A class of recursive optimal stopping problems with applications to stock trading

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joint work with Tiziano De Angelis

Outline

1. Problem formulation
2. Motivation
3. Notation and Assumptions
4. The general case
5. The 2-dimensional case

Outline

The recursive optimal stopping problem

- 2 projects: A and B
- Profit of A is described by φ and Profit of B is ψ.
- If project A is chosen then it is realized P-a.s. instantaneously (i.e. at time τ)
- If project B is chosen then it is realised with probability p at the (delayed) time $\tau + \vartheta$

The recursive optimal stopping problem

- d-dimensional Markov process X
- $\varphi, \psi : \mathbb{R}^d \to \mathbb{R}$, continuous functions such that $\psi(x) > \varphi(x)$ for all $x \in \mathbb{R}^d$
- $p \in [0, 1]$
- ϑ random variable with CDF F
- (τ, α) controls taking values in $\mathbb{R}^+ \times \{0, 1\}$

Problem:

$$v(x) = \sup_{(\tau,\alpha)} \mathsf{E}\Bigg[e^{-r\tau} \varphi(X^x_\tau) \mathbf{1}_{\{\alpha=\mathbf{0}\}} + e^{-r(\tau+\vartheta)} \left(\mathbf{p}\psi(X^x_{\tau+\vartheta}) + (\mathbf{1}-\mathbf{p})\mathbf{v}(\mathbf{X}^{\mathbf{x}}_{\tau+\vartheta}) \right) \mathbf{1}_{\{\alpha=\mathbf{1}\}}, \Bigg]$$

Outline

Motivation

Examples

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- project decision for R&D department

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- in academia: which journal should we submit our papers?

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- project decision for R&D department
- in academia: which journal should we submit our papers?
- trading in the lit market and the dark pool

A 2-dimensional model for trading in the lit and the dark pool

- (Bid) price in the lit market S
- price in the dark pool $S+K$, K is the spread

The problem:

$$v(s,k) = \sup_{(\tau,\alpha)} \mathsf{E}\Big[e^{-r\tau}\gamma S^s_\tau \mathbf{1}_{\{\alpha=0\}} + e^{-r(\tau+\vartheta)}\big(p(S^s_{\tau+\vartheta}+K^k_{\tau+\vartheta})+(1-p)v(S^s_{\tau+\vartheta},K^k_{\tau+\vartheta})\big)\,\mathbf{1}_{\{\alpha=1\}}\Big].$$

Here we take:
$X=(S,K)$, $\varphi(X)=\gamma S$ for $0<\gamma\le 1$ and $\psi(X)=S+K$.

Literature review

The problem itself is new, however there links with

- control problems with recursive utility (initiated by Epstein & Zin (1991), Duffie & Epstein (1992))
- optimal multiple stopping problems (Carmona (2008), De Angelis & Kitapbayev (2017))
- impulse control problems with delay (Bayraktar & Egami (2007), Dayanik & Karatzas (2003))

Optimization in dark pools (Kratz & Schöneborn (2014, 2015, 2018), Crisafi & Macrina (2016)) with a different objective.

Contribution

- Study a general d-dimensional recursive optimal stopping problem;
- Discuss the 2-dimensional case;
- Derive additional properties of the value function and the stopping rule.

Outline

Notation

- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathsf{P})$;
- d-dimensional Markov process X;
- ϑ random variable with CDF F and $p \in [0, 1]$;
- $\mathcal{T}$ set of $\mathbb{F}$-stopping times;
- $\mathcal{D} = \{(\tau, \alpha) : \ \tau \in \mathcal{T}, \ \alpha \in \{0, 1\}, \ \alpha \in \mathcal{F}_\tau\}$;
- Banach space

$$\mathcal{A}_d := \left\{ f : f \in C(\mathbb{R}^d; \mathbb{R}_+), \text{ such that } \|f\|_{\mathcal{A}_d} < +\infty \right\},$$

where $\|f\|^2_{\mathcal{A}_d} := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|^2}{1+|x|^2_d}$.

Assumptions

Assumption

(i) *There exists* $\rho \in (0,1)$ *s.t.*

$$\widehat{X}_t := e^{-2r(1-\rho)t}(1 + |X_t|_d^2),$$

is a P_x*-supermartingale for any* $x \in \mathbb{R}^d$*;*

(ii) *for any compact* $K \subset \mathbb{R}^d$ *we have*

$$\sup_{x \in K} \mathsf{E}_x \left[\sup_{t \geq 0} e^{-rt} |X_t|_d \right] < \infty;$$

(iii) *for any* $x \in \mathbb{R}^d$ *and* $(x_n)_{n \geq 0}$ *s.t.* $x_n \to x$

$$\lim_{n \to \infty} \mathsf{E} \left[\sup_{t \geq 0} e^{-rt} |X_t^{x_n} - X_t^x|_d \right] = 0;$$

(iv) *functions* φ *and* ψ *belong to* $\mathcal{A}_d$ *(with* $\varphi \leq \psi$*).*

Outline

Two equivalent problem formulations

Problem 1. Find a continuous function $v : \mathbb{R}^d \to \mathbb{R}_+$ that satisfies

$$v(x) = \sup_{(\tau,\alpha)\in\mathcal{D}} \mathsf{E}\Big[e^{-r\tau}\varphi(X^x_\tau)\mathbf{1}_{\{\alpha=0\}} + e^{-r(\tau+\vartheta)}\big(p\psi(X^x_{\tau+\vartheta}) + (1-p)v(X^x_{\tau+\vartheta})\big)\mathbf{1}_{\{\alpha=1\}}\Big]. \quad (1)$$

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Problem 2. Find a continuous function $\tilde{v} : \mathbb{R}^d \to \mathbb{R}_+$ that satisfies

$$\tilde{v}(x) = \sup_{\tau\in\mathcal{T}} \mathsf{E}\Big[e^{-r\tau}\max\{\varphi(X_\tau^x), (\Lambda\tilde{v})(X_\tau^x)\}\Big], \quad (2)$$

where $(\Lambda f)(x) := \int_0^\infty e^{-rt}\mathsf{E}\left[p\psi(X_t^x) + (1-p)f(X_t^x)\right]F(\mathrm{d}t)$,
for any continuous function $f : \mathbb{R}^d \to \mathbb{R}_+$.

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Equivalence of the stopping problems

Lemma

A continuous function $v : \mathbb{R}^d \to \mathbb{R}_+$ is a solution of **Problem 1** *if and only if it solves* **Problem 2**.

Sketch of the proof.

$$v(x) = \sup_{(\tau,\alpha)\in\mathcal{D}} \mathsf{E}\Big[e^{-r\tau}\varphi(X^x_\tau)\mathbf{1}_{\{\alpha=0\}} + \int_0^\infty e^{-r(\tau+t)} \big(p\psi(X^x_{\tau+t}) + (1-p)v(X^x_{\tau+t})\big)\, F(\mathrm{d}t)\mathbf{1}_{\{\alpha=1\}}\Big].$$

Use that α is $\mathcal{F}_\tau$-measurable, Fubini's theorem and the strong Markov property of X

$$\mathsf{E}\left[\int_0^\infty e^{-r(\tau+t)} \big(p\psi(X^x_{\tau+t}) + (1-p)v(X^x_{\tau+t})\big)\, F(\mathrm{d}t)\mathbf{1}_{\{\alpha=1\}}\Big|\mathcal{F}_\tau\right] = e^{-r\tau}(\Lambda v)(X^x_\tau)\mathbf{1}_{\{\alpha=1\}}.$$

Finally, the tower property leads to the claim.

The main result

Theorem

- *Problem 2 admits a unique solution* $v \in \mathcal{A}_d$*.*
- *The stopping time*

$$\tau_* = \inf\left\{t \geq 0 : v(X_t) = \max\{\varphi(X_t), (\Lambda v)(X_t)\}\right\}$$

  is optimal for (1)*.*
- *The process*

$$\left(e^{-rt} v(X_t)\right)_{t \geq 0}$$

  is a right-continuous (non-negative) supermartingale
- *The process*

$$\left(e^{-r(t \wedge \tau_*)} v(X_{t \wedge \tau_*})\right)_{t \geq 0}$$

  is a right-continuous (non-negative) martingale.

Sketch of the proof.

Define the operator

$$(\Gamma f)(x) := \sup_{\tau \in \mathcal{T}} \mathsf{E}\left[e^{-r\tau} \max\left\{\varphi(X_\tau^x), (\Lambda f)(X_\tau^x)\right\}\right]. \qquad (3)$$

Objective: v is the unique fixed point of the operator Γ and an optimal stopping time exists.

- **Step 1.** The operator Λ maps $\mathcal{A}_d$ into itself.
- **Step 2.** An optimal stopping time in (3) exists and Γf is lsc for every $f \in \mathcal{C}(\mathbb{R}^d; \mathbb{R}^+)$
- **Step 3.** Γf is usc for every $f \in \mathcal{C}(\mathbb{R}^d; \mathbb{R}^+)$
- **Step 4.** Γ is a contraction

No delay

If $\mathsf{P}(\vartheta = 0) = 1$ the optimiser would always choose $\alpha = 1$.

- if $\psi(X)$ is not achieved the investor learns immediately and instantly stop again and choose $\alpha = 1$
- the mechanism continues (instantaneously) until the payoff is attained

Corollary

If $F(0) = 1$ we have

$$v(x) = \sup_{\tau \in \mathcal{T}} \mathsf{E}_x \left[e^{-r\tau} \psi(X_\tau) \right], \quad \textit{for } x \in \mathbb{R}^d.$$

Outline

Application to trading in the lit/dark pool

The model

$$\mathrm{d}S_t = \mu_1 S_t \mathrm{d}t + \sigma_1 S_t \mathrm{d}B_t^1, \qquad S_0 = s > 0,$$
$$\mathrm{d}K_t = \mu_2 K_t \mathrm{d}t + \sigma_2 K_t \mathrm{d}B_t^2, \qquad K_0 = k > 0.$$

- $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$
- $(B_t^1)_{t\geq 0}$, $(B_t^2)_{t\geq 0}$ Brownian motions with correlation $\nu \in [-1, 1]$.

Application to trading in the lit/dark pool

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- $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 > 0$
- $(B_t^1)_{t\geq 0}$, $(B_t^2)_{t\geq 0}$ Brownian motions with correlation $\nu \in [-1, 1]$.

Problem formulation

$$v(s,k) = \sup_{\tau \in \mathcal{T}} \mathsf{E}\left[e^{-r\tau} \max\left\{ \gamma S_\tau^s, (\Lambda v)(S_\tau^s, K_\tau^k) \right\} \right]$$

where

$$(\Lambda f)(s,k) := \int_0^\infty e^{-rt} \mathsf{E}\left[p(S_t^s + K_t^k) + (1-p) f(S_t^s, K_t^k) \right] F(\mathrm{d}t).$$

Can we say more?

With no loss of generality take $\gamma = 1$.

- The value function is positive homogeneous ▸ →
- We can reduce dimension (i.e. we get a 1 dimensional recursive optimal stopping problem): alternative characterization of the stopping time and the value function ▸ →
- The value function u is monotonic, non-decreasing and convex
- The optimal stopping rule can be expressed in terms of two boundaries ▸ →
- The smooth-fit holds (i.e. the value function is $\mathcal{C}^1$)

Thank you for the attention

COLANERI, K. AND DE ANGELIS, T. (2019)
A class of recursive optimal stopping problems with applications to stock trading
ArXiv: https://arxiv.org/pdf/1905.02650.pdf

Positive homogeneity

- For all $(s,k) \in \mathbb{R}^2_+$ we have $v(s,k) = s\, v(1, k/s)$.
- Define the process $\widehat{Z}_t = \frac{K_t}{S_t}$
- Note that

$$
\begin{aligned}
(\Lambda v)(s,k) :=& \int_0^\infty e^{-rt}\mathsf{E}\left[p(S_t^s + K_t^k) + (1-p)v(S_t^s, K_t^k)\right] F(\mathrm{d}t) \\
:=& s\int_0^\infty e^{-rt}\mathsf{E}\left[S_t^1 p(1+\widehat{Z}_t^z) + (1-p)S_t^1 v(1, \widehat{Z}_t^z)\right] F(\mathrm{d}t)
\end{aligned}
$$

- we change the measure "using" the martingale pert of S_t^1

$$
\frac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}}|_{\mathcal{F}_t} = D_t = e^{\sigma^1 B_t^1 - \frac{\sigma_1^2}{2} t}
$$

- Then $D_t = S_t^1 e^{-\mu_1 t}$

- Let $(Z_t)_{t\geq 0}$ be the solution of the SDE

$$\mathrm{d}Z_t = Z_t(\mu_2 - \mu_1)\mathrm{d}t + Z_t\widetilde{\sigma}\mathrm{d}\widetilde{B}_t, \qquad t \in [0,\infty),$$

  where $\widetilde{B}_t$ is a P-Brownian motion
- $\widehat{Z}$ under Q has the same distribution of Z under P
- Then:

$$(\Lambda v)(s,k) := s\int_0^\infty e^{-(r-\mu_1)t}\mathsf{E}\left[p(1+Z_t^z) + (1-p)v(1,Z_t^z)\right]F(\mathrm{d}t)$$

◂ ←

Reduction to optimal stopping in dimension 1

Define

$$u(z) := \sup_{\tau} \mathsf{E}\left[e^{-(r-\mu_1)\tau} \max\{1, (\Pi u)(Z_\tau^z)\}\right] \tag{4}$$

where

$$(\Pi u)(z) := \int_0^\infty e^{-(r-\mu_1)t} \mathsf{E}\left[p(1+Z_t^z) + (1-p)u(Z_t^z)\right] F(\mathrm{d}t).$$

Then $\mathbf{u(z) = v(1, k/s)}$.

◂ ←

The stopping rule

Continuation region:

$$\mathcal{C} := \{z \in \mathbb{R}_+ \ : \ u(z) > \max[1, (\Pi u)(z)]\}$$

Stopping region:

$$\mathcal{S} := \{z \in \mathbb{R}_+ \ : \ u(z) = \max[1, (\Pi u)(z)]\}$$

Theorem

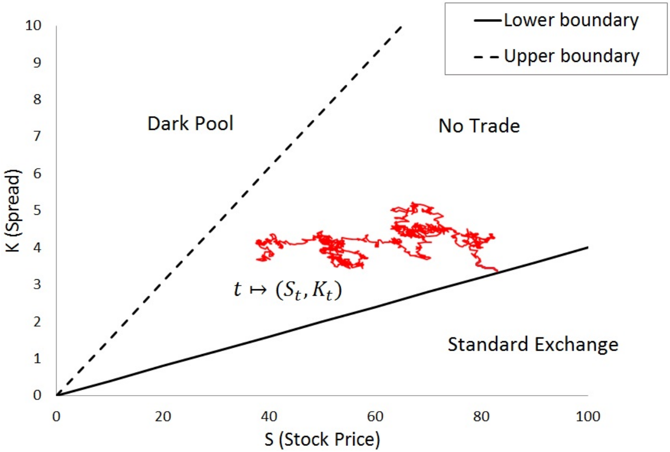
Assume $F(0) < 1$, then there exist two points $0 < a_* < b_* < +\infty$ such that $\mathcal{C} = (a_*, b_*)$.

Corollary

If $F(0) < 1$, then there exists optimal $(\tau_*, \alpha^*) \in \mathcal{D}$ and

$$\tau_* = \inf\{t \geq 0 : K_t \notin (S_t \cdot a_*, S_t \cdot b_*)\} \quad \textit{and} \quad \alpha^* = \mathbf{1}_{\{K_{\tau_*} \geq S_{\tau_*} \cdot b_*\}}.$$

The Continuation and the Stopping regions



←