Testing the maximal rank of the volatility process for continuous diffusions observed with noise

Tobias Fissler, jointly with Mark Podolskij

University of Bern, Aarhus University

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Outline

The statistical problem

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- The process of interest X is a d-dimensional Itô semimartingale of the form

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s, \qquad t \in [0, T]$$

  where

  - b is a d-dimensional drift process,
  - σ is a $\mathbb{R}^{d \times q}$-valued volatility process,
  - W is a q-dimensional Brownian motion.

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  where
  - b is a d-dimensional drift process,
  - σ is a $\mathbb{R}^{d \times q}$-valued volatility process,
  - W is a q-dimensional Brownian motion.
- High frequency observations: Not the whole path $t \mapsto X_t(\omega)$ is available, but only equidistant discrete time observations

$$X_0, X_{\Delta_n}, X_{2\Delta_n}, \ldots, X_{[T/\Delta_n]\Delta_n}$$

  with $\Delta_n \to 0$.

The statistical problem (2)

- Interesting question: minimal dimension of W
  - Modelling and simulation purposes.
  - Economic interpretation: Assume X comprises the stocks of an index (e.g. the DAX, so $d = 30$). Is the market complete or not? How many components do we need to explain the volatility of X?

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- This amounts to ask for the maximal rank of the diffusion process $c_t = \sigma_t \sigma_t^*$ in $[0, T)$. We set

$$r_t = \operatorname{rank}(c_t), \qquad R_T = \sup_{s \in [0,T)} r_s.$$

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- The pathwise 'testing hypothesis' will be for $r \in \{0, \dots, d\}$

$$\Omega_T^r = \{\omega \in \Omega : \ R_T(\omega) = r\}.$$

If σ_t is continuous, the random-set $\{t \in [0, T) \mid r_t(\omega) = R_T(\omega)\}$ has positive Lebesgue measure.

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- Sources of the noise ε:
  (a) Rounding errors (prices are given in cents) that amount to microstructure noise.
  (b) Measurement inaccuracies that lead to additive white noise.

  We confine ourselves to the latter case.

Situation without noise:
The random perturbation approach

The random perturbation approach – deterministic setting

- Let $A, B \in \mathbb{R}^{d\times d}$, $\mathrm{rank}(A) = r$, $\mathrm{rank}(B) = d$ and $\lambda > 0$. By multilinearity we have:

$$\det(A + \lambda B) = \sum_{j=0}^{d} \lambda^{d-j}\gamma_j(A, B) = \lambda^{d-r}\gamma_r(A, B) + O(\lambda^{d-r+1}),$$

$$\gamma_j(A, B) = \sum_{G\in\mathcal{M}^j_{A,B}} \det(G),$$

$$\mathcal{M}^j_{A,B} = \{G \in \mathbb{R}^{d\times d} \,|\, G_i = A_i \text{ or } G_i = B_i, \; A \text{ and } G \text{ share } j \text{ joint columns}\}.$$

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- Provided that $\gamma_r(A, B) \neq 0$ this yields

$$\frac{\det(A + 2\lambda B)}{\det(A + \lambda B)} = \frac{(2\lambda)^{d-r}\gamma_r(A, B) + O(\lambda^{d-r+1})}{\lambda^{d-r}\gamma_r(A, B) + O(\lambda^{d-r+1})} \to 2^{d-r}, \quad \text{as } \lambda \downarrow 0.$$

The random perturbation approach – stochastic setting

- For the increment

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$$

we need to establish a stochastic analogon of a Taylor expansion up to order 3 such that we can write

$$\Delta_i^n X = \alpha_i^n + \beta_i^n + \gamma_i^n,$$

with $\alpha_i^n = O_{\mathbb{P}}(\Delta_n^{1/2})$, $\beta_i^n = O_{\mathbb{P}}(\Delta_n)$, $\gamma_i^n = O_{\mathbb{P}}(\Delta_n^{3/2})$.
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  Reason: The main test statistic is degenerate on $\{R_T < d\}$.
- We impose the additional regularity condition (H) that

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t \sigma_s \, dWs,$$
$$\sigma_t = \sigma_0 + \int_0^t a_s \, ds + \int_0^t v_s \, dWs,$$

  and that b and v are also continuous Itô semimartingales.

The random perturbation approach – stochastic setting (2)

- We use matrix notation and obtain that under (H)

$$\Delta_n^{-1/2}\left(\Delta_{i+1}^n X, \cdots, \Delta_{i+d}^n X\right) = A_i^n + \Delta_n^{1/2} B_i^n + \Delta_n C_i^n,$$

where $A_i^n = \left(A_{i,1}^n, \cdots, A_{i,d}^n\right)$, $B_i^n = \left(B_{i,1}^n, \cdots, B_{i,d}^n\right)$ and $C_i^n = \left(C_{i,1}^n, \cdots, C_{i,d}^n\right)$.

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- The expansion has the form

$$\begin{aligned}
A_{i,j}^n &= \sigma_{i\Delta_n}\Delta_n^{-1/2}\Delta_{i+j}^n W \sim MN(0, c_{i\Delta_n}), \\
B_{i,j}^n &= b_{i\Delta_n} + \Delta_n^{-1} v_{i\Delta_n}\int_{(i+j-1)\Delta_n}^{(i+j)\Delta_n}(W_s - W_{i\Delta_n})dW_s, \\
C_{i,j}^n &= rest.
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$$C_{i,j}^n = rest.$$

- We obtain that $A_{i,j}^n, B_{i,j}^n, C_{i,j}^n$ are $O_{\mathbb{P}}(1)$.

The random perturbation approach – stochastic setting (3)

- In order to identify the correct limit we use squared determinants as test functions. If $r = \operatorname{rank}(c_{i\Delta_n})$, we have the approximation

$$\det^2\left(A_i^n + \Delta_n^{1/2} B_i^n + \Delta_n C_i^n\right) \approx \Delta_n^{d-r} \gamma_r (A_i^n, B_i^n)^2 \,.$$

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- Let $\widehat{W}$ be a d-dimensional Brownian motion independent of X (and all its ingredients). Then we work with the perturbated processes

$$Z_t^{n,1} = X_t + \sqrt{\Delta_n}\widehat{W}_t,$$
$$Z_t^{n,2} = X_t + \sqrt{2\Delta_n}\widehat{W}_t.$$

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$$\Delta_n^{-1/2}\left(\Delta_{i+1}^n Z^{n,1}, \cdots, \Delta_{i+d}^n Z^{n,1}\right) = A_i^n + \Delta_n^{1/2}\widehat{B}_i^n + \Delta_n C_i^n,$$

where

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- One can show that $\gamma_r(A_i^n, \widehat{B}_i^n) \neq 0$ a.s.

The random perturbation approach – The main statistics

We define the main statistics

$$S_t^{n,1} = 2d\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \det{}^2 \Big(\Delta_n^{-1/2}(Z^{n,1}_{(2id+1)\Delta_n} - Z^{n,1}_{2id\Delta_n}), \cdots , \Delta_n^{-1/2}(Z^{n,1}_{(2id+d)\Delta_n} - Z^{n,1}_{(2id+d-1)\Delta_n}) \Big),$$

$$S_t^{n,2} = 2d\Delta_n \sum_{i=0}^{[t/2d\Delta_n]-1} \det{}^2 \Big((2\Delta_n)^{-1/2}(Z^{n,2}_{(2id+2)\Delta_n} - Z^{n,2}_{2id\Delta_n}), \cdots , (2\Delta_n)^{-1/2}(Z^{n,2}_{(2id+2d)\Delta_n} - Z^{n,2}_{(2id+d-2)\Delta_n}) \Big).$$

The random perturbation approach – Law of Large Numbers

- We obtain the following Law of Large Numbers on Ω_T^r

$$\frac{1}{\Delta_n^{d-r}} S_T^{n,1} \xrightarrow{\mathbb{P}} S(r)_T = \int_0^T \Gamma_r(\sigma_s, v_s, b_s) ds > 0,$$
$$\frac{1}{(2\Delta_n)^{d-r}} S_T^{n,2} \xrightarrow{\mathbb{P}} S(r)_T.$$

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- Since the limits are the same we can deduce that

$$S_T^{n,2} / S_T^{n,1} \xrightarrow{\mathbb{P}} 2^{d-R_T}.$$

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$$S_T^{n,2}/S_T^{n,1} \xrightarrow{\mathbb{P}} 2^{d-R_T}.$$

- We can construct a consistent 'estimator' for the maximal rank R_T

$$\widehat{R}(n,T) = d - \frac{\log\left(S_T^{n,2}/S_T^{n,1}\right)}{\log 2} \xrightarrow{\mathbb{P}} R_T.$$

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- We have the following 2-dimensional stable convergence on Ω_T^r

$$\Delta_n^{-1/2}\left(\frac{1}{\Delta_n^{d-r}}S_T^{n,1} - S(r)_T, \frac{1}{(2\Delta_n)^{d-r}}S_T^{n,2} - S(r)_T\right)$$

$$\xrightarrow{\mathcal{L}-s} MN\Big(0, \int_0^T \Theta_r(\sigma_s, v_s, b_s)ds\Big),$$

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$$\xrightarrow{\mathcal{L}-s} MN\Big(0, \int_0^T \Theta_r(\sigma_s, v_s, b_s)ds\Big),$$

- and

$$\Delta_n^{-1/2}\left(\widehat{R}(n,T) - r\right) \xrightarrow{\mathcal{L}-s} MN\Big(0, \int_0^T V_r(\sigma_s, v_s, b_s)ds\Big).$$

The random perturbation approach – Central Limit Theorem (2)

- The conditional variance $\int_0^T V_r(\sigma_s, v_s, b_s)ds$ can be consistently estimated by V_n such that we obtain a feasible version of the stable convergence:

$$\frac{\Delta_n^{-1/2}\left(\widehat{R}(n,T) - R_T\right)}{\sqrt{V_n}} \xrightarrow{\mathcal{L}-s} \Phi \sim \mathcal{N}(0,1).$$

Accounting for the noise: The pre-averaging approach

Accounting for the noise: The intuition

- There is empirical evidence that – especially at very high frequencies – we cannot observe X directly, but only a noisy version

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- We assume a rather simple structure of the noise. We assume that ε is additive Gaussian noise that means
  (i) $\varepsilon_t \sim \mathcal{N}(0, \Sigma)$ for all t,
  (ii) ε_s is independent of ε_t for all $s \neq t$,
  (iii) The noise ε is independent of the semimartingale X (and all its ingredients).

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  (ii) ε_s is independent of ε_t for all $s \neq t$,
  (iii) The noise ε is independent of the semimartingale X (and all its ingredients).
- What happens if we use the same statistics in the presence of noise meaning that we substitute X by Y?

Accounting for the noise: The intuition (2)

- To get an intuition, we assume for simplicity that we have no drift and constant volatility meaning that $X_t = \sigma W_t$. Then

$$\frac{\Delta_i^n X}{\sqrt{\Delta_n}} \sim \mathcal{N}(0, \sigma\sigma^*), \qquad \frac{\Delta_i^n \varepsilon}{\sqrt{\Delta_n}} \sim \mathcal{N}(0, 2\Delta_n^{-1}\Sigma).$$

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- If we work with the non-normalized increments $\Delta_i^n X$ and $\Delta_i^n \varepsilon$, the noise would completely dominate the statistic.

Accounting for the noise: The pre-averaging approach

- Podolskij and Vetter (2006) were the first to introduce the pre-averaging approach. Jacod, Li, Mykland, Podolskij and Vetter (2009) enhanced the approach.
- We consider the weighted average of $k_n \in \mathbb{N}$ successive increments.

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Definition 1

We call $g : [0,1] \to \mathbb{R}$ a weight function if it is continuous, piecewise C^1 with a piecewise Lipsvarepsilontz derivative g', and satisfies

$$g(0) = g(1) = 0, \qquad \int_0^1 g^2(x)dx > 0.$$

Example: $g(x) = \min(x, 1-x)$.

Accounting for the noise: The pre-averaging approach (2)

Definition 2

Let g be a weight function and k_n be a sequence of integers such that $k_n \to \infty$ and $\Delta_n k_n \to 0$ as $\Delta_n \to 0$.
For any d-dimensional process V we define the pre-averaged increments

$$\overline{V}(g)_i^{n,1} = \sum_{j=1}^{k_n-1} g\Big(\frac{j}{k_n}\Big) \left(V_{(i+j)\Delta_n} - V_{(i+j-1)\Delta_n}\right),$$

$$\overline{V}(g)_i^{n,2} = \sum_{j=1}^{k_n-1} g\Big(\frac{j}{k_n}\Big) \left(V_{(i+2j)\Delta_n} - V_{(i+2(j-1))\Delta_n}\right).$$

Accounting for the noise: The pre-averaging approach (3)

Example

(i) For $V = W$ a d-dimensional Brownian motion, $\overline{W}(g)_i^{n,\kappa}$, $\kappa = 1, 2$, is a centered Gaussian variable with covariance matrix

$$\kappa \Delta_n \sum_{j=1}^{k_n - 1} g^2\Big(\frac{j}{k_n}\Big) I_d = \kappa \Delta_n k_n \int_0^1 g^2(s) ds \, I_d + O(\Delta_n).$$

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(ii) For $V = \varepsilon$ where ε is centered additive Gaussian noise with covariance matrix Σ, then $\overline{\varepsilon}(g)_i^{n,\kappa}$ is a centered Gaussian variable with covariance matrix

$$\sum_{j=1}^{k_n} \bigg(g\Big(\frac{j}{k_n}\Big) - g\Big(\frac{j-1}{k_n}\Big)\bigg)^2 \Sigma = k_n^{-1} \int_0^1 \big(g'(s)\big)^2 ds \, \Sigma + O(k_n^{-2}).$$

Accounting for the noise: The pre-averaging approach (4)

- The pre-averaged increments have the following orders

$$\overline{X}(g)_i^{n,\kappa} = O_{\mathbb{P}}\left((\Delta_n k_n)^{1/2}\right),$$
$$\overline{\varepsilon}(g)_i^{n,\kappa} = O_{\mathbb{P}}\left(k_n^{-1/2}\right).$$

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- We are free to choose the window size k_n as long as $k_n \to \infty$ and $\Delta_n k_n \to 0$. So we can give the noise any order we want.
- Consider the perturbated processes

$$Z_t^{n,1} = Y_t + \sqrt{\Delta_n}\widehat{W}_t,$$
$$Z_t^{n,2} = Y_t + \sqrt{2\Delta_n}\widehat{W}_t.$$

Then the expansion has the form

$$(\Delta_n k_n)^{-1/2}\left(\overline{Z}(g)_i^{n,1}, \cdots, \overline{Z}(g)_{i+(d-1)k_n}^{n,1}\right)$$
$$= A(g)_i^n + (\Delta_n k_n)^{1/2}\widehat{B}(g)_i^n + \Delta_n k_n C(g)_i^n + (\Delta_n k_n)^{\nu/2} E(g)_i^n$$

Accounting for the noise: The pre-averaging approach (5)

$$\underbrace{A(g)_i^n}_{O_{\mathbb{P}}(1)} + (\Delta_n k_n)^{1/2} \underbrace{\widehat{B}(g)_i^n}_{O_{\mathbb{P}}(1)} + \Delta_n k_n \underbrace{C(g)_i^n}_{O_{\mathbb{P}}(1)} + (\Delta_n k_n)^{\nu/2} E(g)_i^n.$$

- Recall: We are free to choose the window size k_n as long as $k_n \to \infty$ and $\Delta_n k_n \to 0$ as $\Delta_n \to 0$.
  $\rightsquigarrow$ For $\nu = 0, 1, 2$ we can choose k_n such that $E(g)_i^n$ is $O_{\mathbb{P}}(1)$!
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- Drawback of the pre-averaging approach: Only the weighted averages over a window of size k_n enter into the main statistic.
  $\rightsquigarrow$ The number of data-points decreases from $[T/\Delta_n]$ to $[T/\Delta_n k_n]$.
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- Incentive to choose k_n as small as possible.

Accounting for the noise: The convergence rate

$$\underbrace{A(g)_i^n}_{O_{\mathbb{P}}(1)} + (\Delta_n k_n)^{1/2} \underbrace{\widehat{B}(g)_i^n}_{O_{\mathbb{P}}(1)} + \Delta_n k_n \underbrace{C(g)_i^n}_{O_{\mathbb{P}}(1)} + (\Delta_n k_n)^{\nu/2} \underbrace{E(g)_i^n}_{O_{\mathbb{P}}(1)}.$$

$\nu = 2$ The noise will not appear in the limit and the proofs are rather easy.
$k_n = O(\Delta_n^{-3/4}) \rightsquigarrow$ convergence rate of $\Delta_n^{1/8}$.

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$\nu = 1$ The noise will affect the variance in the CLT. We need two different weight functions for the two statistics $S^{n,1}$ and $S^{n,2}$.
$k_n = O(\Delta_n^{-2/3}) \rightsquigarrow$ convergence rate of $\Delta_n^{1/6}$.

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- $\nu = 2$ The noise will not appear in the limit and the proofs are rather easy.
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- $\nu = 1$ The noise will affect the variance in the CLT. We need two different weight functions for the two statistics $S^{n,1}$ and $S^{n,2}$.
  $k_n = O(\Delta_n^{-2/3}) \rightsquigarrow$ convergence rate of $\Delta_n^{1/6}$.
- $\nu = 0$ The noise enters the CLT as a BIAS. We need BIAS-correction which is rather involved (and practically not feasible).
  $k_n = O(\Delta_n^{-1/2}) \rightsquigarrow$ optimal convergence rate of $\Delta_n^{1/4}$.

Accounting for the noise: The main statistic

We confine ourselves to the case $\nu = 1$ and put the formal assumption.

For $\theta \in (0, \infty)$ let k_n be a sequence of integers satisfying

$$k_n = \frac{1}{\theta \Delta_n^{2/3}} \left(1 + o\big(\Delta_n^{1/6}\big)\right) = O(\Delta_n^{-2/3}).$$

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Let g be a weight function. Then we define the main statistics

$$S(g)_t^{n,1} = 2d\Delta_n k_n \sum_{i=0}^{[t/2d\Delta_n k_n]-1} \det{}^2 \Big((\Delta_n k_n)^{-1/2} \overline{Z}(g)^{n,1}_{2idk_n}, \cdots, (\Delta_n k_n)^{-1/2} \overline{Z}(g)^{n,1}_{(2id+(d-1))k_n} \Big),$$

$$S(g)_t^{n,2} = 2d\Delta_n k_n \sum_{i=0}^{[t/2d\Delta_n k_n]-1} \det{}^2 \Big((2\Delta_n k_n)^{-1/2} \overline{Z}(g)^{n,2}_{2idk_n}, \cdots, (2\Delta_n k_n)^{-1/2} \overline{Z}(g)^{n,2}_{(2id+2(d-1))k_n} \Big).$$

Accounting for the noise: The Law of Large Numbers

- Again, one can show that on Ω_T^r

$$\frac{1}{(\Delta_n k_n)^{d-r}} S(g)_T^{n,1} \xrightarrow{\mathbb{P}} S(r,g)_T^1 = \int_0^T \Gamma_r^1(\sigma_s, v_s, b_s, \Sigma, g) ds > 0,$$

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- For our method it is crucial that the limits coincide. However, $S(r,g)_T^1 = S(r,g)_T^2$.
- Using the same weight function for the two different rates, the equality does not hold!
  Reason: The semimartingale part satisfies a scaling property whereas the noise part does not. (For the case $\nu = 2$ that does not matter since the noise part disappears in the limit.)

Accounting for the noise: The Law of Large Numbers (2)

- A careful inspection of the influence of the weight function to the limit yields the following:
  The maps $g \mapsto S(r,g)^1_T$ and $g \mapsto S(r,g)^2_T$ are real-valued functionals that factorize in a functional mapping g to $\mathbb{R}^4$ and a polynomial mapping from $\mathbb{R}^4$ to $\mathbb{R}$.

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- The part associated to the semimartingale depends on the functionals

$$\int_0^1 g^2(s)ds \qquad \text{(associated with } \sigma\text{)},$$

$$\int_0^1 g^2(s)s\,ds \qquad \text{(associated with } v\text{)},$$

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$$\int_0^1 g(s)ds \qquad \text{(associated with } b\text{)}.$$

- The noise part depends on the functional

$$\int_0^1 \left(g'(s)\right)^2 ds.$$

Accounting for the noise: The Law of Large Numbers (3)

Solution: We use two different weight functions g and h such that

$$\int_0^1 h^2(s)ds = \int_0^1 g^2(s)ds, \qquad \int_0^1 h^2(s)s\,ds = \int_0^1 g^2(s)s\,ds,$$

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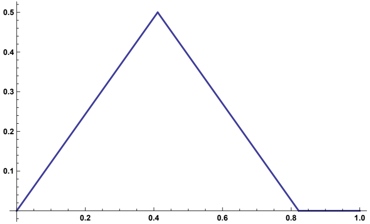
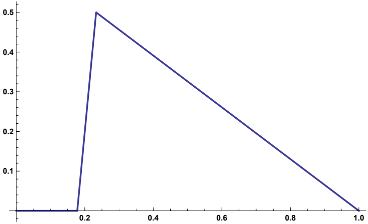
Example: A pair g, h satisfying the above relations:





Accounting for the noise: The Law of Large Numbers (4)

- We obtain that

$$S(h)^{n,2}_T / S(g)^{n,1}_T \xrightarrow{\mathbb{P}} 2^{d-R_T}.$$

Accounting for the noise: The Law of Large Numbers (4)

- We obtain that

$$S(h)_T^{n,2}/S(g)_T^{n,1} \xrightarrow{\mathbb{P}} 2^{d-R_T}.$$

- We can construct a consistent 'estimator' for the maximal rank R_T

$$\widehat{R}(n,T,g,h) = d - \frac{\log\left(S(h)_T^{n,2}/S(g)_T^{n,1}\right)}{\log 2} \xrightarrow{\mathbb{P}} R_T.$$

Accounting for the noise: Central Limit Theorem

- We can also derive associated CLTs with the rate $(\Delta_n k_n)^{1/2} \approx \Delta_n^{1/6}$.
- We have the following 2-dimensional stable convergence on Ω_T^r

$$(\Delta_n k_n)^{-1/2} \left(\frac{1}{(\Delta_n k_n)^{d-r}} S(g)_T^{n,1} - S(r,g)_T^1, \frac{1}{(2\Delta_n k_n)^{d-r}} S(h)_T^{n,2} - S(r,h)_T^2 \right) \xrightarrow{\mathcal{L}-s} MN\Big(0, \int_0^T \Theta_r(\sigma_s, v_s, b_s, \Sigma, g, h) ds\Big),$$

and

$$(\Delta_n k_n)^{-1/2} \left(\widehat{R}(n, T, g, h) - r \right) \xrightarrow{\mathcal{L}-s} MN\Big(0, \int_0^T V_r(\sigma_s, v_s, b_s, \Sigma, g, h) ds\Big).$$

Accounting for the noise: Central Limit Theorem (2)

- The conditional variance $\int_0^T V_r(\sigma_s, v_s, b_s, \Sigma, g, h)ds$ can be consistently estimated by $V(n, T, g, h)$ such that we obtain a feasible version of the stable convergence:

$$\frac{(\Delta_n k_n)^{-1/2}\left(\widehat{R}(n, T, g, h) - R_T\right)}{\sqrt{V(n, T, g, h)}} \xrightarrow{\mathcal{L}-s} \Phi \sim \mathcal{N}(0, 1).$$

The testing procedure

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- For $r \in \{0, \dots, d\}$ we can test the null hypothesis

$$H_0 : \Omega_T^r = \{\omega \in \Omega : \; R_T(\omega) = r\}$$

against the alternative

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- Let $\alpha \in (0,1)$ and c_α denote the symmetric α-quantile of $\mathcal{N}(0,1)$ defined by $\mathbb{P}(|\Phi| > c_\alpha) = \alpha$ when $\Phi \sim \mathcal{N}(0,1)$. Then we obtain an asymptotic level α test in the sense that

$$\mathbb{P}_{H_0}\left(\left| \frac{(\Delta_n k_n)^{-1/2}\left(\widehat{R}(n,T,g,h) - R_T\right)}{\sqrt{V(n,T,g,h)}} \right| > c_\alpha \right) \to \alpha.$$

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- It is also consistent for the alternative in the sense that

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Simulation study & real data example

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- For estimating the maximal rank, one can consider truncated and rounded versions of the estimator (consistency okay, but no CLT for this version!).

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Simulation study

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- In order to improve the finite sample performance and the asymptotic variance, one can use an estimator with overlapping increments. An LLN is straight forward. However, a CLT seems to be more involved.
- Empirical results:
  - The empirical counterparts of our statistics seem to converge to the correct limits.
  - The speed of convergence is rather slow – in line with the convergence rate of $\Delta_n^{1/6}$.
  - Performance is better for smaller dimensions (rate is rather $[T/2dk_n\Delta_n]^{-1/2}$ than $\Delta_n^{1/6}$.
  - In particular, the speed of convergence depends on the complexity of the respective model of the semimartingale.
  - Working with overlapping increments decreases the variance while the bias remains the same.

Real data example

- Consider 8 American banks between 2006 and 2009 (1007 trading days).
  $\rightsquigarrow$ Homogeneous market.
  $\rightsquigarrow$ Period includes crisis.
- Pre-cleaning to exclude jumps (details in the paper).

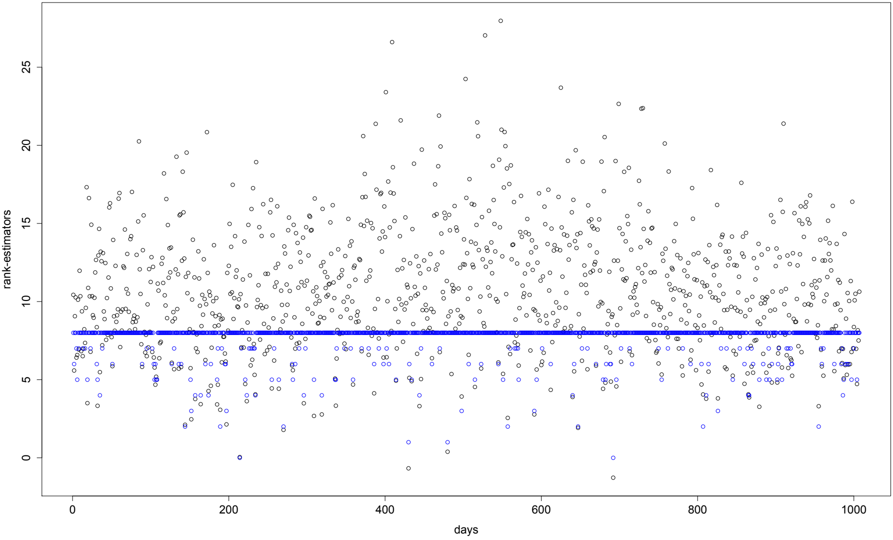

Figure: Estimators $\widehat{R}(g,h)_1^n$ (black) and $\widehat{R}^{\text{int}}(g,h)_1^n$ (blue) over a one-day time window. (Non-overlapping increments)

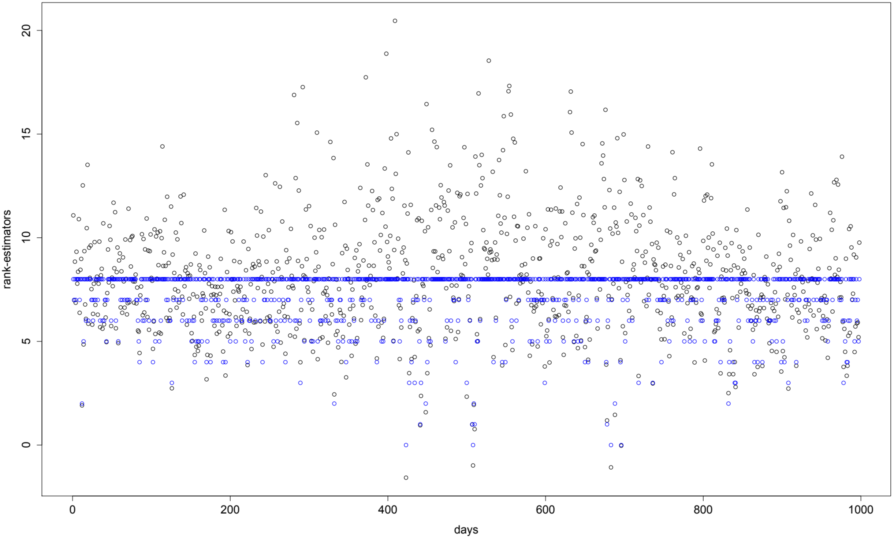

Figure: Estimators $\widehat{R}(g,h)^n_{10}$ (black) and $\widehat{R}^{\mathrm{int}}(g,h)^n_{10}$ (blue) over a 10-days rolling time window. (Non-overlapping increments)

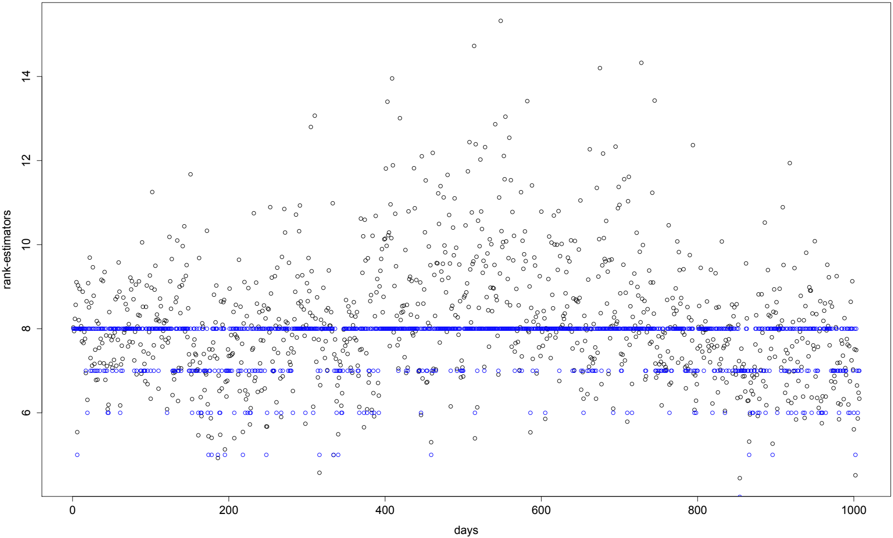

Figure: Estimators $\widetilde{R}(g,h)_1^n$ (black) and $\widetilde{R}^{\text{int}}(g,h)_1^n$ (blue) over a one-day time window. (Overlapping increments)

Real data example – summary

Table: Sample mean and variance for the non-overlapping and overlapping approach.

overlapping	T	mean	variance
no	1	10.85	16.11
no	10	8.20	8.12
yes	1	8.22	2.31

Extension

In theory, one could test the local volatility assumption: $c_t = h(X_t)$ for $f \in C^2(\mathbb{R}^d)$:

- Illustration for $d = 1$. One considers the semimartingale $\begin{pmatrix} X_t \\ c_t \end{pmatrix}$. Then, one can ask for the maximal rank of the co-volatility of $\begin{pmatrix} X_t \\ c_t \end{pmatrix}$.
- Usually, one cannot observe the volatility process. So one needs an estimator of the spot volatility. Since the test-statistic consists of a ratio of two degenerate statistics, one obtains a rate of $\Delta_n^{1/6}$ – even in the absence of noice!

References

T. Fissler and M. Podolskij. Testing the maximal rank of the volatility process for continuous diffusions observed with noise.
Bernoulli. Volume 23, Number 4B (2017), 3021–3066.
http://projecteuclid.org/euclid.bj/1495505084

All further references can be found there.

Thank you for your attention!

Stable convergence

Definition 3

Let $(X_n)_{n \geqslant 1}$ be a sequence of random elements on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a Polish space. X_n converges stably to a random element X,

$$X_n \xrightarrow{\mathcal{L}-s} X$$

where X is defined on an extension $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, if and only if

$$\lim_{n \to \infty} \mathbb{E}\left[\phi(X_n) Z\right] = \widetilde{\mathbb{E}}\left[\phi(X) Z\right]$$

for any bounded and continuous function ϕ and any bounded and $\mathcal{F}$-measurable random variable Z.

Stable convergence II

Proposition 4

The following properties are equivalent:

(i) $X_n \xrightarrow{\mathcal{L}-s} X$;

(ii) $(X_n, Z) \xrightarrow{d} (X, Z)$ *for any* $\mathcal{F}$*-measurable random variable* Z*;*

(iii) $(X_n, Z) \xrightarrow{\mathcal{L}-s} (X, Z)$ *for any* $\mathcal{F}$*-measurable random variable* Z*.*

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Proposition 5

$$X_n \xrightarrow{\mathbb{P}} X \quad \Longrightarrow \quad X_n \xrightarrow{\mathcal{L}-s} X \quad \Longrightarrow \quad X_n \xrightarrow{d} X.$$

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$$X_n \xrightarrow{\mathbb{P}} X \quad \Longrightarrow \quad X_n \xrightarrow{\mathcal{L}-s} X \quad \Longrightarrow \quad X_n \xrightarrow{d} X.$$

Proposition 6

(i) *If* $X_n \xrightarrow{\mathcal{L}-s} X$ *and* $V_n \xrightarrow{\mathbb{P}} V$*, then* $(X_n, V_n) \xrightarrow{\mathcal{L}-s} (X, V)$.

(ii) *If* $X_n \xrightarrow{\mathcal{L}-s} X \sim MN(0, V^2)$ *where* V *is* $\mathcal{F}$*-measurable and* $V_n \xrightarrow{\mathbb{P}} V > 0$*, then* $\frac{X_n}{V_n} \xrightarrow{\mathcal{L}-s} N(0, 1)$.