A mixed-integer programming approach for computing systemic risk measures

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joint work with Nurtai Meimanjan

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- Related to failures in a financial system
- Triggered by correlated external shocks
- Spreads through the system via contractual obligations
- Undesirable consequences for internal/external members

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- Resource for payments: operating cash flow (exogenous)
- A random shock on the operating cash flow $\rightarrow$ (in)ability to meet liabilities

Random shocks and aggregation functions

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  - $\Lambda(\boldsymbol{X}(\omega))$ is an aggregate quantity associated to operating cash flow under scenario $\omega \in \Omega$
  - Simple examples: $\Lambda(\boldsymbol{x}) = \sum_{i=1}^n x_i$, $\Lambda(\boldsymbol{x}) = -\sum_{i=1}^n x_i^-$
  - More realistic examples: $\Lambda(\boldsymbol{x})$ depends on the clearing mechanism.

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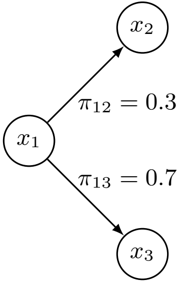
- Relative liabilities matrix: $\boldsymbol{\pi} = (\pi_{ij})_{i,j \in \mathcal{N}} \in \mathbb{R}^{n \times n}_+$ stochastic matrix with $\pi_{ii} = 0$



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- We will work with more general optimization aggregation functions.

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  - Today: more realistic / less regular Λ, linear/polyhedral ρ, computational aspects

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   - Arbitrary convex ρ is out-of-reach for solvers due to integer variables.

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  - In the spirit of extensive formulations in two-stage stochastic programming

The extensive formulation

- Compute $R(\boldsymbol{X}) := \big\{\boldsymbol{z} \in \mathbb{R}^G \mid \mathbb{E}[\Lambda(\boldsymbol{X} + \boldsymbol{B}^\mathsf{T}\boldsymbol{z})] \geq \gamma\big\}$.
- Generally a nonconvex upper set!
- Theorem: Suppose that $R(\boldsymbol{X}) \neq \emptyset$. Then,

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  - $L^\infty(\mathcal{D})$: Essentially bounded measurable selections of random set $\mathcal{D}$
  - In the spirit of extensive formulations in two-stage stochastic programming
  - Almost immediate for finite Ω by monotonicity
  - Needs a careful justification for general Ω by measurable selection arguments
  - Yields mixed-integer programming formulations for scalarizations.

Weighted-sum scalarization

- Let $\boldsymbol{w} \in \mathbb{R}^G_+ \setminus \{\mathbf{0}_G\}$ and consider

$$\mathcal{P}_1(\boldsymbol{w}) = \inf_{z \in R(\boldsymbol{X})} \boldsymbol{w}^\mathsf{T} \boldsymbol{z} = \inf_{\boldsymbol{z} \in \mathcal{Z}} \left\{ \boldsymbol{w}^\mathsf{T} \boldsymbol{z} \mid \mathbb{E}[\Lambda(\boldsymbol{X} + \boldsymbol{B}^\mathsf{T} \boldsymbol{z})] \geq \gamma \right\}.$$

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- Corollary: Let $\boldsymbol{w} \in \mathbb{R}^G_+ \setminus \{\mathbf{0}_G\}$ and assume that $R(\boldsymbol{X}) \neq \emptyset$. Then, $\mathcal{P}_1(\boldsymbol{w})$ equal the optimal value of the problem

$$\begin{aligned} \text{minimize} \quad & \boldsymbol{w}^{\mathsf{T}} \boldsymbol{z} && (\mathsf{P}_1(\boldsymbol{w})) \\ \text{subject to} \quad & \mathbb{E}[f(\boldsymbol{P})] \geq \gamma, \\ & (\boldsymbol{P}, \boldsymbol{S}) \in L^{\infty}(\mathcal{Y}(\boldsymbol{X} + \boldsymbol{B}^{\mathsf{T}} \boldsymbol{z})), \\ & \boldsymbol{z} \in \mathbb{R}^G, \quad \boldsymbol{P} \in L^{\infty}(\mathbb{R}^n), \quad \boldsymbol{S} \in L^{\infty}(\mathbb{Z}^d). \end{aligned}$$

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- When f is linear and Ω is finite, $(\mathsf{P}_1(\boldsymbol{w}))$ is an MILP problem.

Pascoletti-Serafini scalarization

- Let $\boldsymbol{v} \in \mathbb{R}^G$ and consider

$$\begin{aligned}\mathcal{P}_2(\boldsymbol{v}) &:= \inf\left\{\mu \in \mathbb{R} \mid \boldsymbol{v} + \mu \mathbf{1}_G \in R(\boldsymbol{X})\right\} \\ &= \inf\left\{\mu \in \mathbb{R} \mid \mathbb{E}[\Lambda(\boldsymbol{X} + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{v} + \mu \mathbf{1}_n)] \geq \gamma\right\},\end{aligned}$$

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- When f is linear and Ω is finite, $(\mathsf{P}_2(\boldsymbol{v}))$ is an MILP problem.
- Scalarizations can be embedded into an algorithm for nonconvex multiobjective optimization, e.g., (Nobakhtian, Shafiei '17).

Back to clearing systems

- Which $\mathcal{Y}$ to work with?
- EN model: $\mathcal{Y}(\boldsymbol{x}) = \{\boldsymbol{p} \in [\mathbf{0}_n, \bar{\boldsymbol{p}}] \mid \boldsymbol{p} \le \boldsymbol{x} + \boldsymbol{\pi}^\mathsf{T}\bar{\boldsymbol{p}}\} \to d = 0$, no integer variables!
- Interested in adding two additional features:
  - Default costs: Rogers-Veraart model
  - Operating costs: $x_i < 0$ is allowed
  - We can have both features: signed Rogers-Veraart model

Signed Rogers-Veraart model

As in EN model:

- Financial network with nodes $\mathcal{N} = \{1, \ldots, n\}$
- Total obligations: $\bar{\boldsymbol{p}} \in \mathbb{R}^n_+$, relative liabilities: $\boldsymbol{\pi} = (\pi_{ij})_{i,j \in \mathcal{N}} \in \mathbb{R}^{n \times n}_+$

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- Operating cash flow vector: $\boldsymbol{x} \in \mathbb{R}^n$
  $\rightarrow$ External liabilities (e.g. operating costs) are allowed.

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- With $\boldsymbol{x} = \boldsymbol{X}(\omega) + \boldsymbol{B}^\top \boldsymbol{z}$, changes network structure randomly and depending on capital. $\rightarrow$ intractable

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- Major drawback: Interbank and external liabilities are equally senior.

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- Major drawback: Interbank and external liabilities are equally senior.
- Instead, assume that operating costs have seniority over interbank liabilities.

Default costs:

- A defaulting node can use only a fraction $\alpha \in (0, 1]$ of positive operating cash flow, and a fraction $\beta \in (0, 1]$ of cash inflow from other nodes.

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- Definition: $\boldsymbol{p} \in [\mathbf{0}_n, \bar{\boldsymbol{p}}]$ is called a clearing vector if, for every $i \in \mathcal{N}$,
  - Immediate default: If $(\boldsymbol{x} + \boldsymbol{\pi}^{\mathsf{T}} \boldsymbol{p})_i < 0$, then $p_i = 0$.

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- A fixed point characterization is immediate.
- How about an optimization formulation?

Signed Rogers-Veraart model

- Theorem: Consider

$$\begin{aligned} &\text{maximize} \quad && f(\boldsymbol{p}) && (\mathrm{RV}(\boldsymbol{x}))\\ &\text{subject to} \quad && \bar{p}_i s_i - M_i t_i \le x_i + (\boldsymbol{\pi}^\mathsf{T}\boldsymbol{p})_i, && i \in \mathcal{N},\\ & && \bar{p}_i s_i \le p_i \le \bar{p}_i(1-t_i), && i \in \mathcal{N},\\ & && p_i \le (\alpha x_i) \wedge x_i + \beta(\boldsymbol{\pi}^\mathsf{T}\boldsymbol{p})_i + (M_i + \bar{p}_i)(s_i + t_i), && i \in \mathcal{N},\\ & && 0 \le p_i \le \bar{p}_i, \quad s_i, t_i \in \{0,1\}, && i \in \mathcal{N}, \end{aligned}$$

where $M_i \in [x_i^-, +\infty)$ is a constant (big-M). Then, $(\mathrm{RV}(\boldsymbol{x}))$ has an optimal solution. If $(\boldsymbol{p}, \boldsymbol{s}, \boldsymbol{t})$ is an optimal solution, then $\boldsymbol{p}$ is a clearing vector.

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- t_i checks if there is immediate default.
- s_i checks if default costs are active (non-immediate default).
- Now: We can define $\mathcal{Y}^{\mathsf{RV}}(\boldsymbol{x})$ via the constraints, set $\Lambda^{\mathsf{RV}}(\boldsymbol{x})$ as the optimal value.

Systemic risk measure for signed Rogers-Veraart model

- Compute $R^{\mathsf{RV}}(\boldsymbol{X}) := \{\boldsymbol{z} \in \mathbb{R}^G \mid \mathbb{E}[\Lambda^{\mathsf{RV}}(\boldsymbol{X} + \boldsymbol{B}^{\mathsf{T}}\boldsymbol{z})] \geq \gamma\}$.

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- Take $f(\boldsymbol{p}) = \mathbf{1}_n^{\mathsf{T}}\boldsymbol{p}$.
- Combining the MILP for Λ^{RV} with the mixed-integer formulations for scalarizations yields extensive MILPs that can be solved by commercial solvers for finite Ω.

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- Take $f(\boldsymbol{p}) = \mathbf{1}_n^{\mathsf{T}}\boldsymbol{p}$.
- Combining the MILP for Λ^{RV} with the mixed-integer formulations for scalarizations yields extensive MILPs that can be solved by commercial solvers for finite Ω.
- Then, we can run the nonconvex Benson-type algorithm of Nobakhtian, Shafiei '17.

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- $n = 45$ banks: $n_1 = 15$ big banks (“core”) + $n_2 = 30$ small banks (“periphery”)

Computational study: Rogers-Veraart model

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- $n = 45$ banks: $n_1 = 15$ big banks ("core") + $n_2 = 30$ small banks ("periphery")
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- Connectivity probabilities:
  core $\rightarrow$ core: 0.5, core $\rightarrow$ periphery: 0.1,
  periphery $\rightarrow$ core: 0.3, periphery $\rightarrow$ periphery: 0.5

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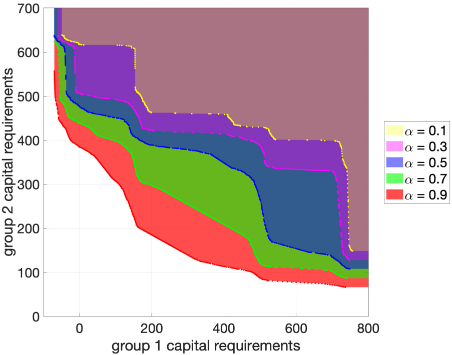
  core $\rightarrow$ core: 0.5, core $\rightarrow$ periphery: 0.1,
  periphery $\rightarrow$ core: 0.3, periphery $\rightarrow$ periphery: 0.5
- Risk measure: negative expectation with threshold $\gamma = \gamma^{\mathsf{p}}(\mathbf{1}_n^{\mathsf{T}}\bar{\boldsymbol{p}})$ with $\gamma^{\mathsf{p}} \in [0, 1]$
- γ^{p}: average target fraction of total liabilities that should be met at clearing
- Base case parameters: $\alpha = 0.7$, $\beta = 0.9$, $\gamma^{\mathsf{p}} = 0.9$

Sensitivity with respect to α

α	$\lvert\mathrm{vert}(\mathcal{L}^T)\rvert$	T	$\overline{\text{time}}(\mathcal{P}_2)$ (sec.)	time(total) (min.)
0.1	273	333	12.2	67.5
0.3	461	484	10.6	85.3
0.5	592	602	5.2	52.5
0.7	583	584	3.9	37.7
0.9	589	589	3.4	33.3

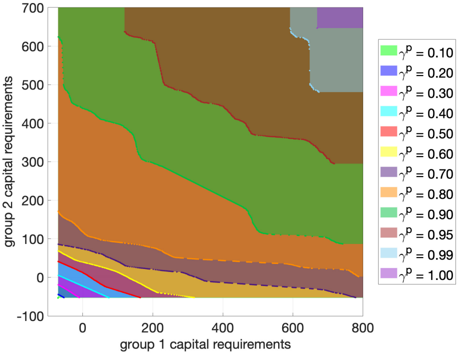
- As α increases, the discontinuity in fixed point formulation decreases, $\mathcal{P}_2$ becomes easier.



Sensitivity with respect to γ^{p}

γ^{p}	$\lvert \mathrm{vert}(\mathcal{L}^T) \rvert$	T	$\overline{\text{time}}(\mathcal{P}_2)$ (sec.)	time(total) (min.)
0.2	13	13	13.8	3
0.3	51	51	30.3	25.7
0.4	94	94	36.6	57.4
0.5	165	165	98.6	271.2
0.6	223	223	138.5	514.9
0.7	389	389	204.3	1324.5
0.8	395	395	91.6	603.0
0.9	583	584	3.9	37.7

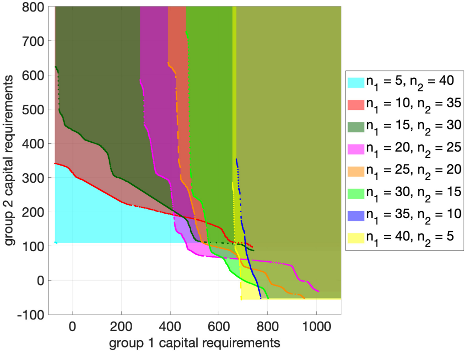
- $\mathcal{P}_2$ and overall problem get more difficult around $\gamma^{\mathrm{p}} = 0.7$.



Sensitivity with respect to n_1

- Change the distribution of nodes among groups while $n = n_1 + n_2 = 45$ is fixed.

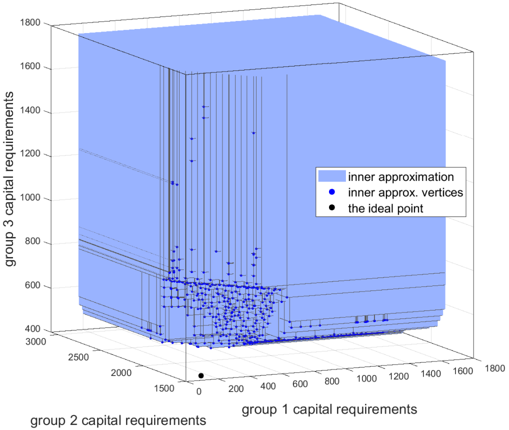
n_1	$\lvert\mathrm{vert}(\mathcal{L}^T)\rvert$	T	$\overline{\text{time}}(\mathcal{P}_2)$ (sec.)	time(total) (min.)
5	6	6	1.0	0.1
10	436	436	4.0	29.0
15	583	584	3.9	37.7
20	516	517	7.9	68.0
25	557	557	6.1	56.8
30	371	371	5.8	35.8
35	187	187	6.1	19.0
40	106	108	5.2	9.4



A three-group Rogers-Veraart network

- $n = 60$ banks: $n_1 = 10$, $n_2 = 20$, $n_3 = 30$
- $\alpha = \beta = 0.9$, $\gamma^{\mathsf{p}} = 0.99$
- Highly nonconvex!

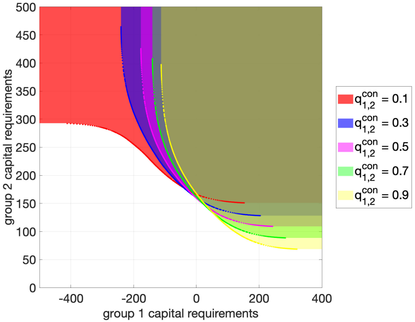
$\lvert \mathrm{vert}(\mathcal{L}^T) \rvert$	T	$\overline{\text{time}}(\mathcal{P}_2)$ (sec.)	time(total) (min.)
975	19382	0.4	138.1



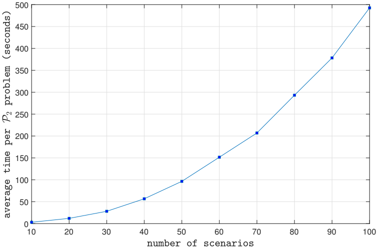
Computational study: signed Eisenberg-Noe model

- Random shock: $K = 100$ instances of a Gaussian random vector
- Two groups with sizes $n_1 = 15$ and $n_2 = 35$
- Computations take more time. The sets look smoother, sometimes convex-looking.
- e.g., sensitivity with respect to connectivity probability $1 \to 2$:

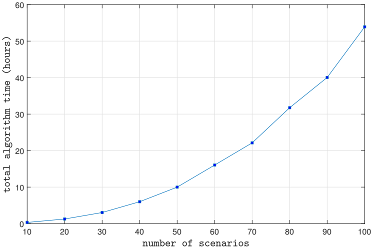
$q^{\text{con}}_{1,2}$	$\lvert \mathrm{vert}(\mathcal{L}^T) \rvert$	T	$\overline{\text{time}}(\mathcal{P}_2)$ (sec.)	time(total) (hr.)
0.1	279	358	294.0	29.2
0.3	394	394	492.6	53.9
0.5	360	360	556.8	55.7
0.7	364	364	633.6	64.1
0.9	377	377	772.8	80.9



- Difficulty increases drastically with K.



(a) $\overline{\text{time}}(\mathcal{P}_2)$ (sec.)



(b) time(total) (hr.)

This is a joint work with Nurtai Meimanjan.

An up-to-date preprint will soon be available at
www.arxiv.org/abs/1903.08367.

Thank you!