# Backward Stochastic Differential Equations and Applications

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Stochastic (ordinary, forward) differential equations (driven by a Brownian motion $W$):

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0$$

or

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \leq t \leq T.$$

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- $x_0 = X_0$ is deterministic
- $X_t$ is $\mathcal{F}_t$-measurable

Consider the same situation backward in time:

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- Is $X_0$ deterministic?
- Is $X_t$ $\mathcal{F}_t$-measurable?
- In general: NO! Everything is $\mathcal{F}_T$-measurable. Problem is not well-posed.

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- $Z$ is part of the solution.

Rename $b =: f$, $X =: Y$ to write

$$X_t = \xi + \int_t^T b(s, X_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T$$

as

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s$$

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A solution to a BSDE is a pair of processes $(Y, Z)$ such that the equation is satisfied.

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$$dY_t = \frac{\pi_t}{S_t} dS_t + r_t(Y_t - \pi_t)dt = (\pi_t(\mu_t - r_t) + r_t Y_t)dt + \pi_t \sigma_t dW_t$$

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If $\lambda$ exists, such that $\mu - r = \sigma\lambda$, we get

$$Y_t = \xi - \int_t^T (Z_s \lambda_s + r_s Y_s)\, ds - \int_t^T Z_s dW_s,$$

which is a BSDE.

The above problem has an explicit solution:

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But: If the rates for borrowing and lending are different, the wealth process satisfies

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corresponding BSDE:

$$\begin{aligned} Y_t =& \xi - \int_t^T \left(Z_s\frac{\mu_s}{\sigma_s} + \frac{r_s^l}{\sigma_s}(\sigma_s Y_s - Z_s)^+ - \frac{r_s^b}{\sigma_s}(\sigma_s Y_s - Z_s)^-\right) ds \\ & - \int_t^T Z_s dW_s \end{aligned}$$

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No more explicit solutions in the above manner.

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  'Reflected' BSDEs, RBSDEs.
- Solution: $(Y, Z, A)$

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A nonlinear expectation is an operator $\mathcal{E}\colon L^2 \to \mathbb{R}$ such that

- $X' \geq X \Rightarrow \mathcal{E}(X') \geq \mathcal{E}(X)$, equality only if $X' = X$.
- $\mathcal{E}(c) = c$ for constants
- for each $X, t$ there is $\eta_t^X$ such that for all $A \in \mathcal{F}_t$:
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Theorem:
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In the case $f = 0$ we get back the ordinary conditional expectation $\mathcal{E}_t(X) = \mathbb{E}[X|\mathcal{F}_t]$ (This will serve as an easy example later).

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If $\mathcal{E}$ is a nonlinear expectation such that

$$\mathcal{E}(X+X') \leq \mathcal{E}(X) + \mathcal{E}^{f_\mu}(X'),$$

with $f_\mu(y,z) = \mu|z|$, and if

$$\mathcal{E}_t(X+X') = \mathcal{E}_t(X) + X' \text{ for } X' \in L^2(\mathcal{F}_t),$$

then there exists a generator $f$, not depending on $y$ such that $\mathcal{E} = \mathcal{E}^f$.

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Assume, there is a (nice) solution $v$ to the PDE

$$0 = \mathcal{L}\phi + f(\cdot, \cdot, \phi, \sigma\partial_x \phi), \quad \phi(T, x) = G(x).$$

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Then, the couple $Y := v(\cdot, X), Z := \partial_x v(\cdot, X)$ solves the backward equation

$$Y_t = G(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

'forward-backward SDE' (decoupled)
Proof: Itô formula.

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If the BSDE has at most one solution, then solving the BSDE and the PDE are equivalent.

Example:

- $\mathcal{L} = \partial_t + \frac{\sigma^2}{2}\partial_{xx} + \mu\partial_x$
- $\mathcal{L}u(t,x) + -|u(t,x) + \sigma\partial_x u(t,x)| = 0$
- $u(T,x) = \sin(x)$

translates into

- $dX_t = \mu dt + \sigma dW_t, X_0 = 1$
- $dY_t = |Y_t + Z_t| + Z_t dW_t$
- $Y_T = \sin(X_T)$

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- Solving the BSDE gives rise (in general) to a viscosity solution.
- Numerical schemes for BSDEs as an alternative to solve PDEs by MC methods (especially in higher dimensions).
- Similar approaches exist for SPDEs. They lead to DSBSDEs (doubly stochastic backward SDEs).

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Goal: Maximize an expected gain of the form

$$J(\nu) = \mathbb{E}\left[ g(X_T^\nu) + \int_0^T f_t(X_t^\nu, \nu(t)) dW_t \right],$$

with respect to $\nu$. Here, $X^\nu$ is the solution of

$$dX_t^\nu = b_t(X_t, \nu_t)dt + \sigma_t(X_t, \nu_t)dW_t$$

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Sufficient/necessary conditions for optimality given by BSDEs:

Let us define the Hamiltonian

$$\mathcal{H}_t(x, u, p, q) = b_t(x, u)p + \sigma_t(x, u)q + f_t(x, u).$$

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Find $\mathrm{argmax}_u(\mathcal{H}_t(x,u,p,q)) = \hat{\nu}_t(x,p,q)$.

Solve an associated BSDE

$$P_t = \partial_x g(X_t^{\hat{\nu}}) + \int_t^T \partial_x \hat{\mathcal{H}}_t(X_s^{\hat{\nu}}, P_s, Q_s)ds - \int_t^T Q_s dW_s.$$

$$J(\nu) = \mathbb{E}\left[g(X_T^{\nu}) + \int_0^T f_t(X_t^{\nu}, \nu(t))dW_t\right]$$

Sufficient/necessary conditions for optimality given by BSDEs:

Let us define the Hamiltonian

$$\mathcal{H}_t(x, u, p, q) = b_t(x, u)p + \sigma_t(x, u)q + f_t(x, u).$$

Find $\mathrm{argmax}_u(\mathcal{H}_t(x, u, p, q)) = \hat{\nu}_t(x, p, q)$.

Solve an associated BSDE

$$P_t = \partial_x g(X_t^{\hat{\nu}}) + \int_t^T \partial_x \hat{\mathcal{H}}_t(X_s^{\hat{\nu}}, P_s, Q_s)ds - \int_t^T Q_s dW_s.$$

get optimal control $\hat{\nu}$ by $\mathrm{argmax}_u(\mathcal{H}_t(X_t, u, P_s, Q_s))$

Utility maximization:
Given:

- Stock: $S_t = S_0 + \int_0^t \mu_r d_r + \int_0^t \sigma_r dW_r$
- Wealth up to now: $X_t^\pi = x + \int_0^t \pi_s dS_s, \quad x > 0.$
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Maximize the expected utility

$$\sup \mathbb{E}\left[U\left(x + \int_0^T \pi_s dS_s - F\right)\right]$$

Utility functions for example:

- logarithmic: $U(x) = \log(x)$
- power: $U(x) = \frac{x^p}{p}, \quad p \in ]0, 1[.$
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BSDE approach:

- some numerics available (Lipschitz, quadratic generators)

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- 'Meta-theorem': Any problem in mathematical finance can be reduced (in some sense) to a (certain type of) BSDE.

Suppose that $\xi = \mathbb{E}\xi + \int_0^T Z_s dW_s$ (predictable representation property.)

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Then, with $Y_t = \mathbb{E}\xi + \int_0^t Z_s dW_s$, we have

$$Y_t = \xi - \int_t^T Z_s dW_s,$$

which is a BSDE with $f = 0$.

Note also that $Y_t = \mathbb{E}\left[\xi | \mathcal{F}_t\right].$

1 What is a BSDE?
- SDEs - the differential dynamics approach to BSDEs

2 Applications - Why do we need BSDEs?
- Pricing of contingent claims
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4 My field within BSDE theory

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equivalent to

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- Show convergence of $(Y^n, Z^n)_{n \geq 0}$ by Banach's fixed-point theorem

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To obtain a numerical scheme for

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we rewrite the equation for one step in a time-net:

$$Y_{t_{i-1}} = Y_{t_i} + \int_{t_{i-1}}^{t_i} f(s, Y_s, Z_s)ds - \int_{t_{i-1}}^{t_i} Z_s dW_s,$$

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and find $\hat{Y}_{t_{i-1}}$ by taking the conditional expectation

$$\hat{Y}_{t_i-1} = \mathbb{E}\left[ \hat{Y}_{t_i} + (\Delta t_i) f(t_i, \hat{Y}_{t_i}, \hat{Z}_{t_i}) \middle| \mathcal{F}_{t_{i-1}} \right]$$

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Multiply

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by $\Delta W_{t_i}$ to get

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Taking the conditional expectation brings us to

$$Z_{t_{i-1}} = \mathbb{E}\left[ (\Delta W_{t_i}) \hat{Y}_{t_i} + (\Delta W_{t_i})(\Delta t_i) f(t_i, \hat{Y}_{t_i}, \hat{Z}_{t_i}) \middle| \mathcal{F}_{t_i-1} \right] \frac{1}{\Delta t_i}$$

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- Theoretical rate of convergence since calculations of conditional expectations are involved!
- Other type of numerical schemes: Involve Picard iterations of the equations and chaos decompositions of random variables.
- Applicable codes/schemes do exist!

Basically I follow three main topics (with Ch. Geiss, University of Jyväskylä):

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- BSDEs with jumps (or Lévy driven BSDEs, BSDEJ, BSDEL):

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\begin{aligned}
Y_t = & \xi + \int_t^T f(s, Y_s, Z_s, U_s) ds \\
& - \int_t^T Z_s dW_s - \int_{]t,T] \times \mathbb{R}_0} U_s(x) \tilde{N}(ds, dx)
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  e.g. if Brownian motion in models is replaced by a Lévy process.
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  Existence and uniqueness for non-Lipschitz generators (one-sided Lipschitz, locally Lipschitz, quadratic growth and beyond)

- Application of Malliavin calculus to BSDEs.

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  The identity $D_t Y_t = Z_t$ allows explicit access to the $Z$-process (trading strategy,...).

- Numerical improvements for BSDEs (with G. Leobacher, KFU Graz)

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