Utility Indifference Pricing for Incomplete Preferences via Convex Vector Optimization

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April 7, 2017

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"Of all the axioms of utility theory, the completeness axiom is perhaps the most questionable. Like others of the axioms, it is inaccurate as a description of real life; but unlike them, we find it hard to accept even from the normative viewpoint. Does "rationality" demand that an individual make definite preference comparisons between all possible lotteries (even on a limited set of basic alternatives)?"
[Aumann 1962]

Incomplete Preferences

Incompleteness of Preferences:

- Some outcomes might be incomparable for the decision maker.
  [Ok, Dubra, Maccheroni 2004]: Vector valued utility representations
- Indecisiveness on the likelihood of the states of the world.
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  [Bewley 1986, 2002]: Bewley's model of Knightian uncertainty .

- [Ok, Ortoleva, Riella 2012]: Under some assumptions an incomplete preference relation accepts
  - either a *single-prior expected multi-utility representation*
  - or a *multi-prior expected single-utility representation*.

- [Galaabaatar, Karni 2013]: Characterization of preferences that admits a *multi-prior expected multi-utility representation*

Utility Representations of Incomplete Preferences

$(\Omega, \mathcal{F}, \mathbb{P})$: finite probability space, $L^0(\mathcal{F}, \mathbb{R}^d)$: $\mathcal{F}$-measurable $\mathbb{R}^d$-valued random vectors,
$\mathcal{M}_1(\Omega)$: probability measures on Ω, $\mathcal{C}(\mathbb{R}^d)$: continuous functions on $\mathbb{R}^d$.

Definition

A preference relation $\succsim$ on $L^0(\mathcal{F}, \mathbb{R}^d)$ is said to admit a multi-prior expected multi-utility representation if there exist $\mathcal{U}$ with $\emptyset \neq \mathcal{U} \subseteq \mathcal{C}(\mathbb{R}^d)$ and $\mathcal{Q}$ with $\emptyset \neq \mathcal{Q} \subseteq \mathcal{M}_1(\Omega)$ such that, for $Y, Z \in L^0(\mathcal{F}, \mathbb{R}^d)$, we have

$$Y \succsim Z \iff \forall u \in \mathcal{U}, \forall Q \in \mathcal{Q}: \quad \mathbb{E}^Q u(Y) \geq \mathbb{E}^Q u(Z).$$

Multivariate Utility Functions:

Definition ([Campi, Owen 2011])

A proper concave function $u : \mathbb{R}^d \to \mathbb{R} \cup \{-\infty\}$ is a multivariate utility function if

(i) $C_u := \mathrm{cl}\,(\mathrm{dom}\, u)$ is a convex cone such that $\mathbb{R}^d_+ \subseteq C_u \neq \mathbb{R}^d$; and

(ii) u is increasing with respect to the partial order $\leq_{C_u}$.

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For complete preferences represented by a single utility function:

- [Benedetti, Campi 2012]: Utility indifference buy and sell prices under proportional transacation costs where p_j^b, p_j^s are defined in terms of a single currency $j \in \{1, \dots, d\}$.

Assumption

a) *The preference relation admits a multi-prior expected multi-utility representation where* $\mathcal{U} = \{u^1, \dots, u^r\}$; $\mathcal{Q} = \{Q^1 \dots Q^s\}$ *for some* $r, s \geq 1$ *with* $q := rs$.

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b) *Any* $u \in \mathcal{U}$ *is a multivariate utility function.*

Notation: $U(\cdot) : L^0(\mathcal{F}, \mathbb{R}^d) \to \mathbb{R}^q$

$$U(\cdot) := (\mathbb{E}^{Q^1} u^1(\cdot), \dots, \mathbb{E}^{Q^s} u^1(\cdot), \dots\dots, \mathbb{E}^{Q^1} u^r(\cdot), \dots, \mathbb{E}^{Q^s} u^r(\cdot))^T.$$

Utility Maximization Problem

$$\text{maximize } U(V_T + C_T) \;\; \text{subject to } V_T \in \mathcal{A}(x),$$

$x \in \mathbb{R}^d$: initial endowment;
$\mathcal{A}(x) \subseteq L^0(\mathcal{F}_T, \mathbb{R}^d)$: wealth that can be generated from x;
$C_T \in L^0(\mathcal{F}_T, \mathbb{R}^d)$: some payoff that is received at time T.

Assumption

$\mathcal{A}(x)$ is a convex set for all $x \in \mathbb{R}^d$.

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Convex Vector Optimization Problem (CVOP).

Convex Vector Optimization

$$\begin{aligned} &\text{maximize} && f(x) \quad \text{(with respect to } \leq_K) \\ &\text{subject to} && g(x) \leq 0, \end{aligned} \tag{P}$$

where

- $K \subseteq \mathbb{R}^q$ is a solid, pointed, polyhedral convex ordering cone,
- $f : \mathbb{R}^n \to \mathbb{R}^q$ is K-concave,
- $g : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex.

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- $\mathcal{X} := \{x \in X : g(x) \leq 0\}$ is convex.
- $\mathcal{P} := \operatorname{cl}(f(\mathcal{X}) - K)$ is called the **lower image** of (P).

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- $\bar{x} \in \mathcal{X}$ is a **weak maximizer** for (P) if $f(\bar{x}) \in \operatorname{bd} \mathcal{P}$.
- (P) is said to be **bounded** if there is $y \in \mathbb{R}^q$ with $\{y\} - K \supseteq \mathcal{P}$.

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Definition ([Löhne, Rudloff, U., 2014])

Let (P) be bounded. A finite subset $\bar{\mathcal{X}}$ of $\mathcal{X}$ is called a **finite (weak) ϵ-solution** to (P) if it consists of only (weak) maximizers; and

$$\operatorname{conv} f(\bar{\mathcal{X}}) - K + \epsilon\{k\} \supseteq \mathcal{P} \supseteq \operatorname{conv} f(\bar{\mathcal{X}}) - K.$$

$k \in \operatorname{int} K$ is fixed.

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$$\max \left\{ w^T f(x) : g(x) \leq 0 \right\}. \tag{P(w)}$$

Proposition

Let $w \in K^+ \setminus \{0\}$. An optimal solution $\bar{x}$ of $(\mathrm{P}(w))$ is a weak maximizer of (P).

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Proposition

Let $w \in K^+ \setminus \{0\}$. An optimal solution $\bar{x}$ of $(\text{P}(w))$ is a weak maximizer of (P).

Theorem

If $\mathcal{X} \subseteq \mathbb{R}^n$ is a non-empty closed set and (P) is a **bounded** *problem, then for each weak maximizer $\bar{x}$ of (P), there exists $w \in K^+ \setminus \{0\}$ such that $\bar{x}$ is an optimal solution to $(\text{P}(w))$.*

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- The lower image:

$$V(x, C_T) := \text{cl} \bigcup_{V_T \in \mathcal{A}(x)} \left(U(V_T + C_T) - \mathbb{R}^q_+ \right).$$

Buy and Sell Prices

For a buy price we need to 'compare' $V(x_0 - p^b, C_T)$ and $V(x_0, 0)$.

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Buying claim C_T at price $p^b \in \mathbb{R}^d$ is 'more preferred' than not buying it if

$$V(x_0, 0) \precsim V(x_0 - p^b, C_T) \iff V(x_0, 0) \subseteq V(x_0 - p^b, C_T)$$

holds.

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Similarly, if

$$V(x_0, 0) \subseteq V(x_0 + p^s, -C_T).$$

then $p^s \in \mathbb{R}^d$ is a sell price.

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$$P^b(C_T) := \{p \in \mathbb{R}^d \mid V(x_0 - p, C_T) \supseteq V(x_0, 0)\}$$

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Assumption

Let $x, y \in \mathbb{R}^d$, $\lambda \in [0,1]$.

a. *$\mathcal{A}(x)$ is a convex set.*

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c. $\lambda\mathcal{A}(x) + (1-\lambda)\mathcal{A}(y) \subseteq \mathcal{A}(\lambda x + (1-\lambda)y)$.
d. *If $V_T \in \mathcal{A}(x)$, then $V_T + r \in \mathcal{A}(x + r)$ for any $r \in \mathbb{R}^d$.*

Buy and Sell Prices

$$P^b(C_T) = \{p \in \mathbb{R}^d \mid V(x_0 - p, C_T) \supseteq V(x_0, 0)\}$$
$$P^s(C_T) = \{p \in \mathbb{R}^d \mid V(x_0 + p, -C_T) \supseteq V(x_0, 0)\}$$

Proposition

$P^b(C_T)$ is a convex lower set and $P^s(C_T)$ is a convex upper set.

$P^b(C_T) = P^b(C_T) - \mathbb{R}^q_+$ and $P^s(C_T) = P^s(C_T) + \mathbb{R}^q_+$

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Proposition

Under the Assumptions on $\mathcal{A}(\cdot)$, we have $\operatorname{int} P^b(C_T) \cap \operatorname{int} P^s(C_T) = \emptyset$.

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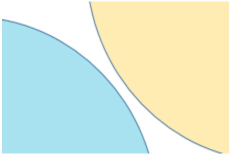

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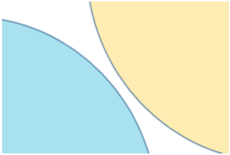

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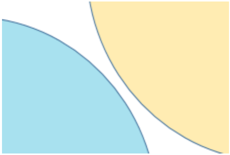

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Recovery of the standard case:

$P(C_T) = [p^b, p^s]$, where the preference relation is complete and $d = 1$.

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Proposition

$P^b(\cdot)$ and $P^s(\cdot)$ are increasing with respect to the partial order $\leq_{C_\mathcal{U}}$, in the sense of set orders $\preccurlyeq$ and $\preccurlyeq$, respectively.

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$$C_T^1 \leq_{C_\mathcal{U}} C_T^2 \implies P^b(C_T^1) \subseteq P^b(C_T^2) \text{ and } P^s(C_T^1) \supseteq P^s(C_T^2)$$

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Proposition

$P^b(\cdot)$ is concave with respect to $\preccurlyeq$; and $P^s(\cdot)$ is convex with respect to $\preccurlyeq$.

For $C_T^1, C_T^2 \in L(\mathcal{F}_T, \mathbb{R}^d)$ and $\lambda \in [0,1]$ we have

$$\lambda P^b(C_T^1) + (1-\lambda)P^b(C_T^2) \subseteq P^b(\lambda C_T^1 + (1-\lambda)C_T^2);$$
$$P^s(\lambda C_T^1 + (1-\lambda)C_T^2) \supseteq \lambda P^s(C_T^1) + (1-\lambda)P^s(C_T^2).$$

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- Both sets are lower images!!!
- In the case of LVOPs, there are ways to compute this set exactly.
- In the case of CVOPs, we can only approximate!

How to Compute?

- Using algorithms in [Löhne, Rudloff, U. 2014] we solve

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- We find a finite weak ϵ-solution $\mathcal{V} = \{V^1, \ldots, V^k\}$ such that

$$\operatorname{conv} U(\mathcal{V}) - \mathbb{R}^q_+ + \epsilon\{c\} \supseteq V(x_0, 0).$$

How to Compute?

- Using algorithms in [Löhne, Rudloff, U. 2014] we solve

$$\text{maximize } U(V_T) \text{ subject to } V_T \in \mathcal{A}(x_0).$$

- We find a finite weak ϵ-solution $\mathcal{V} = \{V^1, \dots, V^k\}$ such that

$$\mathrm{conv}\, U(\mathcal{V}) - \mathbb{R}^q_+ + \epsilon\{c\} \supseteq V(x_0, 0).$$

- We obtain a corresponding 'weight' set $W = \{w^1, \dots, w^k\} \subseteq \mathbb{R}^q_+$ such that

$$v^i := \sup_{V_T \in \mathcal{A}(x_0)} (w^i)^T U(V_T) = (w^i)^T U(V^i).$$

Outer Approximation

$$P^b(C_T) = \{p \in \mathbb{R}^d \mid V(x_0 - p, C_T) \supseteq V(x_0, 0)\}.$$

- **If the utility functions are bounded**, we have

$$P^b(C_T) = \{p \in \mathbb{R}^d \mid \forall w \in \mathbb{R}^q_+ : \sup_{V_T \in \mathcal{A}(x_0 - p)} w^T U(V_T + C_T) \geq \sup_{V_T \in \mathcal{A}(x_0)} w^T U(V_T)\}.$$

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- $W = \{w^1, \dots, w^k\}$ is a 'representative' weight set!
- An outer approximation of $P^b(C_T)$:

$$P^b_{\text{out}}(C_T) := \{p \in \mathbb{R}^d |\ \forall i \in \{1, \dots, k\} : \sup_{V_T \in \mathcal{A}(x_0 - p)} (w^i)^T U(V_T + C_T) \geq \sup_{V_T \in \mathcal{A}(x_0)} (w^i)^T U(V_T)\}.$$

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Lower image of:

$$\begin{aligned} &\text{maximize} && p \quad \text{with respect to} \quad \leq_{\mathbb{R}^d_+} \\ &\text{subject to} && (w^i)^T U(V^i_T + C_T) \geq v^i \quad \text{for} \;\; i = 1, \ldots, k; \\ & && V^i_T \in \mathcal{A}(x_0 - p) \qquad\qquad \text{for} \;\; i = 1, \ldots, k. \end{aligned}$$

Outer Approximation

An outer approximation of $P^s(C_T)$:

$$P^s_{\text{out}}(C_T) := \{p \in \mathbb{R}^d \,|\, \forall i = 1, \dots, k : \sup_{V_T \in \mathcal{A}(x_0+p)} (w^i)^T U(V_T - C_T) \geq v^i\}$$

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Upper image of:

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Lower image of:

$$\begin{array}{lll} \text{maximize} & p \quad \text{with respect to} \quad \leq_{\mathbb{R}^d_+} & \\ \text{subject to} & U(V^i_T + C_T) \geq U(V^i) + \epsilon c & \text{for} \quad i = 1, \ldots, k; \\ & V^i_T \in \mathcal{A}(x_0 - p) & \text{for} \quad i = 1, \ldots, k. \end{array}$$

Inner Approximation

- An inner approximation of $P^s(C_T)$:

$$P^s_{\text{in}}(C_T) := \{p \in \mathbb{R}^d | \; V(x_0 + p, -C_T) \supseteq \; \operatorname{conv} U(\mathcal{V}) - \mathbb{R}^q_+ + \epsilon\{c\}\}$$

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Upper image of:

$$\begin{aligned} \text{minimize} \quad & p \quad \text{with respect to} \quad \leq_{\mathbb{R}^d_+} \\ \text{subject to} \quad & U(V^i_T - C_T) \geq U(V^i) + \epsilon c \qquad \text{for } \; i = 1, \dots, k; \\ & V^i_T \in \mathcal{A}(x_0 + p) \qquad\qquad\qquad \text{for } \; i = 1, \dots, k. \end{aligned}$$

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- x_0: initial endowment;
- $\mathcal{A}(x_0) := x_0 - L^0_n(\mathcal{F}_0, K_0) - L^0_n(\mathcal{F}_1, K_1) - \ldots - L^0_n(\mathcal{F}_T, K_T)$.

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The generating vectors of the solvency cones $K_0, K_1(\omega_1)$ and $K_1(\omega_2)$:

$$K_0 = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 1 \end{bmatrix},\ K_1(\omega_1) = \begin{bmatrix} 2 & -1.9 \\ -1 & 1 \end{bmatrix},\ K_1(\omega_2) = \begin{bmatrix} 1 & -1 \\ -2 & 2.1 \end{bmatrix};$$

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$u(x) = 1 - 0.5(e^{-x_1} + e^{-x_2}),\ x_i \geq 0.$

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- $P^b(C_T)$ is the lower image of

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- $P^s(C_T)$ is the upper image of

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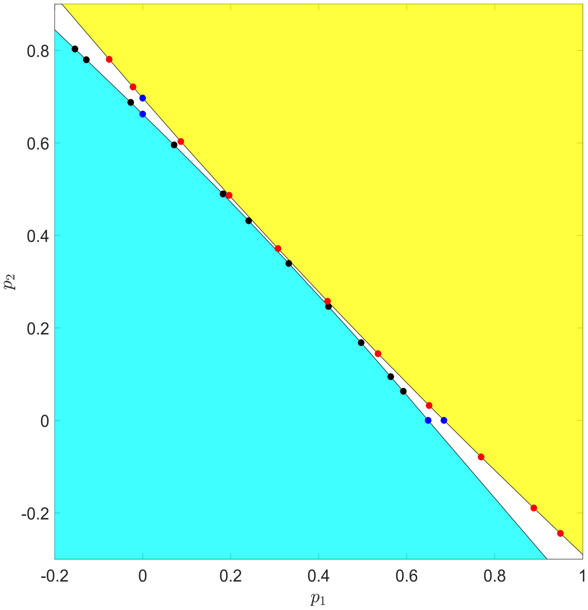
0.8
0.6
0.4
0.2
0
-0.2
p_2
-0.2
0
0.2
0.4
0.6
0.8
1
p_1

Question: Which $p^b \in P^b(C_T)$ and $p^s \in P^s(C_T)$ yield the smallest gap?

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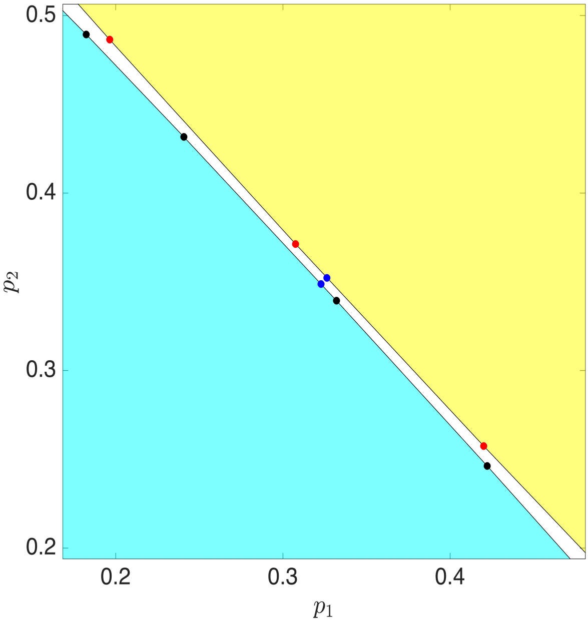
$$\begin{aligned} \text{minimize} \quad & \left\| p^b - p^s \right\| \\ \text{subject to} \quad & \mathbb{E}u(V_T^b - C_T) \geq v^0, \\ & \mathbb{E}u(V_T^s + C_T) \geq v^0, \\ & V_T^b \in \mathcal{A}(x_0 - p^b), \\ & V_T^s \in \mathcal{A}(x_0 + p^s). \end{aligned}$$


0.5
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References

Armbruster B., Delage, E.
Decision Making Under Uncertainty when Preference Relation Information is Incomplete
Management Science, 61: 111–128, 2015.

Benedetti, G., Campi, O.
Multivariate Utility Maximization with Proportional Transaction Costs and Random Endowment
SIAM Journal of Control Optimization, 50(3): 1283–13018, 2012.

Bewley, T. F.
Knightian Decision Theory: Part 1
Decision in Economics and Finance, 25: 79–110, 2002.

Campi, O., Owen M. P.
Multivariate Utility Maximization with Proportional Transaction Costs
Finance and Stochastics, 15(3): 461–499, 2011.

Galaabaatar, T., Karni, E.
Subjective Expected Utility with Incomplete Preferences
Econometrica, 81(1): 255–284, 2013.

Löhne, A., Rudloff, B. and Ulus, F.
Primal and Dual Approximation Algorithms for Convex Vector Optimization Problems
Journal of Global Optimization, 60 (4): 713–736, 2014.

Ok, E., Dubra, J. and Maccheroni, F.
Expected Utility Theory Without the Completeness Axiom
Journal of Economic Theory, 115: 118–133, 2004.

Ok, E., Ortoleva, P. and Riella, G.
Incomplete Preferences under Uncertainty: Indecisiveness in Beliefs versus Tastes
Econometrica, 80(4): 1791–1808, 2012.

Thank you!