Proximal algorithms for nonconvex and nonsmooth minimization problems

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(the talk relies on joint works with
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Contents

The minimization of a nonsmooth plus a smooth function: the convex case

Let $\mathcal{H}$ be a real Hilbert space and

- $f : \mathcal{H} \to \overline{\mathbb{R}}$ a proper, convex, lower semicontinuous function;
- $g : \mathcal{H} \to \mathbb{R}$ a convex and Fréchet differentiable function such that ∇g is $L_{\nabla g}$-Lipschitz continuous.

Consider the convex optimization problem

$$\min_{x \in \mathcal{H}} \{f(x) + g(x)\}. \tag{1}$$

Proximal-gradient splitting

Proximal-gradient algorithm

$$(\forall n \geq 0) \;\; x_{n+1} = \operatorname{prox}_{\gamma f}(x_n - \gamma \nabla g(x_n))$$

Proximal operator

If $f \in \Gamma(\mathcal{H}) := \{k : \mathcal{H} \to \overline{\mathbb{R}} : k$ is proper, convex and lower semicontinuous$\}$ and $\gamma > 0$, then

$$\operatorname{prox}_{\gamma f}(x) := \operatorname{argmin}_{u \in \mathcal{H}} \left\{ f(u) + \frac{1}{2\gamma}\|u - x\|^2 \right\} \;\; \forall x \in \mathcal{H}.$$

Convergence of the proximal-gradient algorithm

If $\gamma \in \left(0, \frac{2}{L_{\nabla g}}\right)$, $x_0 \in \mathcal{H}$ and (1) is solvable, then $(x_n)_{n \geq 0}$ converges weakly to an optimal solution of (1).
If x^* is an optimal solution of (1) and $\gamma := \frac{1}{L_{\nabla g}}$, then

$$0 \leq (f+g)(x_n) - (f+g)(x^*) \leq \frac{L_{\nabla g}\|x_0 - x^*\|^2}{2n} \;\; \forall n \geq 1.$$

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Accelerated proximal-gradient splitting

Accelerated proximal-gradient splitting (FISTA)

$$(\forall n \geq 1) \quad \left\lfloor \begin{array}{l} x_n = \operatorname{prox}_{\frac{1}{L_{\nabla g}} f} \left(y_n - \frac{1}{L_{\nabla g}} \nabla g(y_n) \right) \\ y_{n+1} = x_n + \alpha_n (x_n - x_{n-1}) \end{array} \right.$$

Convergence of FISTA (Beck, Teboulle, 2009)

Let be $y_1 = x_0 \in \mathcal{H}$ and $\alpha_n = \frac{t_n - 1}{t_{n+1}}$ $\forall n \geq 1$, where $t_1 := 1$ and

$$t_{n+1} = \frac{1 + \sqrt{1 + 4t_n^2}}{2} (\Leftrightarrow t_{n+1}^2 - t_{n+1} = t_n^2).$$

If x^* is an optimal solution of (1), then

$$0 \leq (f+g)(x_n) - (f+g)(x^*) \leq \frac{2L_{\nabla g} \|x_0 - x^*\|^2}{(n+1)^2} \ \forall n \geq 1.$$

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Convergence of the FISTA iterates (Chambolle, Dossal, 2014)

Let be $y_1 = x_0 \in \mathcal{H}$ and $\alpha_n = \frac{t_n - 1}{t_{n+1}}$ $\forall n \geq 1$, where $t_1 := 1$ and for $a > 3$

$$t_n = \frac{n + a - 1}{a} (\Rightarrow t_{n+1}^2 - t_{n+1} \leq t_n^2).$$

Then $(x_n)_{n \geq 0}$ converges weakly to an optimal solution of (1).
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$$0 \leq (f + g)(x_n) - (f + g)(x^*) \leq \frac{L_{\nabla g} a^2 \|x_0 - x^*\|^2}{2(n + a - 1)^2} \ \forall n \geq 1.$$

(Attouch, Peypouquet, 2015)

In the hypotheses of (Chambolle, Dossal, 2014), if x^* is an optimal solution of (1), then

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The minimization of the sum of two nonconvex functions

Consider the optimization problem

$$\min_{x \in \mathcal{H}} \{f(x) + g(x)\}. \qquad (2)$$

- $\mathcal{H}$ is a finite-dimensional real Hilbert space;
- $f : \mathcal{H} \to \overline{\mathbb{R}}$ is proper, lower semicontinuous and bounded from below;
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Inertial proximal-gradient algorithm

For $0 < \underline{\alpha} \leq \alpha_n \leq \overline{\alpha}$ and $0 \leq \beta_n \leq \beta$ consider the iterative scheme:

$$(\forall n \geq 1) \;\; x_{n+1} \in \operatorname{prox}_{\alpha_n f}(x_n - \alpha_n \nabla g(x_n) + \beta_n(x_n - x_{n-1})).$$

General assumption

Let $0 < \underline{\alpha} \leq \overline{\alpha}$ and $\beta > 0$ satisfy

$$1 > \overline{\alpha} L_{\nabla g} + 2\beta \frac{\overline{\alpha}}{\underline{\alpha}}.$$

Then

$$M_1 := \frac{1 - \overline{\alpha} L_{\nabla g}}{2\overline{\alpha}} - \frac{\beta}{2\underline{\alpha}} > M_2 := \frac{\beta}{2\underline{\alpha}}.$$

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Fundamental inequality

$$(f+g)(x_{n+1}) + M_2\|x_n - x_{n+1}\|^2 + (M_1 - M_2)\|x_n - x_{n+1}\|^2$$
$$\leq (f+g)(x_n) + M_2\|x_{n-1} - x_n\|^2 \quad \forall n \geq 1.$$

Consequences I

If $f+g$ is bounded from below, then

- $\sum_{n\geq 1} \|x_n - x_{n-1}\|^2 < +\infty$;
- the sequence $((f+g)(x_n) + M_2\|x_{n-1} - x_n\|^2)_{n\geq 1}$ is monotonically decreasing and convergent;
- the sequence $((f+g)(x_n))_{n\geq 0}$ is convergent.

Consequences II

If $f+g$ is coercive, i.e.

$$\lim_{\|x\|\to+\infty} (f+g)(x) = +\infty,$$

then $(x_n)_{n\geq 0}$ has a convergent subsequence to a critical point of $f+g$. In fact, every cluster point of $(x_n)_{n\geq 0}$ is a critical point of $f+g$.

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The limiting subdifferential of a proper and lower semicontinuous function $h : \mathcal{H} \to \overline{\mathbb{R}}$

- the Fréchet (viscosity) subdifferential at $x \in \operatorname{dom} h$:

$$\hat{\partial} h(x) = \left\{ v \in \mathcal{H} : \liminf_{y \to x} \frac{f(y) - f(x) - \langle v, y - x \rangle}{\|y - x\|} \geq 0 \right\}$$

- the limiting (Mordukhovich) subdifferential at $x \in \operatorname{dom} h$:

$$\partial h(x) = \{ v \in \mathcal{H} : \exists x_n \to x, h(x_n) \to h(x) \text{ and } \exists v_n \in \hat{\partial} h(x_n), v_n \to v \text{ as } n \to +\infty \}$$

Properties of the limiting subdifferential

- if $x \in \mathcal{H}$ is a local minimizer of h, then $x \in \operatorname{crit}(h) := \{z \in \mathcal{H} : 0 \in \partial h(z)\}$;
- if h continuously differentiable around $x \in \mathcal{H}$, then $\partial h(x) = \{\nabla h(x)\}$;
- closedness criterion: $v_n \in \partial h(x_n)$ $\forall n \geq 0$, $(x_n, v_n) \to (x, v)$ and $h(x_n) \to h(x)$ as $n \to +\infty$, then $v \in \partial h(x)$. ;
- sum formula: if $k : \mathcal{H} \to \mathbb{R}$ is continuously differentiable, then $\partial(h + k)(x) = \partial h(x) + \nabla k(x)$ for all $x \in \mathcal{H}$;
- if h is convex, then $\partial h(x) = \{v \in \mathcal{H} : h(y) \geq h(x) + \langle v, y - x \rangle \ \forall y \in \mathcal{H}\}$ $\forall x \in \operatorname{dom} h$.

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Recall that

$$\sum_{n\geq 1} \|x_n - x_{n-1}\|^2 < +\infty.$$

If one can ensure that

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The Kurdyka-Łojasiewicz property

Let $h : \mathcal{H} \to \overline{\mathbb{R}}$ be proper and lower semicontinuous. The function h is said to have the Kurdyka-Łojasiewicz (KL) property at $x \in \operatorname{dom} \partial h = \{z \in \mathcal{H} : \partial h(z) \neq \emptyset\}$ if there exist

- $\eta \in (0, +\infty]$;
- a neighborhood U of x;
- a concave and continuous function $\varphi : [0, \eta) \to [0, +\infty)$ such that $\varphi(0) = 0$, φ is continuously differentiable on $(0, \eta)$ and $\varphi'(s) > 0$ for every $s \in (0, \eta)$

such that

$$\varphi'(h(y) - h(x))\operatorname{dist}(0, \partial h(y)) = \varphi'(h(y) - h(x)) \inf\{\|v\| : v \in \partial h(y)\} \geq 1 \qquad (3)$$

for every

$$y \in U \cap \{z \in \mathcal{H} : h(x) < h(z) < h(x) + \eta\}.$$

If h has the KL property at every point in $\operatorname{dom} \partial h$, then h is called KL function.

The KL property is satisfied at every noncritical point

If $x \in \operatorname{dom} h$ is a noncritical point of h, then there exists $c > 0$ such that

$$\|y - x\| + |h(y) - h(x)| \leq c \Longrightarrow \operatorname{dist}(0, \partial h(y)) \geq c.$$

Then (3) is fulfilled for $\varphi(s) = \frac{1}{c}s$ and every

$$y \in B(x, c/2) \cap \{z \in \mathcal{H} : h(x) - c/2 < h(z) < h(x) + c/2\}.$$

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If h is continously differentiable around x, then (3) becomes

$$\varphi'(h(y) - h(x))\|\nabla h(y)\| = \|\nabla(\varphi \circ (h - h(x)))(y)\| \geq 1 \tag{4}$$

for every

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Łojasiewicz (1963)

If $h : \mathcal{H} \to \mathbb{R}$ is a real-analytic function and $x \in \mathcal{H}$ a critical point, then there exist $\theta \in [1/2, 1)$ and $C, \varepsilon > 0$ such that (Łojasiewicz property)

$$|h(y) - h(x)|^{\theta} \leq C\|\nabla h(y)\| \text{ for every } y \in \mathcal{H} \text{ with } \|y - x\| < \varepsilon.$$

Thus, (4) is fulfilled for $\varphi(s) = \frac{1}{1-\theta} C s^{1-\theta}$ and every

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Examples of KL functions

- semi-algebraic functions, i.e., functions having as graph semi-algebraic sets, namely, sets of the form

$$\bigcup_{j=1}^{p} \bigcap_{i=1}^{q} \{u \in \mathbb{R}^m : g_{ij}(u) = 0 \text{ and } h_{ij}(u) < 0\},$$

  where $g_{ij}, h_{ij} : \mathbb{R}^m \to \mathbb{R}$ are polynomial functions;
- real polynomial functions;
- indicator functions of semi-algebraic sets;
- finite sums and product of semi-algebraic functions;
- compositions of semi-algebraic functions;
- $\|\cdot\|_p$ for $p \in \mathbb{Q}$ (including the case $p = 0$);
- convex functions fulfilling a certain growth condition;
- uniformly convex functions.

Theorem

If $f+g$ is coercive and $H : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}}$,

$$H(x,y) = (f+g)(x) + M_2\|x-y\|^2$$

is a KL function, then there exists $\overline{x} \in \text{crit}(f+g)$ such that $\lim_{n\to+\infty} x_n = \overline{x}$.

- Step 1 (decrease property):

$$H(x_{n+1},x_n) + (M_1 - M_2)\|x_{n+1}-x_n\|^2 \leq H(x_n, x_{n-1}) \; \forall n \geq 1.$$

- Step 2 (subgradient lower bound for the iterates gap):
For every $n \geq 1$ there exists

$$w_{n+1} = (y_{n+1} + 2M_2(x_{n+1}-x_n), 2M_2(x_n - x_{n+1})) \in \partial H(x_{n+1}, x_n),$$

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Here,

$$0 < N = \sup_{n\geq 1}\left\{\frac{1}{\alpha_n} + L_{\nabla g} + 4M_2, \frac{\beta_n}{\alpha_n}\right\} < +\infty.$$

Theorem

If $f+g$ is coercive and $H : \mathcal{H} \times \mathcal{H} \to \overline{\mathbb{R}}$,

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▶ Step 3 (applying the KL property):

Let $x \in \text{crit}(f+g)$ be a cluster point of $(x_n)_{n\geq 0}$ and $H(x_n, x_{n-1}) > H(x,x)$ for every $n \geq 1$. Then there exists $\overline{n} \geq 1$ such that for every $n \geq \overline{n}$

$$\frac{(M_1 - M_2)\|x_{n+1} - x_n\|^2}{N(\|x_n - x_{n-1}\| + \|x_{n-1} - x_{n-2}\|)} \leq \frac{H(x_n, x_{n-1}) - H(x_{n+1}, x_n)}{\|w_{n+1}\|} \leq$$
$$\frac{(H(x_n, x_{n-1}) - H(x_{n+1}, x_n))}{\text{dist}((0,0), \partial H(x_n, x_{n-1}))} \leq$$
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$$\varphi(H(x_n, x_{n-1}) - H(x,x)) - \varphi(H(x_{n+1}, x_n) - H(x,x)).$$

By denoting for every $n \geq 1$

$$\varepsilon_n = \frac{N}{M_1 - M_2}\left(\varphi(H(x_n, x_{n-1}) - H(x,x)) - \varphi(H(x_{n+1}, x_n) - H(x,x))\right)$$
$$a_n = \|x_n - x_{n-1}\|,$$

it holds

$$a_{n+1} \leq \sqrt{\varepsilon_n(a_n + a_{n-1})} \leq \frac{1}{4}(a_n + a_{n-1}) + \varepsilon_n \ \forall n \geq \overline{n}.$$

Since $\sum_{n\geq 1} \varepsilon_n < +\infty$, it follows that

$$\sum_{n\geq 1} a_n = \sum_{n\geq 1} \|x_n - x_{n-1}\| < +\infty.$$

Hence $(x_n)_{n\geq 0}$ is a Cauchy sequence and, consequently, convergent.

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Corollary

If $f + g$ is coercive and semi-algebraic, then

(a) $\sum_{n \geq 0} \|x_{n+1} - x_n\| < +\infty$;

(b) there existsthen there exists $\overline{x} \in \mathrm{crit}(f + g)$ such that $\lim_{n \to +\infty} x_n = \overline{x}$.

Numerical experiment I

Consider the optimization problem

$$\inf_{(x_1,x_2)\in\mathbb{R}^2} |x_1| - |x_2| + x_1^2 - \log(1+x_1^2) + x_2^2$$

- $f:\mathbb{R}^2\to\mathbb{R}, f(x_1,x_2) = |x_1| - |x_2|$ is nonconvex and continuous;
- For $\gamma > 0$ and $x = (x_1,x_2)\in\mathbb{R}^2$ it holds:

$$\operatorname{prox}_{\gamma f}(x) = \operatorname{prox}_{\gamma|\cdot|}(x_1) \times \operatorname{prox}_{\gamma(-|\cdot|)}(x_2),$$

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- $g:\mathbb{R}^2\to\mathbb{R}, g(x_1,x_2) = x_1^2 - \log(1+x_1^2) + x_2^2$, is continuously differentiable, while ∇g is $\frac{9}{4}$-Lipschitz continuous;
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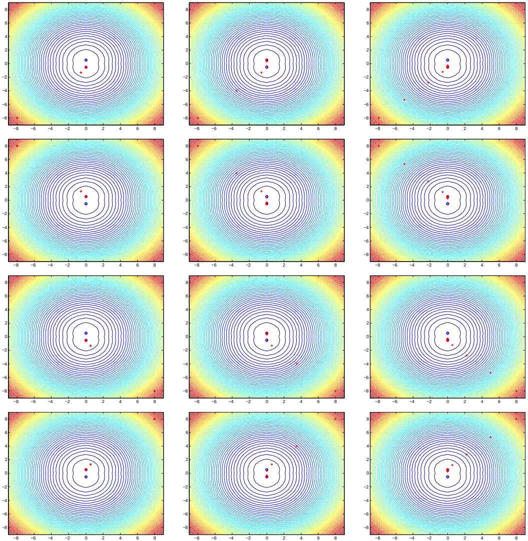

Iterations: 100; Starting points: $(-8,-8), (-8,8), (8,-8)$ and $(8,8)$, respectively;
First column: the non-inertial version ($\beta_n = \beta = 0\ \forall n \geq 1$); Second column: $\beta_n = \beta = 0.199\ \forall n \geq 1$; Third column: $\beta_n = \beta = 0.299\ \forall n \geq 1$.

Numerical experiment II (restoration of noisy blurred images)

For a given matrix $A \in \mathbb{R}^{m\times m}$ describing a blur operator and a given vector $b \in \mathbb{R}^m$ representing the blurred and noisy image, the task is to estimate the unknown original image $\overline{x} \in \mathbb{R}^m$ fulfilling

$$A\overline{x} = b.$$

We solve the regularized nonconvex minimization problem

$$\inf_{x\in\mathbb{R}^m} \left\{ \sum_{k=1}^{M} \sum_{l=1}^{N} \varphi\big((Ax-b)_{kl}\big) + \lambda \|Wx\|_0 \right\},$$

where

- $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \log(1+t^2)$, is derived form the Student t distribution;
- $\lambda > 0$ is a regularization parameter;
- $W : \mathbb{R}^m \to \mathbb{R}^m$ is a discrete Haar wavelet transform with four levels;
- $\|y\|_0 = \sum_{i=1}^{m} |\operatorname{sgn}(y_i)|$, for $y = (y_1, ..., y_m)$.

In this context, $x \in \mathbb{R}^m$ represents the vectorized image $X \in \mathbb{R}^{M\times N}$, where $m = M \cdot N$.

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- Since $WW^T = W^T W = I_m$,

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- For the experiments we used the 256×256 boat test image which we first blurred by using a Gaussian blur operator of size 9×9 and standard deviation 4 and to which we afterward added a zero-mean white Gaussian noise with standard deviation 10^{-6}.
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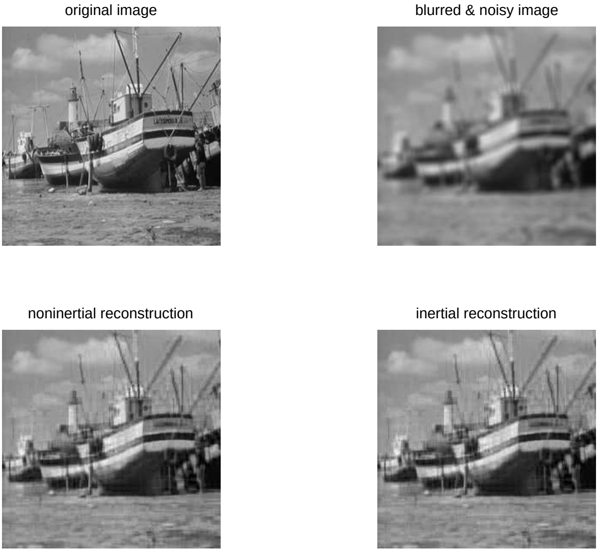

The first row shows the original 256×256 boat test image and the blurred and noisy one and the second row the reconstructed images after 300 iterations.

D.C. programming

Consider the optimization problem

$$\min \{g(x) + \varphi(x) - h(Kx) \mid x \in \mathcal{H}\} \tag{5}$$

- $\mathcal{G}$ and $\mathcal{H}$ are finite-dimensional real Hilbert spaces;
- $g : \mathcal{H} \to \overline{\mathbb{R}}$ and $h : \mathcal{G} \to \overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions;
- $K : \mathcal{H} \to \mathcal{G}$ is a linear mapping;
- $\varphi : \mathcal{H} \to \mathbb{R}$ is convex, Fréchet differentiable and $\nabla\varphi$ is $L_{\nabla\varphi}$-Lipschitz continuous.

Toland dual problem

$$\min \{h^*(y) - (g + \varphi)^*(K^*y) \mid y \in \mathcal{G}\}. \tag{6}$$

Primal-dual formulation

$$\min \{\Phi(x, y) \mid x \in \mathcal{H}, y \in \mathcal{G}\}, \tag{7}$$

$$\Phi : \mathcal{H} \times \mathcal{G} \to \overline{\mathbb{R}}, \quad \Phi(x, y) := g(x) + \varphi(x) + h^*(y) - \langle y, Kx \rangle.$$

Φ is proper and lower semicontinuous.

D.C. programming

Consider the optimization problem

$$\min\{g(x)+\varphi(x)-h(Kx) \mid x\in\mathcal{H}\} \tag{5}$$

- $\mathcal{G}$ and $\mathcal{H}$ are finite-dimensional real Hilbert spaces;
- $g:\mathcal{H}\to\overline{\mathbb{R}}$ and $h:\mathcal{G}\to\overline{\mathbb{R}}$ are proper, convex and lower semicontinuous functions;
- $K:\mathcal{H}\to\mathcal{G}$ is a linear mapping;
- $\varphi:\mathcal{H}\to\mathbb{R}$ is convex, Fréchet differentiable and $\nabla\varphi$ is $L_{\nabla\varphi}$-Lipschitz continuous.

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$$\min\{h^*(y)-(g+\varphi)^*(K^*y) \mid y\in\mathcal{G}\}. \tag{6}$$

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Proposition

1. The optimal values of (5), (6) and (7) are equal.
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3. Let $\bar{x} \in \mathcal{H}$ be an optimal solution of (5). Then

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5. Let $(\bar{x}, \bar{y}) \in \mathcal{H} \times \mathcal{G}$ be an optimal solution of (7). Then $\bar{x}$ is an optimal solution of (5), and $\bar{y}$ is an optimal solution of (6). Furthermore,

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We denote by $\mathrm{crit}\Phi$ the set of critical points of the function Φ.

Critical points of $g + \varphi - h \circ K$

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A double-proximal gradient algorithm

Let $(x_0, y_0) \in \mathcal{H} \times \mathcal{G}$, and let $(\gamma_n)_{n\geq 0}$ and $(\mu_n)_{n\geq 0}$ be sequences of positive numbers. For all $n \geq 0$ set

$$x_{n+1} := \operatorname{prox}_{\gamma_n g}\left(x_n + \gamma_n K^* y_n - \gamma_n \nabla\varphi(x_n)\right),$$
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Important inequalities

For all $n \geq 0$

$$\Phi(x_{n+1}, y_n) - \Phi(x_n, y_n) \leq \left(\frac{L_{\nabla\varphi}}{2} - \frac{1}{\gamma_n}\right) \|x_n - x_{n+1}\|^2,$$
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Proposition

- For all $n \geq 0$, if $0 < \gamma_n \leq \frac{2}{L_{\nabla\varphi}}$, then

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- Let $\inf \{g(x) + \varphi(x) - h(Kx) \mid x \in \mathcal{H}\} > -\infty$ and

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Proposition

Let $\inf \{g(x) + \varphi(x) - h(Kx) \mid x \in \mathcal{H}\} > -\infty$ and (10) be satisfied. If $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$ are bounded, then

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Let

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Theorem (Convergence result)

Let (10) be satisfied and assume that the sequence $(x_n, y_n)_{n \geq 0}$ is bounded. Then the following assertions hold:

1. $\emptyset \neq \omega(x_0, y_0) \subseteq \text{crit}\Phi \subseteq \text{crit}(g + \varphi - h \circ K) \times \text{crit}(h^* - (g + \varphi)^* \circ K^*)$,
2. $\lim_{n\to\infty} \text{dist}((x_n, y_n), \omega(x_0, y_0)) = 0$,
3. if the common optimal value of the problems (5), (6) and (7) is finite, then $\omega(x_0, y_0)$ is a compact and connected set, and so are the sets of cluster points of the sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$,
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Lemma (subgradient estimation)

For each $n \geq 1$ with $\gamma_{n-1} < \frac{2}{L_{\nabla \varphi}}$, there exist

$$\begin{pmatrix} x_n^* \\ y_n^* \end{pmatrix} = \begin{pmatrix} \frac{x_{n-1}-x_n}{\gamma_{n-1}} + K^*(y_{n-1} - y_n) + \nabla\varphi(x_n) - \nabla\varphi(x_{n-1}) \\ \frac{y_{n-1}-y_n}{\mu_{n-1}} \end{pmatrix} \in \partial\Phi(x_n, y_n),$$

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$$\begin{aligned} \|x_n^*\| &\leq \|K\| \, \|y_{n-1} - y_n\| + \frac{1}{\gamma_{n-1}} \|x_{n-1} - x_n\|, \\ \|y_n^*\| &\leq \frac{1}{\mu_{n-1}} \|y_{n-1} - y_n\|. \end{aligned} \tag{11}$$

Theorem (convergence result when Φ is a KL function)

Let

$$0 < \underline{\gamma} := \inf_{n\geq 0} \gamma_n \leq \overline{\gamma} := \sup_{n\geq 0} \gamma_n < \frac{2}{L_{\nabla\varphi}},$$

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Suppose that Φ is in addition a KL function and that the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are bounded. Then $(x_n, y_n)_{n\geq 0}$ is a Cauchy sequence, thus convergent to a critical point of Φ.

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Suppose that Φ is in addition a KL function and that the sequences $(x_n)_{n\geq 0}$ and $(y_n)_{n\geq 0}$ are bounded. Then $(x_n, y_n)_{n\geq 0}$ is a Cauchy sequence, thus convergent to a critical point of Φ.

Theorem (convergence rates)

In the hypotheses of the previous theorem, assume that Φ is a KL function with desingularization function $s \mapsto \frac{1}{1-\theta} C s^{1-\theta}$ for some $C > 0$ and $0 \leq \theta < 1$. Let $\bar{x}$ and $\bar{y}$ be the limit points of the sequences $(x_n)_{n \geq 0}$ and $(y_n)_{n \geq 0}$, respectively. Then the following convergence rates are guaranteed:

1. if $\theta = 0$, then there exists $n_0 \geq 0$, such that $x_n = x_{n_0}$ and $y_n = y_{n_0}$ for $n \geq n_0$;
2. if $0 < \theta \leq \frac{1}{2}$, then there exist $c > 0$ and $0 \leq q < 1$ such that

   $$\|x_n - \bar{x}\| \leq c q^n \qquad \text{and} \qquad \|y_n - \bar{y}\| \leq c q^n$$

   for all $n \geq 0$;
3. if $\frac{1}{2} < \theta < 1$, then there exists $c > 0$ such that

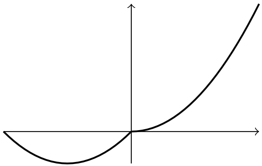
   $$\|x_n - \bar{x}\| \leq c n^{-\frac{1-\theta}{2\theta - 1}} \qquad \text{and} \qquad \|y_n - \bar{y}\| \leq c n^{-\frac{1-\theta}{2\theta - 1}}$$

   for all $n \geq 0$.

An example

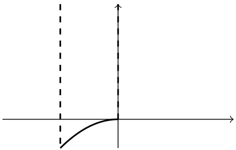
- Primal program

$$\min_{x \in \mathbb{R}} \left\{ \frac{1}{2}x^2 - \max\{-x, 0\} \right\}$$

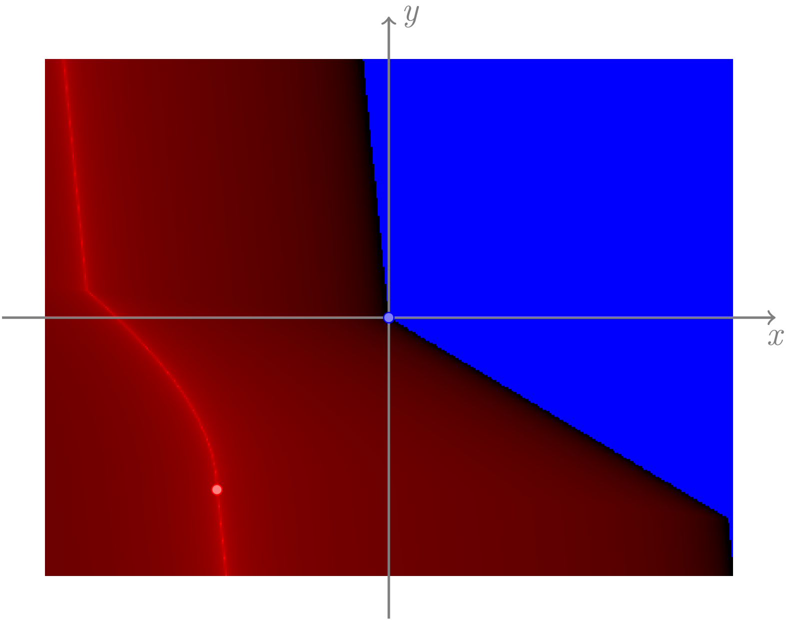


- Dual program

$$\min_{y \in [-1,0]} \left\{ -\frac{1}{2}y^2 \right\}$$



- Primal-dual critical points: $(-1,-1)$ and $(0,0)$.



y
x

Application to image processing

- We represent an image of the size $m \times n$ pixels by a vector $x \in \mathbb{R}^{mn}$ with entries in $[0,1]$ (where 0 represents pure black and 1 represents pure white).
- The original image $x \in \mathbb{R}^{mn}$ is assumed to be blurred by a linear operator $A : \mathbb{R}^{mn} \to \mathbb{R}^{mn}$ and corrupted with noise ν. Knowing $b = Ax + \nu$, we want to reconstruct the original image x by considering the minimization problem

$$\min_{x \in \mathbb{R}^{mn}} \left(\frac{\mu}{2} \|Ax - b\|^2 + J(Dx) \right),$$

where $\mu > 0$ is a regularization parameter, $D : \mathbb{R}^{mn} \to \mathbb{R}^{2mn}$ is the discrete gradient operator given by $Dx = (D_1 x, D_2 x)$,

$$D_1 : \mathbb{R}^{mn} \to \mathbb{R}^{mn}, (D_1 x)_{i,j} := \begin{cases} x_{i+1,j} - x_{i,j}, & i = 1, \dots, m-1; j = 1, \dots, n; \\ 0, & i = m; j = 1, \dots, n \end{cases}$$

$$D_2 : \mathbb{R}^{mn} \to \mathbb{R}^{mn}, (D_2 x)_{i,j} := \begin{cases} x_{i,j+1} - x_{i,j}, & i = 1, \dots, m; j = 1, \dots, n-1; \\ 0, & i = 1, \dots, m; j = n, \end{cases}$$

and $J : \mathbb{R}^{mn} \to \mathbb{R}$ is a regularizing functional penalizing noisy images.

Choices for the functional J:

- Zhang penalty (Zhang, 2009): $\mathrm{Zhang}_a(z) = \sum_{j=1}^{2mn} g_a(z_j)$, where $a > 0$ and

$$g_a(z_j) = \begin{cases} \frac{1}{a}\left|z_j\right| & \text{if } \left|z_j\right| < a, \\ 1 & \text{if } \left|z_j\right| \geq a \end{cases} = \frac{1}{a}\left|z_j\right| - \begin{cases} 0 & \text{if } \left|z_j\right| < a, \\ \frac{1}{a}(\left|z_j\right| - a) & \text{if } \left|z_j\right| \geq a. \end{cases}$$

Denoting the part after the curly brace as $h_a(z_j)$ and $h_a(z) := \sum_{j=1}^{2mn} h_a(z_j)$, we have

$$\mathrm{prox}_{\gamma h_a^*}(z) = \begin{cases} -\frac{1}{a} & \text{if } z \leq -\frac{1}{a} - \gamma a, \\ z + \gamma a & \text{if } -\frac{1}{a} - \gamma a \leq z \leq -\gamma a, \\ 0 & \text{if } -\gamma a \leq z \leq \gamma a, \\ z - \gamma a & \text{if } \gamma a \leq z \leq \frac{1}{a} + \gamma a, \\ \frac{1}{a} & \text{if } z \geq \frac{1}{a} + \gamma a. \end{cases}$$

- LZOX penalty (Lou, Zeng, Osher, Xin, 2009): $\mathrm{LZOX}_a(z) = \|z\|_{\ell^1} - a\|z\|_\times$, where

$$\|(u,v)\|_\times := \sum_{i=1}^{m}\sum_{j=1}^{n}\sqrt{u_{i,j}^2 + v_{i,j}^2}.$$

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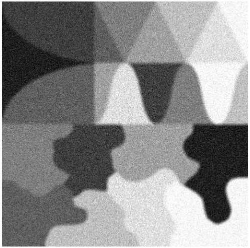
- We tested the MATLAB code on a PC with Intel Core i5 4670S ($4\times$ 3.10GHz) and 8GB DDR3 RAM (1600MHz);
- Stopping criterion: $\|(x_{n+1}, y_{n+1}) - (x_n, y_n)\|_\infty \leq 10^{-4}$;
- Stepsizes: $\mu_n = \gamma_n = \frac{1}{8\mu}$ for all $n \geq 0$;
- Initial values: $x_0 = b$, $y_0 \in \partial h(Kx_0)$.

(b) Original image



(c) Blurry image

- $\text{ISNR}(x_k) = 10 \log_{10} \left(\frac{\|x-b\|^2}{\|x-x_k\|^2} \right)$

	$a = 0.01$	$a = 0.03$	$a = 0.1$	$a = 0.3$	$a = 1.0$	$a = 3.0$
$\mu = 1.0$	−43.708	−33.711	−23.148	−13.846	−3.0288	2.4922
$\mu = 10.0$	−18.781	−9.9406	−3.2070	2.5442	5.9227	**6.97777**
$\mu = 20.0$	−11.270	−4.8428	0.43533	4.7768	6.76613	6.57299
$\mu = 50.0$	−4.8333	−1.05553	2.63959	6.46109	6.81752	3.952101
$\mu = 100.0$	−1.7546	−0.14560	3.16532	6.90202	5.29597	2.129705
$\mu = 200.0$	−0.41418	0.0619477	2.98543	6.38513	3.088196	1.110186
$\mu = 500.0$	0.0077144	0.121807	2.101321	3.816813	1.317390	0.482406
$\mu = 1000.0$	0.0528014	0.127592	1.423684	2.070959	0.692487	0.271777

ISNR values for Zhang after 50 iterations

	$a = 0.00$	$a = 0.2$	$a = 0.4$	$a = 0.5$	$a = 0.6$	$a = 0.8$	$a = 1.0$
$\mu = 1.0$	−3.0288	−4.2266	−3.7637	−3.6569	−3.5150	−4.3590	−13.701
$\mu = 10.0$	5.9227	6.26615	6.414791	6.44871	6.45780	6.28863	4.301090
$\mu = 20.0$	6.76613	6.90005	**6.93064**	6.917926	6.88018	6.61521	5.305623
$\mu = 50.0$	6.81752	6.78308	6.65411	6.4923	6.36250	5.780558	4.741993
$\mu = 100.0$	5.29597	5.23264	5.05189	4.91247	4.739717	4.287092	3.696120
$\mu = 200.0$	3.088196	3.060511	2.985871	2.930448	2.863122	2.693096	2.477708
$\mu = 500.0$	1.317390	1.312168	1.298834	1.288983	1.277010	1.246724	1.208036
$\mu = 1000.0$	0.692487	0.691049	0.687585	0.685057	0.682000	0.674272	0.664401

ISNR values for LZOX after 50 iterations

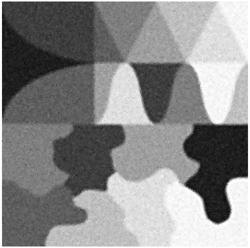
(d) LZOX, $\mu = 20$, $a = 0.4$

(e) LZOX, $\mu = 20$, $a = 1$

(f) LZOX, $\mu = 50$, $a = 0$

(g) Zhang, $\mu = 10$, $a = 3$

(h) Zhang, $\mu = 20$, $a = 1$



(i) Zhang, $\mu = 100$, $a = 0.1$

Reconstructions

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