funded by TÜBİTAK (Scientific & Technological Research Council of Turkey), Project No. 118M479.

June 10, 2022

Outline

Convex Vector Optimization Problems

Convex Vector Optimization Problems

$$\begin{array}{lll} \text{`minimize'} & f(x) & \text{(with respect to } \leq_C) \\ \text{subject to} & x \in \mathcal{X}, & \end{array} \tag{P}$$

where

- $C \subseteq \mathbb{R}^q$ is a solid, pointed, polyhedral convex ordering cone,
- $f : \mathbb{R}^n \to \mathbb{R}^q$ is C-convex:

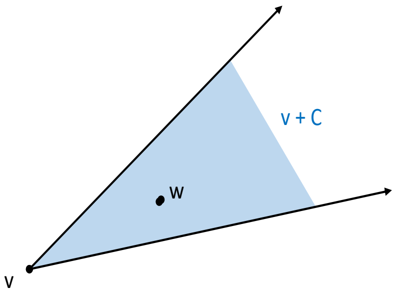
$$f(\alpha x + (1 - \alpha y)) \leq_C \alpha f(x) + (1 - \alpha) f(y)$$

- Feasible region $\mathcal{X}$ is convex.
- $f(\mathcal{X}) = \{f(x) \in \mathbb{R}^q \mid x \in \mathcal{X}\}$.

Order Relation $\leq_C$

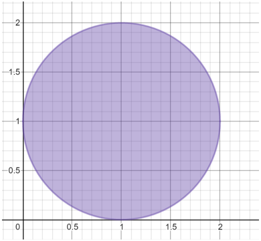
Partial ordering induced by non-trivial convex pointed cone $C \subseteq \mathbb{R}^q$:

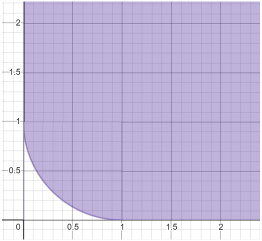
$$v \leq_C w \iff w - v \in C$$



Solution Concepts: Set Optimization point of view

$\mathcal{P} := \mathrm{cl}(f(\mathcal{X}) + C)$ is convex and closed. (*upper image*)



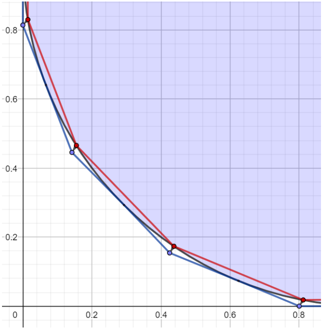


Solution Concepts: Set Optimization point of view

A finite subset $\bar{\mathcal{X}}$ of $\mathcal{X}$ is called a **finite (weak) ϵ-solution (w.r.t. c)** to (P) if it consists of only (weak minimizers) weak efficient solutions; and

$$\operatorname{conv} f(\bar{\mathcal{X}}) + C - \epsilon\{c\} \supseteq \mathcal{P}.$$

($c \in \operatorname{int} C$ is fixed.)



$$\operatorname{conv} f(\bar{\mathcal{X}}) + C - \epsilon\{c\} \supseteq \mathcal{P} \supseteq \operatorname{conv} f(\bar{\mathcal{X}}) + C.$$

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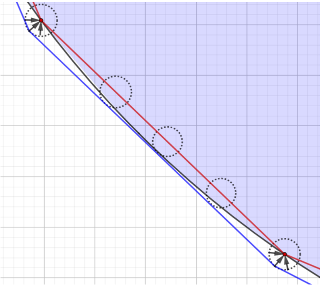
$$\operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0,\epsilon) \supseteq \mathcal{P}.$$

(No direction biasedness.)



$$\operatorname{conv} f(\bar{\mathcal{X}}) + C + B(0,\epsilon) \supseteq \mathcal{P} \supseteq \operatorname{conv} f(\bar{\mathcal{X}}) + C.$$

Literature and Motivation

Outer Approximation Algorithms for CVOPs

Algorithm	**Finiteness / Convergence**	**Choice of Direction**	**Vertex Selection (VS)**	**Scalarization Model**	**Dual Algorithm**
[Klamroth, et al., 2003]	Convergence for biobjective	Inner point (fixed)	Distance to upper image	Gauge-based model	Inner approx. algorithm
[Ehrgott, et al., 2011]	-	Inner point (fixed)	Arbitrary	Pascoletti Serafini **	-
[Löhne, et al., 2014]	-	Fixed	Arbitrary	Pascoletti Serafini	Geometric dual algorithm
[Dörfler, et al., 2021]	-	Inner point (changing)	Distance to inner approximation	Pascoletti Serafini	-
[Keskin, Ulus, 2022]	-	Several variants	Several Variants	Pascoletti Serafini	-
[Ararat, et al., 2022]*	Finiteness	Not Relevant	Arbitrary	Norm minimizing	-

* Ararat, Ç., Ulus, F. and Umer, M. (2022) A Norm Minimization-Based Convex Vector Optimization Algorithm

Journal of Optimization Theory and Applications. DOI: 10.1007/s10957-022-02045-8

** [Pascoletti, Serafini, 1984]

Assumptions

$$\begin{array}{lll} \text{‘minimize’} & f(x) & (\text{with respect to } \leq_C) \\ \text{subject to} & x \in \mathcal{X} & \end{array} \tag{P}$$

- C is a closed convex cone that is also solid, pointed, and nontrivial.
- f is a C-convex and continuous function.
- $\mathcal{X}$ is a compact convex set with $\operatorname{int} \mathcal{X} \neq \emptyset$.

Norm Minimizing Scalarization and Relevant Results

Norm Minimizing Scalarization

$\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^q$ and $v \in \mathbb{R}^q$.

$$\text{minimize } \|z\| \quad \text{subject to} \quad f(x) - z - v \leq_C 0, \quad x \in \mathcal{X},\ z \in \mathbb{R}^q \quad (\mathrm{P}(v))$$

- Convex program
- The optimal value is $d(v, \mathcal{P})$.

Norm Minimizing Scalarization and the Lagrange Dual

$$\text{minimize } \|z\| \quad \text{subject to} \quad f(x) - z - v \leq_C 0, \quad x \in \mathcal{X},\ z \in \mathbb{R}^q \quad (\mathrm{P}(v))$$

$\phi(w) := \inf_{x \in X, z \in \mathbb{R}^q} L(x, z, w), \quad w \in \mathbb{R}^q.$

$$\text{maximize } \phi(w) \text{ subject to } w \in \mathbb{R}^q. \qquad (\mathrm{D}(v))$$

The optimal value of $(\mathrm{D}(v))$:

$$\sup_{w \in \mathbb{R}^q} \phi(w) = \sup \left\{ \inf_{x \in \mathcal{X}} w^{\mathsf{T}} f(x) - w^{\mathsf{T}} v \mid \|w\|_* \leq 1,\ w \in C^+ \right\}$$

Proposition

1. *For every $v \in \mathbb{R}^q$, there exist optimal solutions (x^v, z^v) and w^v to problems* $(\mathrm{P}(v))$ *and* $(\mathrm{D}(v))$, *respectively, and the optimal values coincide.*

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2. *If $v \notin \operatorname{int} \mathcal{P}$, then x^v is a weak minimizer of* (P).
3. *If $w^v \neq 0$, then,*

$$\mathcal{H} = \{y \in \mathbb{R}^q \mid (w^v)^\mathsf{T} y \geq (w^v)^\mathsf{T} \Gamma(x^v)\} \supseteq \mathcal{P}.$$

   Moreover, $\operatorname{bd} \mathcal{H}$ *is a supporting hyperplane of $\mathcal{P}$ both at $\Gamma(x^v)$ and $y^v = v + z^v$.*

The Primal Algorithm

(Algorithm 1)

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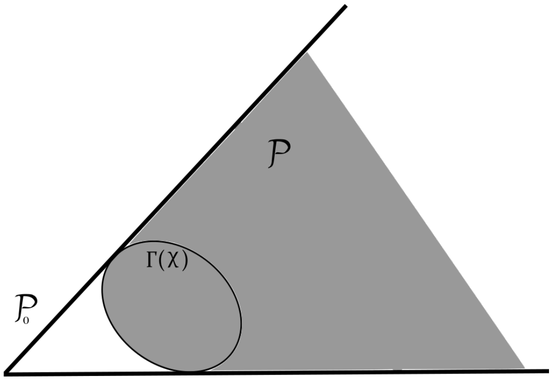
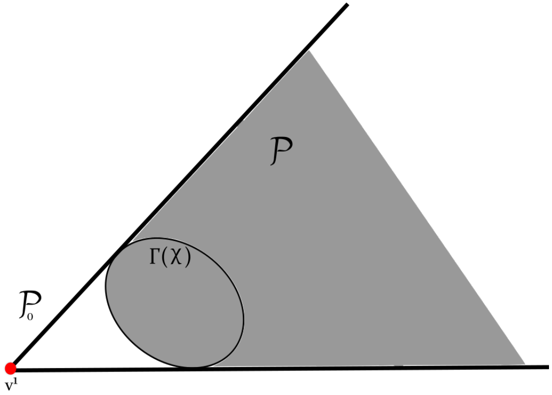
1. Find an initial outer approximation $\mathcal{P}_0^{\text{out}}$ of $\mathcal{P}$.
2. For $k \geq 0$, find the vertices V_k of $\mathcal{P}_k^{\text{out}}$.
3. For $v \in V_k$, solve $(\mathrm{P}(v))$ to find the closest point $v + z^v$ on $\operatorname{bd}\mathcal{P}$.
4. If $\|z^v\| > \epsilon$, find the supporting halfspace $\mathcal{H}_k$ of $\mathcal{P}$.
5. Update $\mathcal{P}_{k+1}^{\text{out}} = \mathcal{P}_k^{\text{out}} \cap \mathcal{H}_k$, go step 2.
6. If $\|z^v\| \leq \epsilon$ for all $v \in V_k$, STOP.

The Primal Algorithm (Algorithm 1)



Algorithm 1



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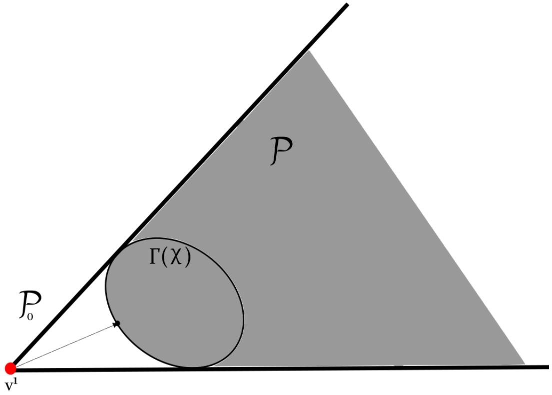

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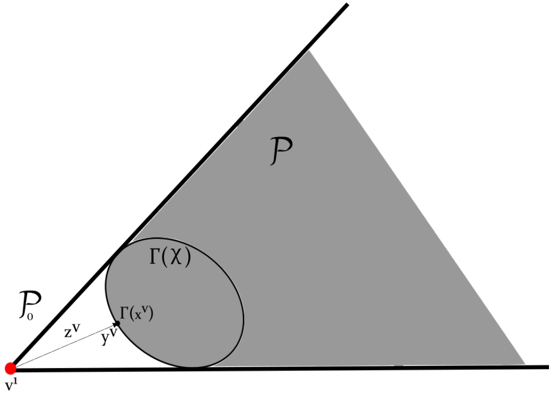

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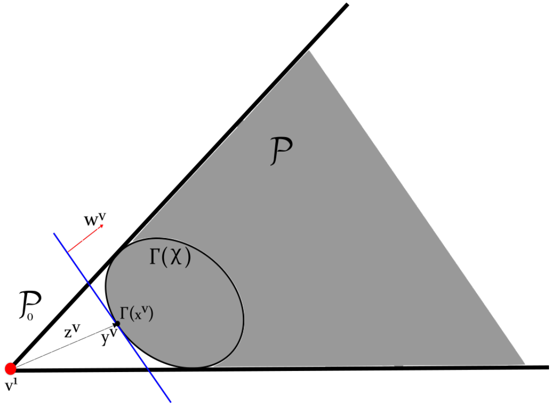

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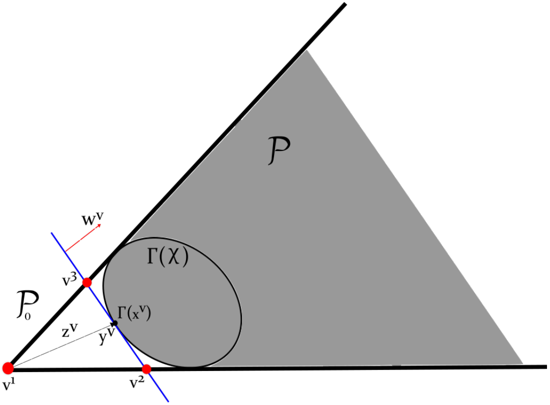

Proposition

If terminates, then Algorithm 1 returns a finite weak ϵ-solution $\bar{\mathcal{X}}$ to (P).
If terminates, then Algorithm 1 returns a finite ϵ-solution $\bar{\mathcal{W}}$ to (D).

Towards Finiteness

Lemma

Let $v \notin \mathcal{P}$ and $\mathcal{H}$ be the supporting halfspace to $\mathcal{P}$. If $\|z^v\| \geq \epsilon$, then $B \cap \mathcal{H} = \emptyset$,

$$B := \left\{ y \in \{v\} + C \mid \|y - v\| \leq \frac{\epsilon}{2} \right\}.$$

Towards Finiteness

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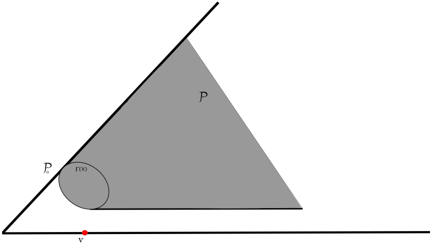
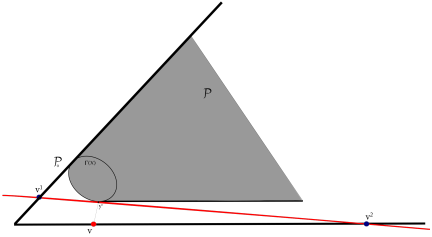
$$B := \left\{ y \in \{v\} + C \mid \|y - v\| \leq \frac{\epsilon}{2} \right\}.$$

At each iteration, we discard a region with positive fixed volume from the current outer approximation!

A limitation of Algorithm 1

The region in which vertices of updated outer approximation can be found may not be compact.





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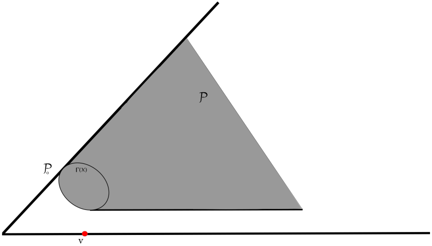
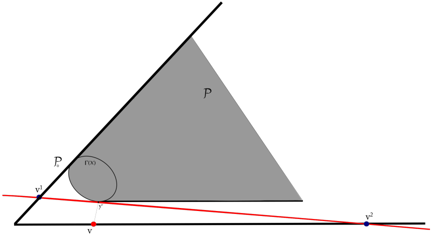
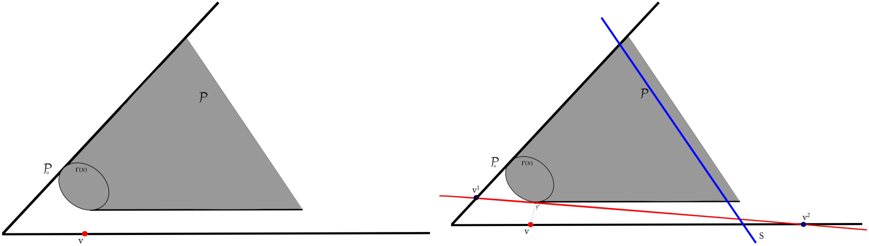




The finiteness of the algorithm is an open question!

Idea: A sufficiently large compact subset!



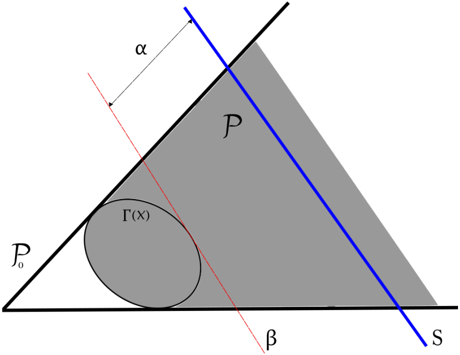
Towards a Modified Algorithm

$$S := \{y \in \mathbb{R}^q \mid \bar{w}^\mathsf{T} y \leq \beta + \alpha\}$$

$$\bar{w} := \frac{\sum_{j=1}^{J} w^j}{\left\|\sum_{j=1}^{J} w^j\right\|_*} \in \operatorname{int} C^+$$

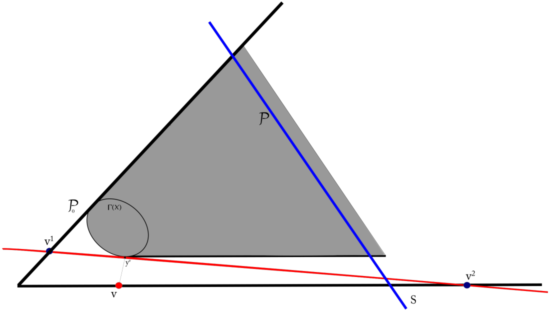
$$\beta \geq \sup_{x \in \mathcal{X}} \bar{w}^\mathsf{T} \Gamma(x)$$

$$\alpha > \max_{v \in \mathcal{V}_0} (\bar{w}^\mathsf{T} v - \beta)^+ + \delta^H(\mathcal{P}_0^{\text{out}}, \mathcal{P})$$



Proposition

Let v be a vertex of $\bar{\mathcal{P}}_k^{out}$ for some $k \geq 1$. If $v \notin \operatorname{int} S$, then $y^v = v + z^v \in \mathrm{wMin}_C(\mathcal{P}) \setminus \mathrm{Min}_C(\mathcal{P})$.



Question: Do we really observe vertices out of S in practice?

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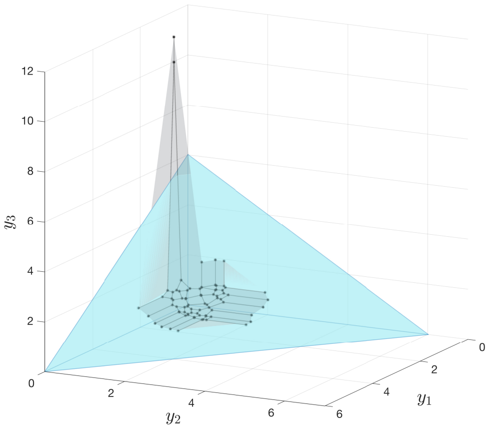
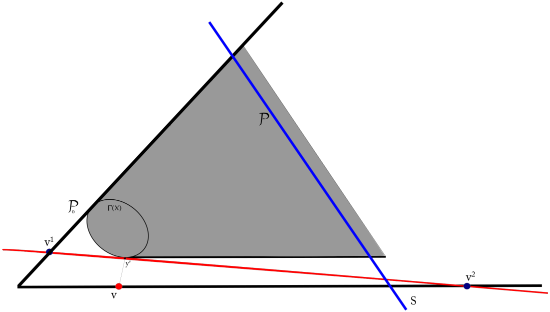


Answer: YES!

Question: Is it sufficient to consider only the vertices within S and ignore the others?



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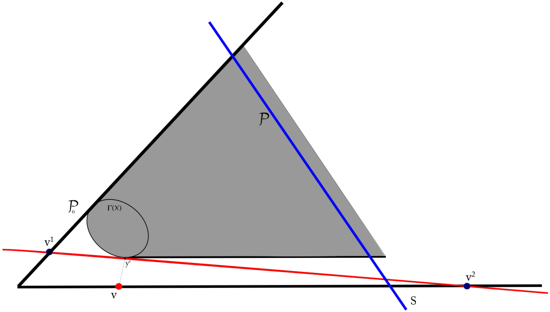


Answer: NO!

Modified Primal Algorithm (Algorithm 2) - initialization

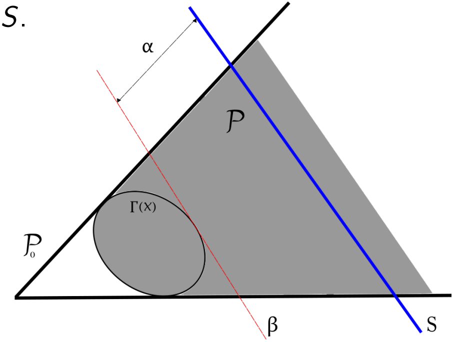
i.1. Find initial outer approximation $\bar{\mathcal{P}}_0^{\text{out}} = \mathcal{P}_0^{\text{out}}$ of $\mathcal{P}$.

i.2. Compute the set $\bar{V}_0$ of vertices of $\bar{\mathcal{P}}_0^{\text{out}}$.

i.3. For all $v \in \bar{\mathcal{V}}_0$, solve (P($v$)), find (x^v, z^v), w^v.

i.4. Compute $\delta^H(\mathcal{P}_0^{\text{out}}, \mathcal{P})$, β and α.

i.5. Store an H-representation of S.



Algorithm 2 - main loop

1. An initial outer approximation $\bar{\mathcal{P}}_0^{\text{out}} = \mathcal{P}_0^{\text{out}}$ of $\mathcal{P}$.
2. For the k^{th} iteration, $k \geq 0$, let $\bar{\mathcal{P}}_k^{\text{out}}$ be the current outer approximation.
3. $\bar{V}_k$ is the set of vertices of $\bar{\mathcal{P}}_k^{\text{out}} \cap S$.
4. For a vertex $v \in \bar{\mathcal{V}}_k$, find a point $y^v = v + z^v$ on the boundary of $\mathcal{P}$.
5. If $\|z^v\| > \epsilon$, find $\mathcal{H}_k$ of $\mathcal{P}$ at y^v.
6. Current approximation is updated as $\bar{\mathcal{P}}_{k+1}^{\text{out}} = \bar{\mathcal{P}}_k^{\text{out}} \cap \mathcal{H}_k$.
7. $\|z^v\| \leq \epsilon$ for all vertices, the algorithm terminates.

Theorem

Algorithm 2 works correctly: if the algorithm terminates, then it returns a finite weak ϵ-solution to (P).

Lemma

Let $v \notin \mathcal{P}$ and $\mathcal{H}$ be the supporting halfspace to $\mathcal{P}$. If $\|z^v\| \geq \epsilon$, then $B \cap \mathcal{H} = \emptyset$,

$$B := \left\{ y \in \{v\} + C \mid \|y - v\| \leq \frac{\epsilon}{2} \right\}.$$

Theorem

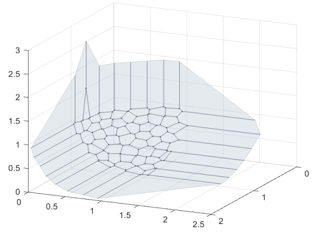
The Algorithm 2 terminates after a **finite number of iterations**.

Computational Results

Example

$$\begin{aligned} &\text{minimize} \;\; \Gamma(x) = x \;\; \text{with respect to} \leq_C \\ &\text{subject to} \;\; \|x - e\|_2 \leq 1, \;\; x \in \mathbb{R}^q, \end{aligned}$$

where $q = \{3, 4\}$ and the ordering cone is $C = \mathbb{R}^q_+$.



Outer approximation obtained from Algorithm 1 using ℓ_2 norm

		q=3							q=4						
p	Alg	ϵ	$\lvert\bar{\mathcal{X}}\rvert$	Opt	T_{opt}	En	T_{en}	T	ϵ	$\lvert\bar{\mathcal{X}}\rvert$	Opt	T_{opt}	En	T_{en}	T
1	1	0.05	33	52	13.29	20	0.31	13.68	0.5	30	41	11.66	12	0.22	11.95
	2		42	59	15.59	17	0.26	15.99		57	69	19.27	11	0.28	19.71
	Lit[1]		56	89	17.32	34	1.02	18.56		33	44	9.71	12	0.20	9.97
2	1		29	45	10.49	17	0.23	10.79		29	34	8.70	6	0.07	8.80
	2		44	61	14.24	17	0.24	14.68		94	99	25.98	5	0.07	26.20
	Lit		32	50	9.76	19	0.27	10.10		31	42	9.18	12	0.19	9.43
∞	1		21	34	8.15	14	0.16	8.35		8	9	2.34	2	0.02	2.38
	2		37	51	12.39	13	0.15	12.61		11	15	4.23	1	0.02	4.39
	Lit		21	34	6.76	14	0.15	6.96		8	9	2.05	2	0.02	2.09
1	1	0.01	175	262	69.13	88	20.17	92.99	0.1	143	177	52.94	34	2.67	56.25
	2		161	235	62.55	73	10.56	75.28		232	273	78.37	38	5.05	84.42
	Lit		256	397	76.54	142	113.29	212.22		412	510	111.49	91	48.25	165.40
2	1		128	196	46.51	69	8.41	56.52		-					
	2		145	209	49.42	64	6.56	57.29		-					
	Lit		139	213	41.49	75	10.39	53.88		208	265	57.93	46	5.08	63.67
∞	1		93	145	35.34	53	3.44	39.47		68	82	22.59	12	0.22	22.87
	2		107	154	37.20	47	2.43	40.15		-					
	Lit		87	137	26.72	51	3.03	30.36		-					

[1] [Löhne, et al., 2014]

Example

$$\begin{aligned} \text{minimize} \quad & \Gamma(x) = (\|x\|_2^2 + b^1 x, \|x\|_2^2 + b^2 x, \|x\|_2^2 + b^3 x)^\mathsf{T} \\ & \text{with respect to } \leq_{\mathbb{R}^3_+} \\ \text{subject to} \quad & \|x\|_2^2 \leq 100, \; 0 \leq x_i \leq 10 \text{ for } i \in \{1, \dots, n\}, \; x \in \mathbb{R}^n, \end{aligned}$$

where $\hat{b}^1 = (0, 10, 120)$, $\hat{b}^2 = (80, -448, 80)$, $\hat{b}^3 = (-448, 80, 80)$ and $b^1, b^2, b^3 \in \mathbb{R}^n$.

(a) Let $n = 3$ and $b^1 = \hat{b}^1, b^2 = \hat{b}^2, b^3 = \hat{b}^3$.

(b) Let $n = 9$ and $b^1 = (\hat{b}^1, \hat{b}^1, \hat{b}^1), b^2 = (\hat{b}^2, \hat{b}^2, \hat{b}^2), b^3 = (\hat{b}^3, \hat{b}^3, \hat{b}^3)$.

[Ehrgott, et al., 2011, Example 5.10]

			n=3						n=9					
ϵ	p	Alg	$\lvert\bar{\mathcal{X}}\rvert$	Opt	$\mathrm{T_{opt}}$	En	$\mathrm{T_{en}}$	T	$\lvert\bar{\mathcal{X}}\rvert$	Opt	$\mathrm{T_{opt}}$	En	$\mathrm{T_{en}}$	T
10	2	1	502	943	285.43	132	100.12	401.95	1561	2754	1159.60	225	753.05	2046.88
		2	958	3924	1194.30	137	122.28	1339.09	1770	4213	1819.70	218	733.58	2682.37
		Lit	305	965	259.85	164	211.23	502.43	2718	4520	1772.08	259	1324.24	3295.96
	inf	1	197	592	179.71	106	41.61	227.61	1231	2106	901.81	152	150.43	1076.14
		2	199	1206	390.99	100	36.67	432.99	3638	9222	4045.45	166	219.60	4301.63
		Lit	180	586	157.46	101	33.60	196.24	2628	5057	1994.39	164	202.33	2231.07
5	2	1	1178	3127	932.65	245	1059.10	2175.25	4461	7968	3371.91	390	8008.67	12795.23
		2	1207	5557	1702.79	259	1488.53	3441.28	7052	15662	6546.29	409	9908.03	18268.17
		Lit	579	3932	1049.79	309	3046.66	4529.37	7046	11149	4307.21	476	16898.76	23603.25
	inf	1	412	1740	526.33	185	325.59	907.19	3153	4538	1889.06	294	2164.45	4390.16
		2	465	2655	837.24	188	374.95	1268.01	3570	8155	3482.44	305	2589.99	6470.71
		Lit	342	1412	380.10	185	352.70	787.48	3146	4712	1845.91	300	2455.39	4663.98

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A Norm Minimization-Based Convex Vector Optimization Algorithm
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- Novel norm minimization-based scalarization, free of direction parameter.
- Algorithms based on the new scalarization
- Comparable performance of the algorithms.
- Finiteness guarantee for the first time.
- Convergence Rate: ongoing
  After slight modifications, we prove $\mathcal{O}(k^{2/1-q})$; and working for a better convergence rate.

Direction Free Geometric Dual Algorithm

Outer Approximation Algorithms for CVOPs

Algorithm	Finiteness / Convergence	Choice of Direction	Vertex Selection (VS)	Scalarization Model	Dual Algorithm
[Klamroth, et al., 2003]	Convergence for biobjective	Inner point (fixed)	Distance to upper image	Gauge-based model	Inner approx. algorithm
[Ehrgott, et al., 2011]	-	Inner point (fixed)	Arbitrary	Pascoletti Serafini	-
[Löhne, et al., 2014]	-	Fixed	Arbitrary	Pascoletti Serafini	Geometric dual algorithm*
[Dörfler, et al., 2021]	-	Inner point (changing)	Distance to inner approximation	Pascoletti Serafini	-
[Keskin, Ulus, 2022]	-	Several variants	Several Variants	Pascoletti Serafini	-
[Ararat, et al., 2022]	Finiteness	Not Relevant	Arbitrary	Norm minimizing	-

The dual algorithms solve weighted sum scalarization:

$$\inf_{x \in \mathcal{X}} w^{\mathsf{T}} f(x)$$

* [Heyde, 2013]

Outer Approximation Algorithms for CVOPs

Algorithm	Finiteness / Convergence	Choice of Direction	Vertex Selection (VS)	Scalarization Model	Dual Algorithm
[Klamroth, et al., 2003]	Convergence for biobjective	Inner point (fixed)	Distance to upper image	Gauge-based model	Inner approx. algorithm
[Ehrgott, et al., 2011]	-	Inner point (fixed)	Arbitrary	Pascoletti Serafini *	-
[Löhne, et al., 2014]	-	Fixed	Arbitrary	Pascoletti Serafini	Geometric dual algorithm
[Dörfler, et al., 2021]	-	Inner point (changing)	Distance to inner approximation	Pascoletti Serafini	-
[Keskin, Ulus, 2022]	-	Several variants	Several Variants	Pascoletti Serafini	-
[Ararat, et al., 2022]	Finiteness	Not Relevant	Arbitrary	Norm minimizing	Geometric dual algorithm*

Ararat, Ç. and Tekgül, S. and Ulus, F.

"Geometric duality and a geometric dual algorithm for CVOPs."

Geometric Dual Problem and Solution Concept

Geometric Dual Problem

$$\begin{array}{lll} \text{`maximize'} & \xi(w) & (\text{with respect to } \leq_K) \\ \text{subject to} & w \in C^+, & \end{array} \tag{D}$$

where

- $\xi \colon \mathbb{R}^q \to \mathbb{R}^{q+1}$,

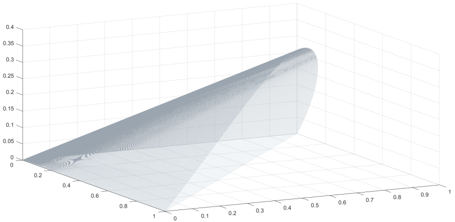
$$\xi(w) := (w_1, \ldots, w_q, \inf_{x \in \mathcal{X}} w^\mathsf{T} f(x))^\mathsf{T}$$

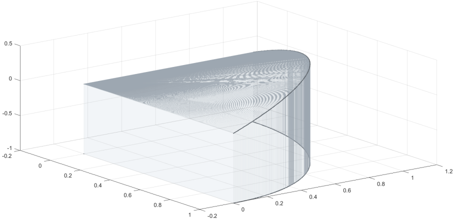
- $e^{q+1} = (0, \ldots, 0, 1)^\mathsf{T} \in \mathbb{R}^{q+1}$,

$$K := \{\lambda e^{q+1} \mid \lambda \geq 0\}$$

Solution Concept for (D)

$$\mathcal{D} := \xi(C^+) - K = \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid w \in C^+,\ \alpha \leq \inf_{x\in\mathcal{X}} w^\mathsf{T} f(x)\}$$





Solution Concept for (D)

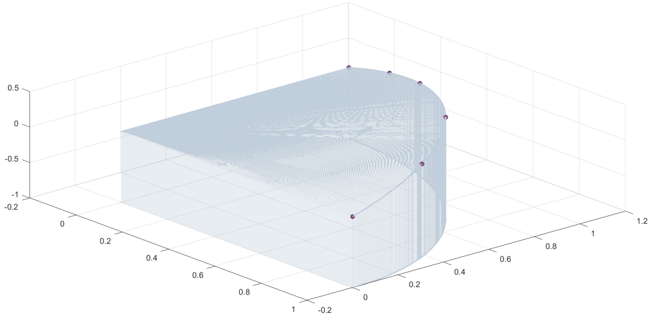
A finite set $\bar{\mathcal{W}} \subseteq C^+ \cap \mathbb{S}^{q-1}$ is called a **finite ϵ-solution** of (D) if it consists of only K-maximizers and

$$\operatorname{cone}(\operatorname{conv} \xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D},$$

where $\mathbb{S}^{q-1} := \{z \in \mathbb{R}^q \mid \|z\|_* = 1\}$.

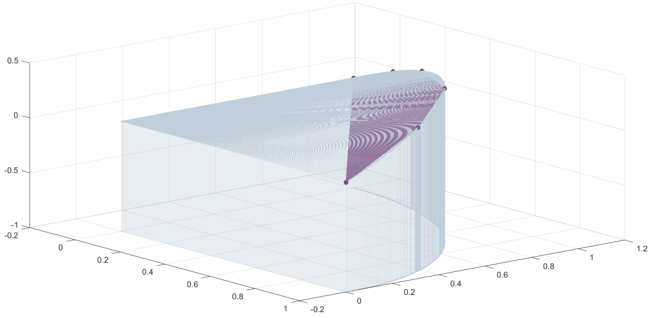
Solution Concept for (D)

$$\mathrm{cone}(\mathrm{conv}\,\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$$



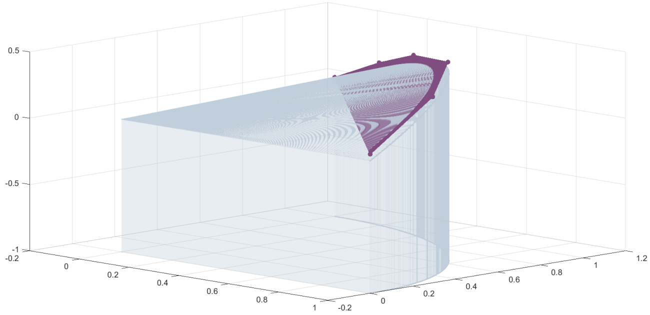
Solution Concept for (D)

$$\operatorname{cone}(\operatorname{conv}\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$$



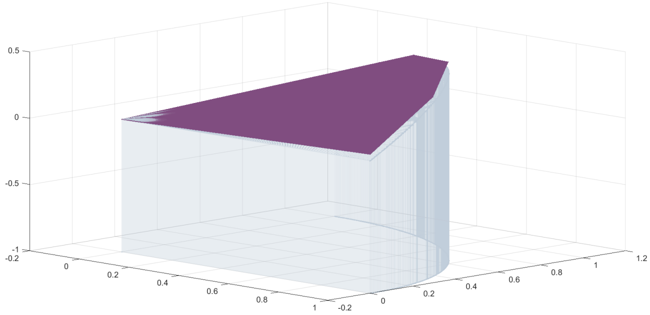
Solution Concept for (D)

$$\text{cone}(\text{conv}\,\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$$



Solution Concept for (D)

$$\text{cone}(\text{conv}\,\xi(\bar{\mathcal{W}}) + \epsilon\{e^{q+1}\}) - K \supseteq \mathcal{D}$$



Solution Concept for (D)

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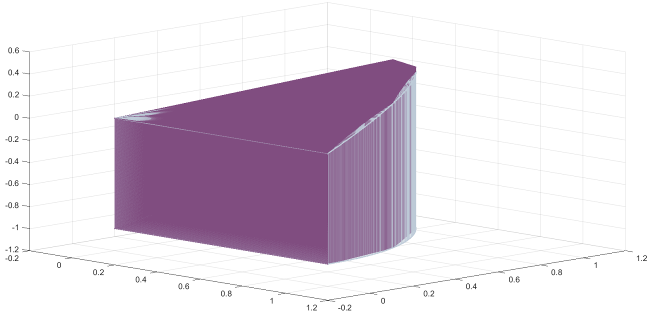
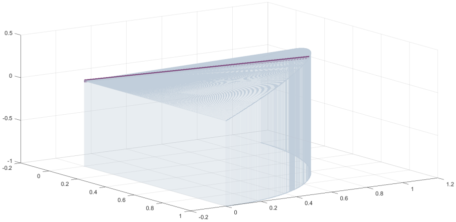
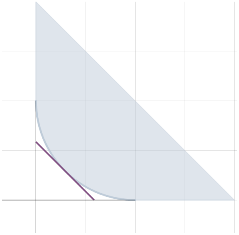


Geometric Duality Results

Supporting Hyperplanes of $\mathcal{P}$

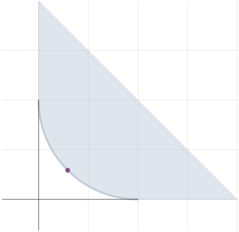
$$\mathcal{H}\colon \mathbb{R}^{q+1} \rightrightarrows \mathbb{R}^q, \qquad \mathcal{H}(w,\alpha) := \{y \in \mathbb{R}^q \mid w^\mathsf{T} y - \alpha \geq 0\}$$
$$H(w,\alpha) := \{y \in \mathbb{R}^q \mid w^\mathsf{T} y - \alpha = 0\}.$$

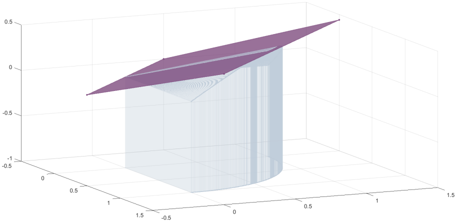




Supporting Hyperplanes of $\mathcal{D}$

$$\mathcal{H}^*\colon \mathbb{R}^q \rightrightarrows \mathbb{R}^{q+1}, \qquad \mathcal{H}^*(y) := \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid w^\mathsf{T} y - \alpha \geq 0\}$$
$$H^*(y) := \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid w^\mathsf{T} y - \alpha = 0\}.$$





Geometric Duality Theorem

$\mathcal{F}_{\mathcal{D}}^*$: set of all K-maximal proper face of $\mathcal{D}$
$\mathcal{F}_{\mathcal{P}}$: set of all C-minimal proper face of $\mathcal{P}$

$$\Psi\colon \mathcal{F}_{\mathcal{D}}^* \rightrightarrows \mathbb{R}^q, \quad \Psi(F^*) := \bigcap_{(w^\mathsf{T},\alpha)^\mathsf{T} \in F^*} H(w,\alpha) \cap \mathcal{P}.$$

Theorem

Ψ is an inclusion-reversing one-to-one correspondence between $\mathcal{F}_{\mathcal{D}}^*$ and $\mathcal{F}_{\mathcal{P}}$. The inverse map is given by

$$\Psi^{-1}(F) := \bigcap_{y \in F} H^*(y) \cap \mathcal{D}.$$

Geometric Duality Results

For closed and convex sets $\bar{\mathcal{P}}$ and $\bar{\mathcal{D}}$, we define

$$\mathcal{D}_{\bar{\mathcal{P}}} := \{(w^\mathsf{T}, \alpha)^\mathsf{T} \in \mathbb{R}^{q+1} \mid \forall y \in \bar{\mathcal{P}} \colon w^\mathsf{T} y - \alpha \geq 0\},$$
$$\mathcal{P}_{\bar{\mathcal{D}}} := \{y \in \mathbb{R}^q \mid \forall (w^\mathsf{T}, \alpha)^\mathsf{T} \in \bar{\mathcal{D}} \colon w^\mathsf{T} y - \alpha \geq 0\}.$$

Proposition

- *We have*

$$\mathcal{D}_{\mathcal{P}} = \mathcal{D}, \quad \mathcal{P}_{\mathcal{D}} = \mathcal{P}$$

  .

- *Let $\emptyset \neq \bar{\mathcal{P}} \subsetneq \mathbb{R}^q$ be a closed convex set. We have*

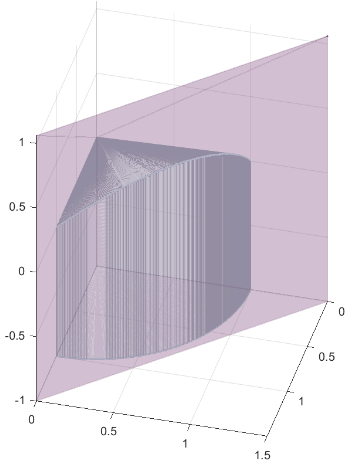
$$\bar{\mathcal{P}} = \mathcal{P}_{\mathcal{D}_{\bar{\mathcal{P}}}}.$$

- *Let $\emptyset \neq \bar{\mathcal{D}} \subseteq \mathbb{R}^{q+1}$ be a closed convex lower set. Suppose further that $\bar{\mathcal{D}}$ is a cone and $\mathcal{P}_{\bar{\mathcal{D}}} \neq \emptyset$. We have*

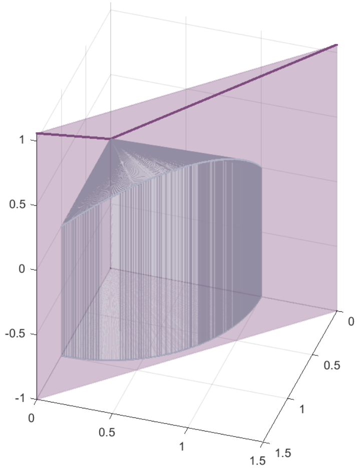
$$\mathcal{D}_{\mathcal{P}_{\bar{\mathcal{D}}}} = \mathcal{D}.$$

The Geometric Dual Algorithm

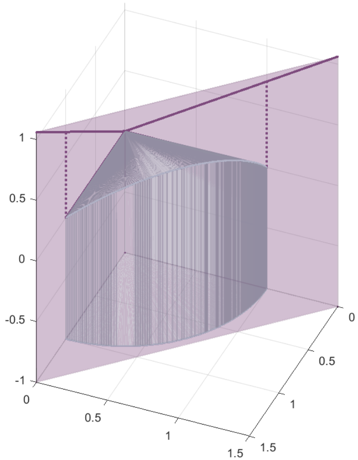
Dual Algorithm



Dual Algorithm

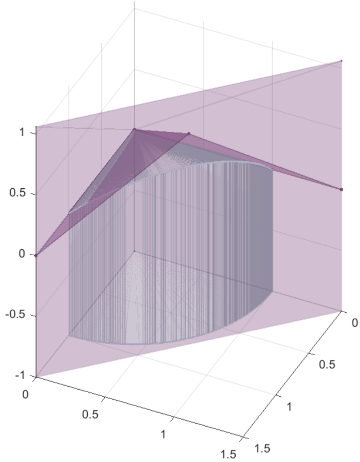


Dual Algorithm

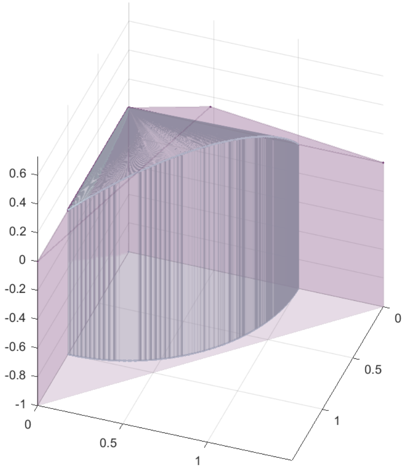


$$\begin{array}{ll} \text{minimize} & w^{\mathsf{T}} f(x) \\ \text{subject to} & x \in \mathcal{X}. \end{array}$$

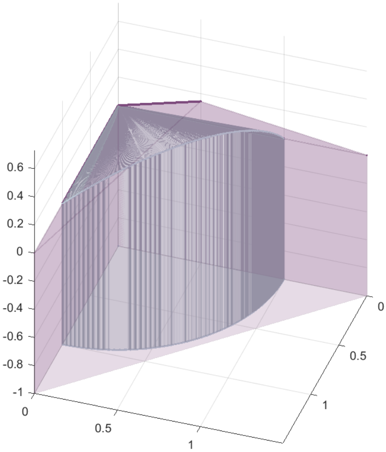
Dual Algorithm



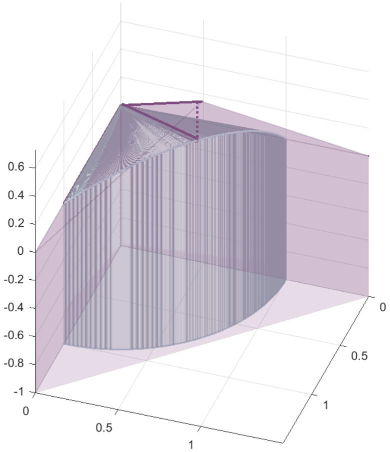
Dual Algorithm



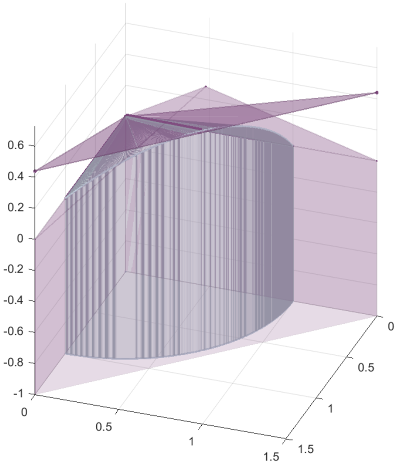
Dual Algorithm



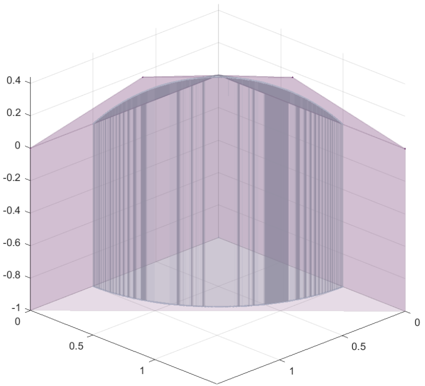
Dual Algorithm



Dual Algorithm



Dual Algorithm



Dual Algorithm

Proposition

When the dual algorithm terminates, it returns

- *a finite ϵ-solution $\bar{\mathcal{W}}$ to* (D)
- *and a finite weak $\tilde{\epsilon}$-solution $\bar{\mathcal{X}}$ to* (P).

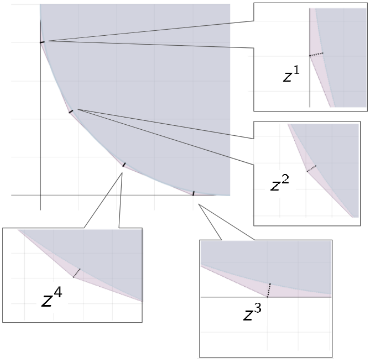
$$\tilde{\epsilon} = \frac{\epsilon}{\min\limits_{\lambda \in \Delta^{J-1}} \left\| \sum_{j=1}^{J} \lambda_j w^j \right\|_*},$$

where $w^1, \ldots w^J$ are the generating vectors of C^+.

Computational Results

- Primal error indicator

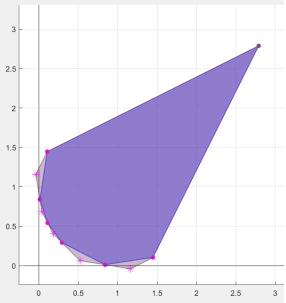
The actual Hausdorff distance



Proximity Measures

- Primal error indicator
- Hypervolume indicator

Hypervolume between the inner and outer approximations



$$\mathrm{HV} := \left(\frac{\mathrm{HV}(\mathcal{V}_o, r) - \mathrm{HV}(\mathcal{V}_i, r)}{\mathrm{HV}(\mathcal{V}_o, r)} \right) \times 100.$$

Test Problems

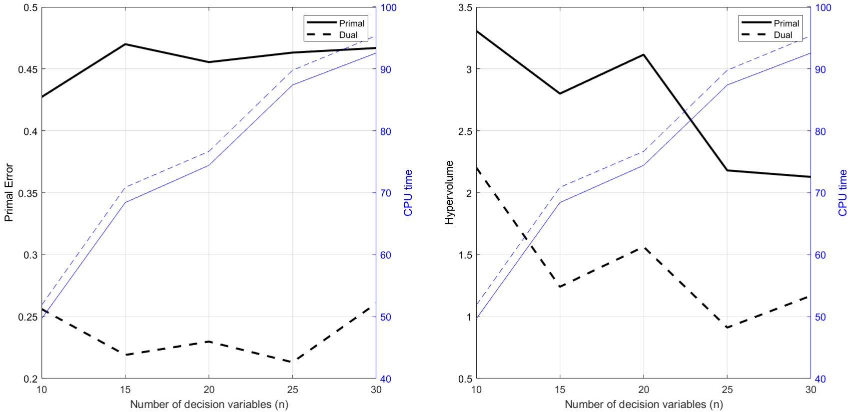
$$\begin{aligned} &\text{minimize} \quad f(x) = A^\mathsf{T} x \quad \text{with respect to } \leq_{\mathbb{R}^q_+} \\ &\text{subject to} \quad x^\mathsf{T} P x - 1 \leq 0 \end{aligned}$$

- $P \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix
- $A \in \mathbb{R}^{n \times q}_+$
- For $q = 2$, we take $n \in \{5, 10, 15, \ldots, 50\}$ (50 randomly generated instance)
- For $q = 3$, we take $n \in \{10, 15, 20, 25, 30\}$ (20 randomly generated instance)

Results for $q = 3$

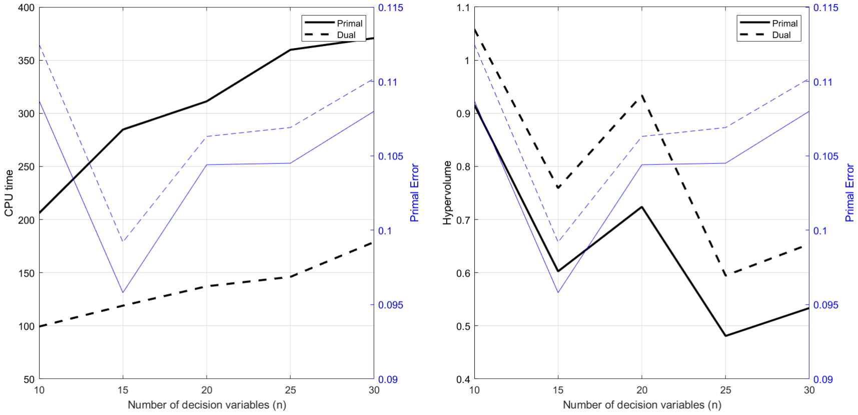
n	Alg	Stop	Opt	T_{opt}	T_{opt}/Opt	En	T_{en}	T	$T/\lvert\bar{\mathcal{X}}\rvert$	PE	HV
10	1	$\epsilon_1 = 0.5000$	52.35	19.92	0.3801	5.05	0.09	46.41	0.91	0.4209	3.5805
	2	$\epsilon_2 = 0.2887$	86.25	29.03	0.3363	4.40	0.20	71.36	70.84	0.1106	1.1973
	1	$\epsilon_3 = 0.1106$	232.80	85.32	0.3655	6.90	0.16	200.13	0.92	0.1052	1.2846
	2	$T = 46.41$	60.70	20.32	0.3336	4.05	0.24	48.43	0.80	0.2244	1.8735
15	1	$\epsilon_1 = 0.5000$	70.85	26.98	0.3777	5.60	0.09	63.54	0.93	0.4694	2.6458
	2	$\epsilon_2 = 0.2887$	105.15	34.87	0.3311	4.70	0.26	84.58	0.80	0.1033	0.7146
	1	$\epsilon_3 = 0.1033$	295.30	108.74	0.3677	7.45	0.21	251.56	0.91	0.0991	0.5690
	2	$T = 63.54$	81.75	27.01	0.3307	4.40	0.36	65.51	0.81	0.2022	1.0703
20	1	$\epsilon_1 = 0.5000$	65.45	24.62	0.3797	5.50	0.09	57.12	0.92	0.4569	3.4845
	2	$\epsilon_2 = 0.2887$	101.95	33.66	0.3302	4.65	0.24	81.31	0.80	0.1052	1.0783
	1	$\epsilon_3 = 0.1052$	284.25	104.18	0.3672	7.35	0.19	238.73	0.91	0.1020	0.7881
	2	$T = 57.12$	74.40	24.63	0.3324	4.30	0.29	59.06	0.80	0.1996	1.5307
25	1	$\epsilon_1 = 0.5000$	85.95	32.63	0.3786	5.95	0.12	82.51	0.98	0.4650	2.3063
	2	$\epsilon_2 = 0.2887$	139.25	46.13	0.3306	4.95	0.34	112.65	0.81	0.1071	0.6154
	1	$\epsilon_3 = 0.1071$	368.60	137.91	0.3736	7.70	0.26	340.28	1.01	0.1034	0.5063
	2	$T = 82.51$	106.10	35.02	0.3297	4.80	0.56	84.55	0.80	0.1878	0.8167
30	1	$\epsilon_1 = 0.5000$	95.70	36.59	0.3818	6.00	0.12	91.27	0.99	0.4690	2.2715
	2	$\epsilon_2 = 0.2887$	150.70	51.13	0.3386	5.15	0.43	131.15	0.87	0.1066	0.6110
	1	$\epsilon_3 = 0.1066$	452.70	170.22	0.3751	8.05	0.32	419.02	1.02	0.1046	0.5420
	2	$T = 91.27$	109.00	36.89	0.3381	4.60	0.48	93.41	0.86	0.2272	0.8947

Results for $q = 3$



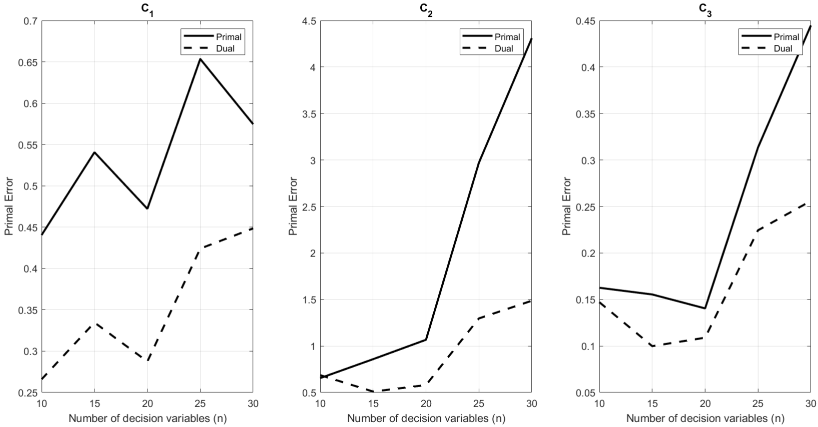
Average PE (left) and HV (right) values under nearly equal runtime (rows one and four of Table).

Results for $q = 3$



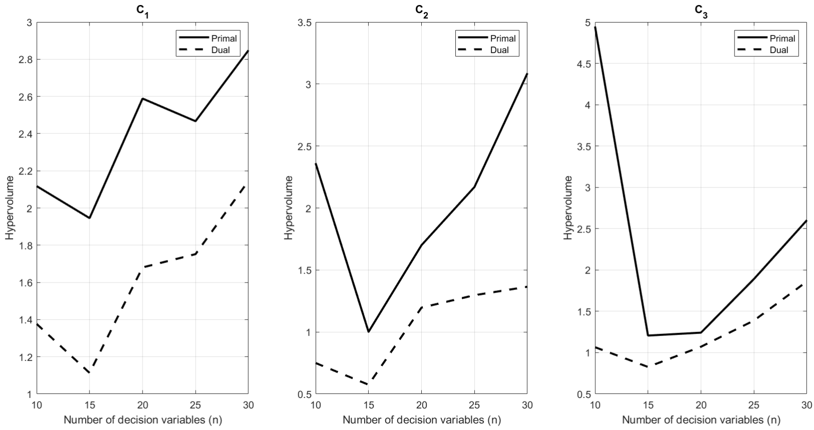
Average CPU time (left) and HV (right) values under nearly equal PE (rows two and three of Table).

Results - different ordering cones



Average primal error values of random instances for $q = 3$ for ordering cones C_1 (left), C_2 (middle) and C_3 (right) when the algorithms are run under time limit of 50 seconds.

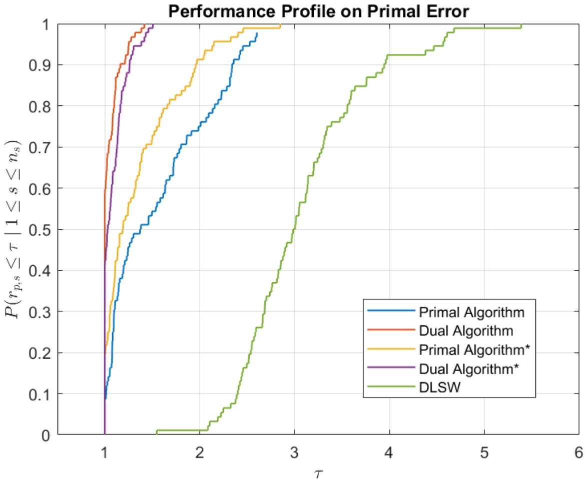
Results - different ordering cones



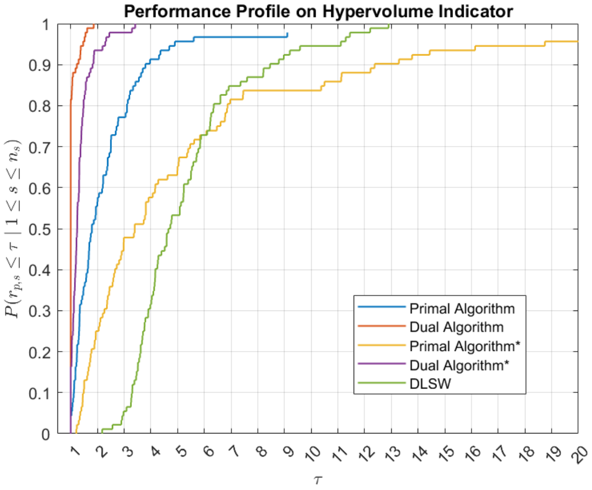
Average HV values of random instances for $q = 3$ for ordering cones C_1 (left), C_2 (middle) and C_3 (right) when the algorithms are run under time limit of 50 seconds.

Performance Profiles

[Dolan, Moré, 2002]



Primal Algorithm [Ararat, et al., 2022], Primal Algorithm* and Dual Algorithm* [Löhne, et al., 2014], DLSW [Dörfler, et al., 2021]



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- Geometric duality relations - free of direction parameter
- Algorithm based on the new results
- Promising performance of the algorithm

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- Geometric duality relations - free of direction parameter
- Algorithm based on the new results
- Promising performance of the algorithm
- Finiteness and convergence rate: **Future Work!**

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Outer approximation algorithms for convex vector optimization problems
Submitted. arxiv preprint no: 2109.07119

Thank you!