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Pricing options on flow forwards by neural networks in Hilbert space

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Problem

- Using a neural network, learn the map

$$x \mapsto \mathbb{E}[\mathscr{P}(X_\tau^{t,x})]$$

$$H \longrightarrow \mathbb{R}$$

- Here, $(X_s^{t,x})_{s\geq t}$ is a stochastic process in a Hilbert space H, starting in $x \in H$ at time t.

- $\mathscr{P} : H \to \mathbb{R}$ is the payoff functional

- Learning on simulated data by exploiting the structure in Hilbert space

Why?

INFINITE DIMENSIONAL OPTION PRICING

Why?

- Pricing options in electricity and gas markets
  - …as well as markets for temperature, wind and freight

- Options written on forwards/futures contracts
  - …where delivery takes place over a period $[T_1, T_2]$

- EEX: calls and puts on flow forwards, $t \leq \tau \leq T_1 < T_2$

$$C(t) = \mathbb{E}[\max(\hat{F}_\tau(T_1, T_2) - K, 0) \mid \mathcal{F}_t]$$

$$\hat{F}_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F_t(T)\,dT$$

Why? … some notation

- Musiela parametrization: $\xi := T - t$, ξ is time-to-maturity

$$X_t(\xi) := F_t(\xi + t)$$

- Let H be a Hilbert space of real-valued functions on $\mathbb{R}_+$,
  - where the evaluation functional $\delta_\xi \in H^*$

$$\hat{F}_t(T_1, T_2) = \hat{F}_t(t + (T_1 - t), t + (T_1 - t) + (T_2 - T_1))$$

$$= \frac{1}{T_2 - T_1} \int_0^\infty 1_{[0, T_2 - T_1]}(u - (T_1 - t)) X_t(u) du$$

$$:= \delta_{T_1 - t} \mathcal{D}_{T_2 - T_1}(X_t)$$

- Assume integral operator $\mathcal{D}_\lambda \in L(H)$

Why? … back to options again

- Let $p : \mathbb{R} \to \mathbb{R}$ be the option's payoff function

$$p(\hat{F}_\tau(T_1, T_2)) = p(\delta_{T_1-\tau}\mathcal{D}_{T_2-T_1}(X_\tau)) := \mathcal{P}(X_\tau)$$

- Model for the forward curve: assume H is a Banach algebra
  - Pointwise multiplication of curves

$$X_t := \exp(Y_t)$$

- Log-forward curves given by H-valued OU-process
  - W is Q-Wiener process in H, N is homogeneous Poisson random measure on H with Levy measure $\nu(dz)$

$$dY_t = \partial_\xi Y_t dt + \alpha(t)dt + \eta(t)dW(t) + \int_H \gamma(t,z)\widetilde{N}(dt,dz)$$

Why ?...

- Time-dependent, non-random coefficients, integrable

$$|\alpha(\cdot)|, \quad \|\eta(\cdot)\|_{HS} \in L^2_{loc}(\mathbb{R}_+), \quad \int_H |\gamma(\cdot, z)|^2 \nu(dz) \in L^1(\mathbb{R}_+)$$

- Mild solution
  - shift semigroup $\mathcal{S}_t f := f(\cdot + t)$ strongly continuous with densely defined generator ∂_ξ

$$Y_t = \mathcal{S}_t Y_0 + \int_0^t \mathcal{S}_{t-s}\alpha(s)ds + \int_0^t \mathcal{S}_{t-s}\eta(s)dW(s) + \int_0^t \int_H \mathcal{S}_{t-s}\gamma(s,z)\widetilde{N}(ds,dz)$$

- No-arbitrage drift condition,

$$\alpha(t,\cdot) = -\frac{1}{2}|Q^{1/2}\eta^*(t)(\delta_\cdot^* 1)|^2 - \int_H \{\exp(\gamma(t,z)) - 1 - \gamma(t,z)\}\,\nu(dz)$$

Why?

- Using algebra-property and semigroup

$$X_\tau = \exp(Y_\tau) = \exp(\mathcal{S}_{\tau-t}Y_t)\exp(Z_{t,\tau}) = (\mathcal{S}_{\tau-t}X(t))\exp(Z_{t,\tau})$$

$$Z_{t,\tau} = \int_t^\tau \mathcal{S}_{\tau-s}\alpha(s)ds + \int_t^\tau \mathcal{S}_{\tau-s}\eta(s)dW(s) + \int_t^\tau\int_H \mathcal{S}_{\tau-s}\gamma(s,z)\widetilde{N}(ds,dz)$$

- Option price (zero risk free rate)

$$V(t) = \mathbb{E}[p(\hat{F}_\tau(T_1,T_2)) \mid \mathcal{F}_t] = \mathbb{E}[\mathcal{P}(X_\tau) \mid \mathcal{F}_t]$$

- Or, $V(t) := V(t, X_t)$, with

$$V(t,x) = \mathbb{E}[\mathcal{P}((\mathcal{S}_{\tau-t}x)\exp(Z_{t,\tau}))]$$

$$H \ni x \mapsto V(t,x) \in \mathbb{R}$$

…and now, really why!

- The option price is *not* a function of $\hat{F}_t(T_1, T_2)$

$$\delta_{T_1-\tau}\mathscr{D}_{T_2-T_1}((\mathcal{S}_{\tau-t}x)\exp(Z_{t,\tau})) = \frac{1}{T_2 - T_1}\int_{T_1-t}^{T_2-t} x(v)e^{Z_{t,\tau}(v-(\tau-t))}dv$$

- The option price is depending on the *term structure* $T \mapsto F_t(T)$

- **Proposition:** Suppose p is Lipschitz. Then V is well-defined and Lipschitz continuous.

  Proof: well-defined comes from exponential moments of Wiener process (Fernique) and exponential integrability in no-arbitrage condition of jumps. Lipschitz follows by direct calculation.

Convenient minimization

- Let μ be a measure on H supported on a compact
- Define for $g \in L^2(\mu)$

$$I(g) = \mathbb{E}[\int_H |\mathscr{P}(\mathcal{S}_{\tau-t} x \exp(Z_{t,\tau})) - g(x)|^2 \mu(dx)]$$

- **Proposition**: $I(V) = \inf_{g \in L^2(\mu)} I(g)$

  Proof: A direct calculation shows

$$I(g) = \int_H \mathrm{Var}(\mathscr{P}(\mathcal{S}_{\tau-t} x \exp(Z_{t,\tau})))\mu(dx) + \int_H |V(t,x) - g(x)|^2 \mu(dx) \geq I(V)$$

BACKGROUND AND OTHER APPLICATIONS

Background

- Jentzen et al.: neural net to learn the map

$$\mathbb{R}^d \ni x \mapsto \mathbb{E}[\mathscr{P}(X_\tau^{t,x})]$$

- $X_\tau^{t,x}$ diffusion process on $\mathbb{R}^d$
- Options on very-high dimensional baskets of assets
  - I.e., x is high dimensional
  - Numerical solution of PDEs in high dimensions

- Cheridito, Teichmann et al.: optimal stopping and American options

- Bayer, Cuchiero, Horvath et al.: implied rough volatility, local volatility

- Weinan E et al.: stochastic control

Other applications

- Hedging volume and price risk by quantos
    - Joint payoff on price and temperature

$$\mathbb{E}[\mathscr{P}(F_\tau^E(T_1, T_2))\mathscr{Q}(F_\tau^T(T_1, T_2))]$$

- Virtual power plants and swing options
    - User-time contracts (volume-flexibility at strikes)
    - Gas and coal-fired power plants (strip of calls)

- Fixed-income: call and puts on SOFR-futures
    - SOFR: secured overnight rates, US substitute for LIBOR
    - CME: trades in SOFR-futures, “flow forwards” on SOFR rates

NEURAL NETWORK IN HILBERT SPACE

Classical neural networks (one layer)

- Given continuous function $f \in C(\mathbb{R}^d, \mathbb{R})$, find neural network that approximates it on a compact $K \subset \mathbb{R}^d$

- Neural network
  - Fix continuous activation function $\sigma : \mathbb{R} \to \mathbb{R}$
  - For $a \in \mathbb{R}^d, \ell, b \in \mathbb{R}$ define a neuron $\mathcal{N}_{\ell,a,b} \in C(\mathbb{R}^d, \mathbb{R})$

$$x \mapsto \ell \sigma(a^\top x + b)$$

- One-layer neural network (NN)

$$\sum_{i=1}^{N} \mathcal{N}_{\ell_i,a_i,b_i}(x) = \sum_{i=1}^{N} \ell_i \sigma(a_i^\top x + b_i)$$

Universal approximation theorem

- For given activation function σ, define linear space generated by neurons

$$\mathfrak{N}(\sigma) := \text{span}\left\{\mathcal{N}_{\ell,a,b} \,:\, \ell, b \in \mathbb{R}, a \in \mathbb{R}^d\right\}$$

- **Universal approximation**: under mild conditions on σ
  - $\mathfrak{N}(\sigma)$ is dense with respect to the topology of uniform convergence on compacts.
  - For every $f \in C(\mathbb{R}^d, \mathbb{R})$ and compact $K \subset \mathbb{R}^d$, given $\epsilon > 0$ there exists $N \in \mathbb{N}$ and $\ell_i, b_i \in \mathbb{R}, a_i \in \mathbb{R}^d$ such that

$$\sup_{x \in K} \left| f(x) - \sum_{i=1}^{N} \mathcal{N}_{\ell_i,a_i,b_i}(x) \right| < \epsilon$$

Neural network in infinite dimensions

- Extend network from $\mathbb{R}^d$ to $\mathbb{R}^\infty$, i.e., to infinite dimensional topological vector space $\mathfrak{X}$
  - Approximate functions $f \in C(\mathfrak{X}, \mathbb{R})$

- Why?
  - Compute efficiently option prices on flow forwards

- $\mathfrak{X} =$ Hilbert space: **exploit structure of basis functions**
  - Train network using data AND structural information!
  - Flexibility on activation function across all dimensions

Neural network

- A neuron is defined by
  - Fixed activation function $\sigma \in C(\mathfrak{X}, \mathfrak{X})$
  - Affine map $x \mapsto Ax + b$, with $A \in L(\mathfrak{X}), b \in \mathfrak{X}$
  - $\ell \in \mathfrak{X}^*$, with $\mathfrak{X}^*$ topological dual of $\mathfrak{X}$

$$\mathcal{N}_{\ell,A,b}(x) = \langle \ell, \sigma(Ax + b) \rangle$$

- Let $\mathfrak{X}$ be a Frechet space
  - Complete metrizable locally convex topological vector space
  - Topology generated by seminorms $(p_k)_{k \in \mathbb{N}}$

- Consider $C(\mathfrak{X}, \mathbb{R})$ with locally convex topology generated by family of seminorms $\{q_K \ : \ K \subset \mathfrak{X}, \text{compact}\}$
  - Riesz representation theorem

Universal approximation

- Activation function σ is called *discriminatory* is for any fixed pair (μ, K)

$$\int_K \langle \ell, \sigma(Ax + b) \rangle \mu(dx) = 0$$

  for all $\ell \in \mathfrak{X}^*, A \in L(\mathfrak{X}), b \in \mathfrak{X}$ implies that $\mu = 0$

- μ is a regular (signed) Borel measure on K

- **Proposition**: Let σ be discriminatory. Then $\mathfrak{N}(\sigma)$ is dense in $C(\mathfrak{X}, \mathbb{R})$

  Proof: Following Cybenko's classical proof using Hahn-Banach. Riesz representation for linear functionals on $C(\mathfrak{X}, \mathbb{R})$

Discriminatory activation function?

- Restrict to activation functions being *von-Neumann bounded*
  - For any $k \in \mathbb{N}$ there exists $c_k > 0$ such that

$$\sup_{y \in \sigma(\mathfrak{X})} p_k(y) \leq c_k$$

- Fix a $\psi \in \mathfrak{X}^* \backslash \{0\}$, define hyperplane Ψ_0 and half-spaces Ψ_+, Ψ_-

$$\Psi_+ := \{x \in \mathfrak{X} \ : \ \langle \psi, x \rangle > 0\}$$

$$\Psi_0 := \mathrm{Ker}(\psi)$$

$$\Psi_- := \{x \in \mathfrak{X} \ : \ \langle \psi, x \rangle < 0\}$$

Discriminatory by separating property

- Activation function σ is *separating* if there exists $\psi \in \mathfrak{X}^*\backslash\{0\}$ and $u_0+, u_-, u_0 \in \mathfrak{X}$ such that either $u_+ \notin$ span$\{u_0, u_-\}$ or $u_- \notin$ span$\{u_0, u_+\}$ and

$$\lim_{\lambda\to\infty} \sigma(\lambda x) = u_*, \text{if } x \in \Psi_*$$

for $* \in \{+,-,0\}$

- **Proposition**: If σ is von Neumann-bounded and separating, then it is discriminatory.

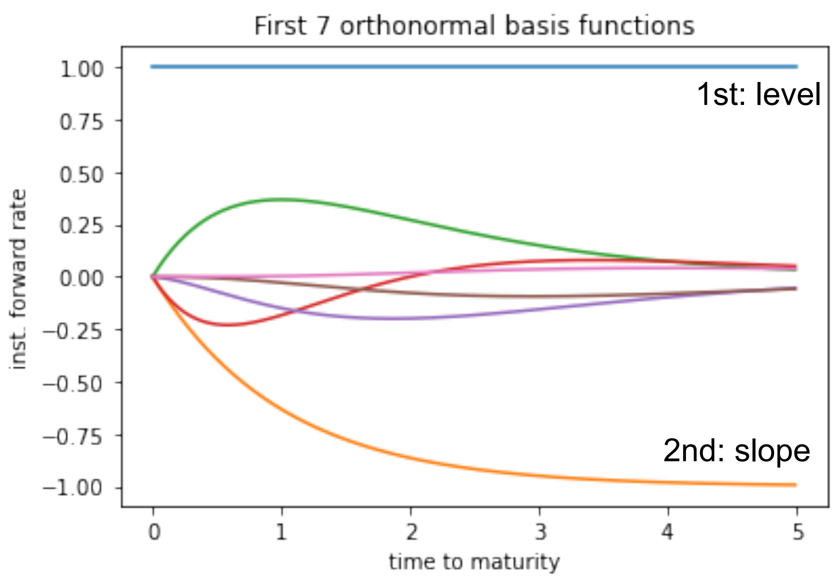
  Proof: Using the Hahn-Jordan decomposition of μ, and playing around wth the flexible choice of the neural network parameters ℓ, A, b and separation of σ.

APPLICATION

Pricing call options on monthly forwards

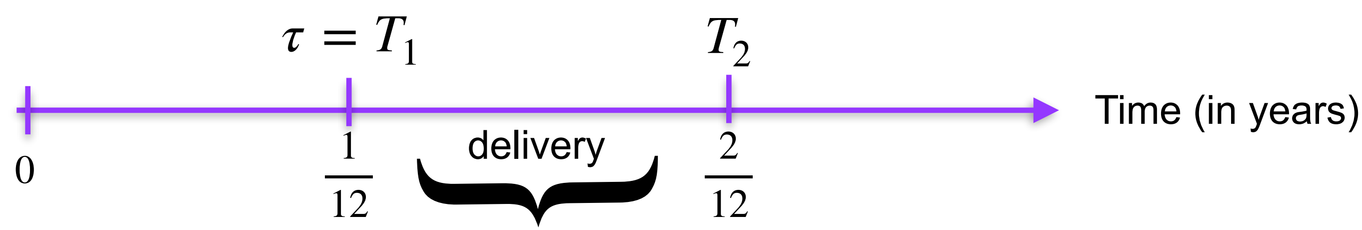
- Focus on Wiener case (no jumps) for X_τ, defined on $\mathfrak{X} = H_w$
  - Volatility is identity operator $\eta = \mathrm{Id}$
  - H_w Hilbert space of absolutely continuous functions on $\mathbb{R}_+$ with weak derivative decaying to zero faster than some monotonly increasing function w
  - Filipovic space, "weighted Sobolev space".
  - Separable and well-suited for forward dynamics

- Neural network may exploit information from basis functions



Pricing calls….

- Delivery next month, and strike in one month at strike price 1



- Train a neural network for $x \mapsto V(0,x)$, where $x \in K$ compact

- Training using *simulated* data
  - μ probability distribution on K (we use uniform distribution)
  - Draw M samples $(x^{(m)}, z^{(m)})_{m=1}^{M}$, $x^{(m)} \sim \mu$ and $z^{(m)} \sim Z_{0,\tau}$

$$\inf_{N,\ell_i,A_i,b_i} \frac{1}{M} \sum_{m=1}^{M} \left| \mathcal{P}((\mathcal{S}_\tau x^{(m)}) \exp(z^{(m)})) - \sum_{i=1}^{N} \mathcal{N}_{\ell_i,A_i,b_i}(x^{(m)}) \right|^2$$

Choice of activation function

- Consider Lipschitz continuous function $\beta : \mathbb{R} \to \mathbb{R}$

$$\lim_{y\to\infty} \beta(y) = 1, \lim_{y\to-\infty} \beta(y) = 0, \quad \beta(0) = 0$$

- We use

$$\beta(y) = \max\{0,1 - \exp(-y)\}$$

- Let $\psi \in H_w^* \setminus \{0\}$ and $z \in H_w, z \neq 0$, and define

$$\sigma(x) := \beta(\psi(x))z$$

- σ is Lipschitz, **von Neumann-bounded** and **separating**!

More specifications in training

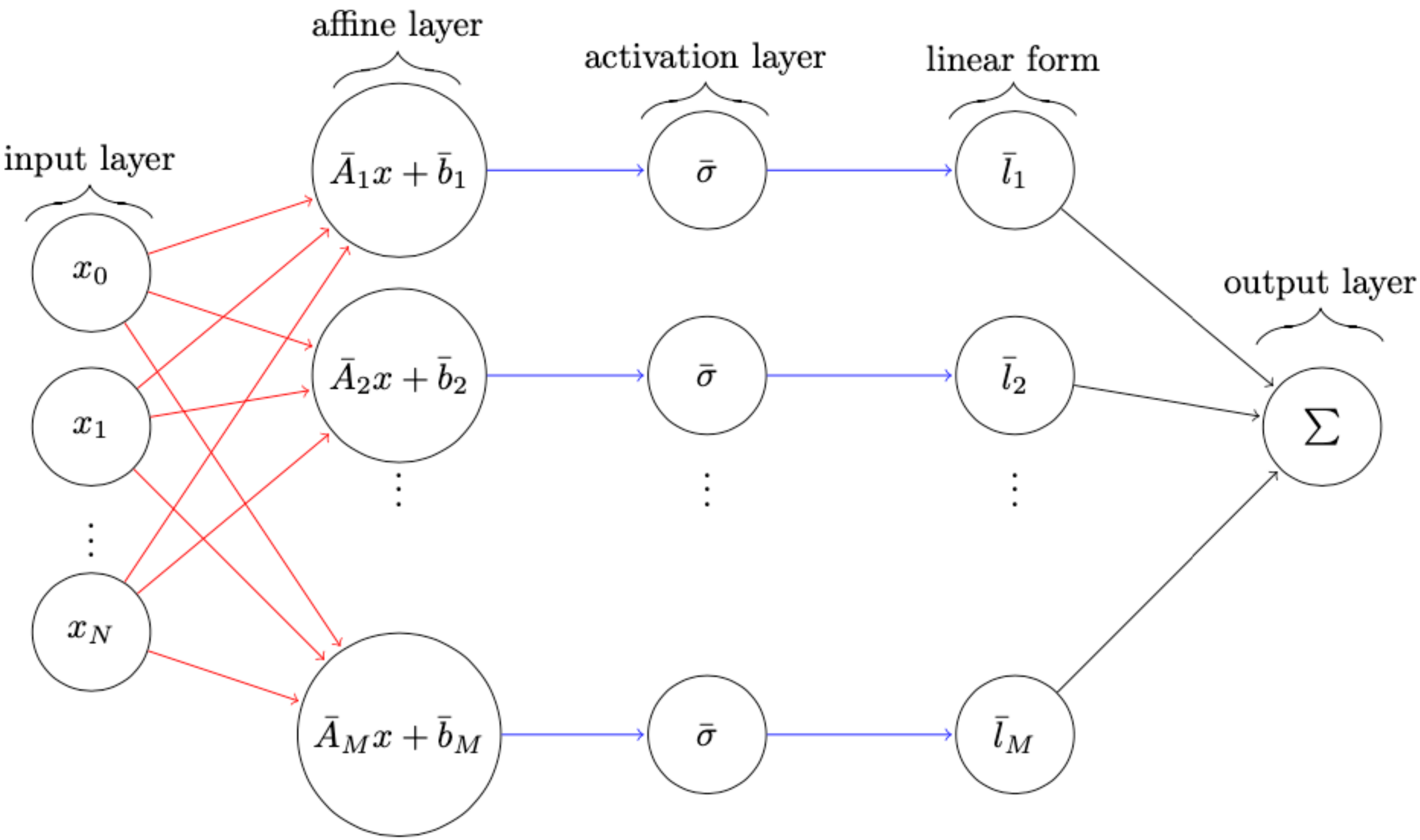
- Compact set K:

$$K_O := \left\{ \sum_{k=1}^{O} x_k e_k \ : \ x_k \in [-r, r] \right\}$$

- Training is done for a finite-dimensional projection of the neural network

- Trained several networks with hidden nodes (neurons) ranging from 1 to 30
    - Stochastic gradient descent in training
    - M=10 million Monte Carlo samples
    - Implemented in Python using TensorFlow and Keras libraries

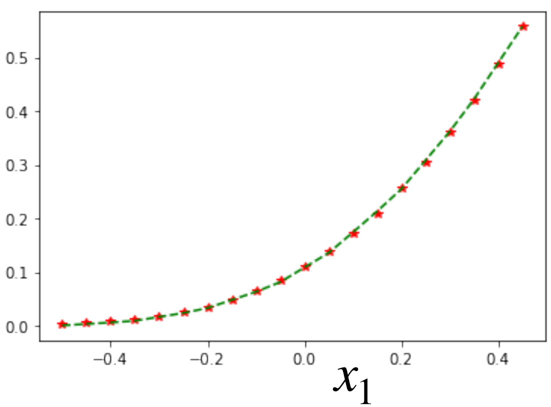
Neural network architecture



Validation

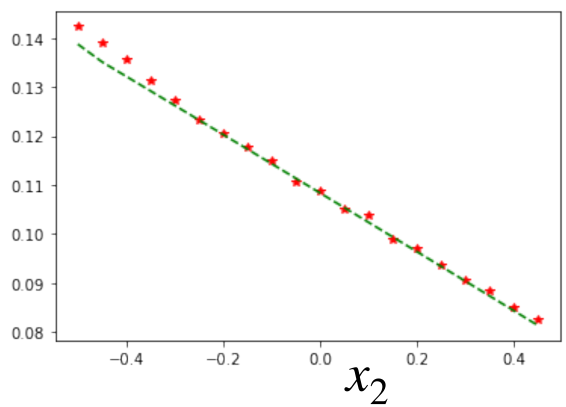
- Validated on 10.000 initial curves x with corresponding “true” option prices
  - x’s are randomly drawn from μ
  - “True” option prices calculated by Monte Carlo simulations, using 100.000 samples

Option price as a function of initial curve



Level, first coordinate

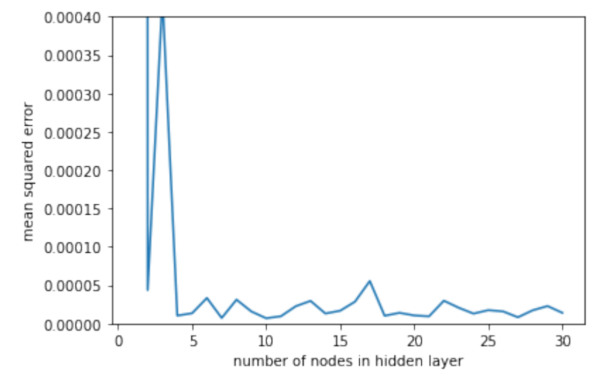
$x_2 = x_3 = \ldots = x_7 = 0$



Slope, second coordinate

$x_1 = x_3 = \ldots = x_7 = 0$

MSE

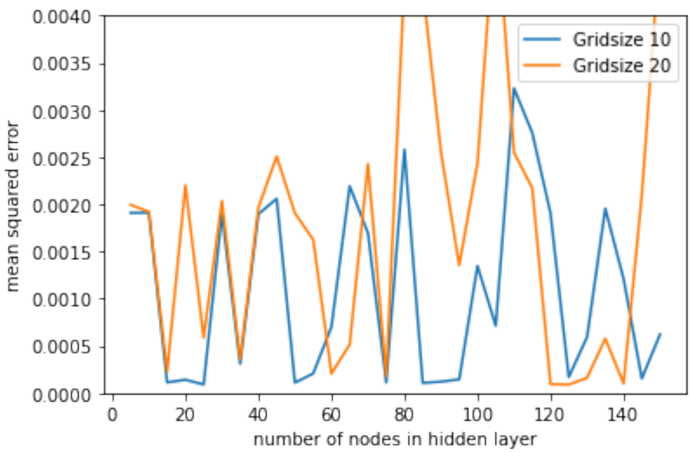


Nodes

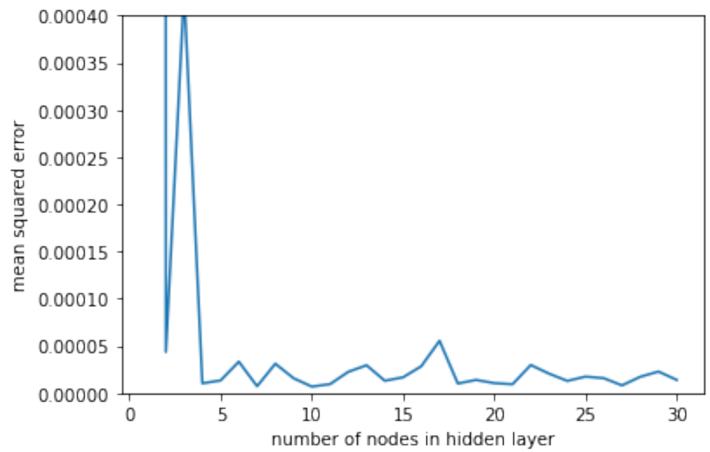
Comparison

- Comparing with classical neural network
  - Samples the initial forward curve $\xi \mapsto x(\xi)$
- Use networks with similar complexity
  - 10 and 20 discrete sample points of curves
  - 60-1800 parameters for 5-50 nodes in classical network, vs. 63-1890 parameters for 1-30 nodes in Hilbert network

MSE classical



MSE Hilbert



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