# Affine Models for Energy Markets

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# Outline

- Recap: No-arbitrage principle
- Introduction: Modelling in electricity markets
- Dynamic, FDR and affine models

# No-Arbitrage and Risk neutral measure

- Random market dynamics are mathematically modelled relative to some measure $P$. (Physical measure)
- No Arbitrage (NA) means that one cannot systematically take advantage of a market.
- FTAP: A market does not allow for arbitrage if there is an equivalent measure $Q$ that turns all discounted **traded** assets into $Q$-martingales. (Pricing measure)
- (NA) Pricing: (NA) compatible prices are derived via $Q$-expectation for some artificial measure.

Note: **Traded means here** that buy-hold-sell strategies are possible.

# Market specifics of electricity markets

- Electricity is a flow commodity, i.e. it is not delivered instantaneously but with a rate over time.
- Electricity is difficult to store. **It's not traded in the previous mentioned sense.**
- Physical contracts are only open to physical participants.
- Spot prices are observed prices between physical participants.
- Futures prices are observed prices between (possibly non-physical) participants. However, their 'payoffs' are linked to aggregated realised spot prices.

# Spots and futures in electricity markets



## Spot:

- 'Spot market for electricity' refers to a daily auction for energy delivery on the next day.
- We will (somewhat ignorantly) denote the spot price of one unit of electricity at time $t$ by $S_t$. (Usually, Euro/MWh) (some prices are reported on epexspot.com)

## Futures:

- Electricity futures refer to standardised period (e.g. calender days, weeks, month, quarters, years) and 'deliver' the aggregated spot price equivalence over this period. (some European prices are reported on EEX.com)
- We denote delivery periods simply as intervals $[T_1, T_2]$ with $T_1 < T_2$ and the price to enter a futures contract at time $t$ with delivery period $[T_1, T_2]$ by $F_t(T_1, T_2)$.

There are other electricity markets, e.g. so-called intra-day market and reserve markets.

# Fundamental principles for energy spot prices and futures

- Spot price: Discounted spot prices **does not** need to be a $Q$-martingale or anything related.
- Futures price: Discounted futures prices **does** need to be a $Q$-martingale.
- Price relation if risk free rate is zero and both spots and futures are present on a market: (see [Benth, Benth, Koekebakker 08])

$$F_t(T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \mathrm{E}_Q[S_u | \mathcal{F}_t] du$$

# Summary of spot models

1. Any model for the spot satisfies the (NA) assumption.
2. $\Rightarrow$: Any equivalent measure $Q$ is a candidate to be the risk neutral measure.
3. Unless, $Q$ is chosen as well, **futures prices are not determined** by the spot prices $S$.

# Futures models — how to model?

1. For each delivery period $[T_1, T_2]$ make a price model for $F_t(T_1, T_2)$. (Direct model)
2. Make the assumption that there is a random curve $x \mapsto f_t(x)$ for every point of time $t$ and define (HJM-type model)

$$F_t(T_1, T_2) := \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} f_t(x - t) dx.$$

# Remarks on Direct and HJM-type models:

Direct models:

- Models only those futures that are actually traded.
- No information on the spot price but can be included as well.
- FTAP works as always.

HJM-type model:

- Implicitly models prices for all possible delivery periods.
- $f_t(x)$ can be thought of a price of electricity with instantaneous delivery in $x$ units of time.
- Another view on $f_t$ is that $\mathrm{E}_Q[S_{t+x}|\mathcal{F}_t] = f_t(x)$ holds.
- In particular, $S_t = f_t(0)$ is modelled along the way.

$\longrightarrow$ In this talk I will from now on ignore the direct modelling approach.

# Futures models — continued

Three approaches for HJM-type models.

- Dynamic model: $df_t(x) = \beta_t(x)dt + \sigma_t(x)dW_t$
- Finite dimensional realisation (FDR): $f_t(x) = h(x+t) + g(x, Y_t)$ for deterministic curves $g, h$ and $Y$ a finite dimensional Itô-process ($dY_t = b_t dt + c_t dW_t$).
- Affine models: $f_t(x) = h(x+t) + \sum_{j=1}^{d} u_j(x) Y_t^j$ for $Y$ an Itô-process.

Observations:

1. Dynamic models are very flexible. However, they might be infinite dimensional.
2. FDR models are as flexible as functions $g, h$ are. If $g, h$ are smooth, then they are dynamic models.
3. Affine models are special types of FDR models where $f_t$ stays in the affine space $h(\cdot + t) + \mathrm{Span}(u_1, \dots, u_d)$.

# HJM-drift condition

The following result was derived in a different setting but all the arguments apply for FDR model as well.

**Theorem (Bühler (2006))**

*(Under some mild technical condition) an FDR model $f_t(x) = g(x, Y_t) + h(x+t)$, $t, x \geq 0$ is free of arbitrage[1] if and only if*

1. *$f_t(x)$ is continuously differentiable in $x$ and*
2. *$f_t'(x)$ is the drift coefficient of $f_t(x)$ under $Q$.*

*If the FDR model is understood as a dynamic model, then necessarily*

$$\beta_t(x) = f_t'(x) + \gamma_t \sigma_t(x).$$

*This is 'sort of' true for any dynamic model with no arbitrage.*

[1] in the sense that a market with all hypothetical futures is free of arbitrage

# All dynamic models are approximately affine models

**Theorem (Benth, K. (2017))**

*Let $df_t(x) = \beta_t(x)dt + \sigma_t(x)dW_t$ be an arbitrage-free (Markovian) dynamic model*[2]*. Then there is a sequence $f^{(n)}$ of arbitrage-free (Markovian) affine models such that*

$$f^{(n)} \to f.$$

Isn't this obvious by a projection method?

[2] such that $f_t$ has values in a Hilbert space of functions where point evaluations are continuous (RKHS)

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- Recall that Bühler's result tells us that $f^{(n)}$ can only be arbitrage free if it is $C^1$ in $x$.
- Non-careful projections might violate this! Apparently, care has to be taken.

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2. Let us pretend from now on that we believe in an FDR model:

$$f_t(x) = g(x, Y_t) + h(x + t)$$

with $g, h$ already chosen. (and $dY_t = b_t dt + c_t dW_t$).

$\longrightarrow$ How do I 'get' $Y$?.

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3. Wait, what kind of data does one expect?
4. There are finitely many time-points and for any of these time-points we have finitely many futures prices.
5. $\longrightarrow$ implicit information on $Y$.

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3. Wait, what kind of data does one expect?
4. There are finitely many time-points and for any of these time-points we have finitely many futures prices.
5. $\longrightarrow$ implicit information on $Y$.
6. Let's just assume that we have a method to estimate the diffusion coefficient $c_t$ of $Y$.

# 'Getting' $Y$: An example

1. Let us consider the following explicit choices for $g, h$; namely $h = 0$ and
$$g(x, y) = \Phi\left(\frac{1 - y}{\sqrt{1 + x}}\right), \quad x \geq 0, y \in \mathbb{R}.$$
2. Also let $Y$ be an Itô-process ($dY_t = b_t dt + c_t dW_t$) such that $f_t(x) = g(x, Y_t)$ satisfies (NA).

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3. Then,
$$c_t = 1.$$
4. Is that a problem? Well, an estimator for the diffusion coefficient will rarely turn out to be 1.

# Volatility estimator and NA consistency for FDR models

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**Theorem (K. Shu, (To appear 22?))**

*Let* $g \in C^{(1,2)}(\mathbb{R}_+ \times \mathbb{R}^d, \mathbb{R})$ *and* $h : \mathbb{R}_+ \to \mathbb{R}$ *continuous such that for any possible diffusion coefficient* $c$ *there is an Itô process* $dY_t^c = b_t^c dt + c_t dW_t$ *such that*

$$f_t^c(x) := g(x, Y_t^c) + h(t, x), \quad t, x \geq 0$$

*is an (NA) model.*



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*is an (NA) model.*
*Then there is* $A \in C^2(\mathbb{R}^d, \mathbb{R}^d)$ *and* $u : C^\infty(\mathbb{R}_+, \mathbb{R})$ *such that*

$$g(x, y) = \sum_{j=1}^{d} u_j(x) A_j(y), \quad x \geq 0, y \in \mathbb{R}^d$$

*i.e. the model is affine in the new variable* $z := A(y)$.



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# Summary

1. (NA) Dynamic model ≈ (NA) Affine model.

2. (NA) FDR model + ’freedom to estimate/choose diffusion coefficient’ ⇒ (NA) Affine model.

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# The End!

**Thank you for your attention!**