Measuring systemic risk via model uncertainty

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based on a joint work with Birgit Rudloff

March 4, 2016
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Systemic risk

- Interconnected financial system
- Failures affecting multiple entities
  - e.g. chain of defaults
- Important in the event of financial crisis
- Systemic vs. institutional risk

Measuring systemic risk

1. Aggregation function Λ
2. Acceptance set $\mathcal{A}$
3. Systemic risk measure R^{sys}

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  - Like a utility function but multivariate!

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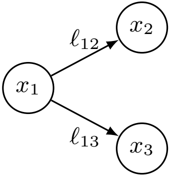

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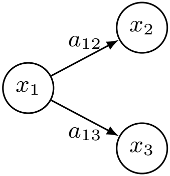
- Liability matrix: $(\ell_{ij})_{0 \leq i,j \leq d}$
- Total liability of entity i: $\bar{p}_i = \sum_{j=0}^{d} \ell_{ij}$
- Relative liabilitiy of i to j: $a_{ij} = \frac{\ell_{ij}}{\bar{p}_i}$



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- Clearing/realized payment vector (equilibrium): $p(x) = (p_1(x), \ldots, p_d(x))$
- $p(x) \in \mathbb{R}_+^d$ is the solution of the fixed point problem

$$p_i(x) = \bar{p}_i \wedge \left(x_i + \sum_{j=1}^{d} p_j(x) a_{ji} \right), \quad i \in \{1, \ldots, d\}\,.$$

- Equilibrium: Pay either what you owe or what you have.
- There exists a unique $p(x)$ under mild conditions.

Example: Eisenberg, Noe ('01) (cont'd)

- Equity/loss of entity i after clearing:

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- Aggregation function: equity of the society

$$\Lambda(x) := e_0(x) = \sum_{j=1}^{d} p_j(x) a_{j0}.$$

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- The impact of wealth vector X on society is $\Lambda(X)$.
- More features could be modeled:
  - Liquid and illiquid assets (e.g. Cifuentes, Shin, Ferrucci ('05))
  - Random liability matrix (e.g. Amini, Filipovic, Minca ('15))
  - Impact on a group of entities: $\Lambda\colon \mathbb{R}^d \to \mathbb{R}^m$ with $\Lambda(x) = (e_1(x), \ldots, e_m(x))$, e.g. $\{1, \ldots, m\}$ are the small banks

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- Chen, Iyengar, Moallemi ('13), Kromer, Overbeck, Zilch ('14)
- More naive choices as they ignore the network structure

- Which values of $\Lambda(X)$ are acceptable?
- $\mathcal{A} \subseteq L^\infty$ acceptance set of a scalar convex risk measure ρ
- $\mathcal{A} = \{Y \in L^\infty \mid \rho(Y) \leq 0\}$
- $\rho(Y) = \inf \{y \in \mathbb{R} \mid Y + y \in \mathcal{A}\}$
- If $\mathcal{A}$ is weak*-closed and convex, then ρ admits the dual representation

$$\rho(Y) = \sup_{\mathbb{S} \in \mathcal{M}(\mathbb{P})} \left(\mathbb{E}^{\mathbb{S}} [-Y] - \alpha(\mathbb{S}) \right),$$

where α can be chosen as

$$\alpha(\mathbb{S}) = \sup_{Y \in L^\infty} \left(\mathbb{E}^{\mathbb{S}} [-Y] - \rho(Y) \right).$$

- A measure of systemic risk is the set of all capital allocations that make the impact to the society acceptable.

1. Aggregation mechanism insensitive to capital levels

$$R^{\text{ins}}(X) = \left\{ z \in \mathbb{R}^d \mid \Lambda(X) + \sum_{i=1}^{d} z_i \in \mathcal{A} \right\}$$

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    - Feinstein, Rudloff, Weber ('15): grid search algorithm
    - Biagini, Fouque, Fritelli, Meyer-Brandis ('15): similar structure

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- ... under mild assumptions:
  - ρ is a convex weak*-lsc risk measure.
  - Λ is concave and increasing (with respect to componentwise ordering).
  - $\rho(0) \in \operatorname{int} \Lambda(\mathbb{R}^d)$.

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- Recall scalar case:

$$\rho(X) = \sup_{\mathbb{S} \in \mathcal{M}(\mathbb{P})} \left(\mathbb{E}^{\mathbb{S}}\left[-X\right] - \alpha(\mathbb{S}) \right)$$

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Dual representations for systemic risk measures

- $R^{\mathsf{sen}}(X) = \left\{z \in \mathbb{R}^d \mid \Lambda(X+z) \in \mathcal{A}\right\}$

- The systemic risk measure R^{sen} has the dual representation

$$R^{\mathsf{sen}}(X) = \bigcap_{\mathbb{Q} \in \mathcal{M}_d(\mathbb{P}), w \in \mathbb{R}^d_+ \setminus \{0\}} \mathbb{E}^{\mathbb{Q}}\left[-X\right] + \left\{z \in \mathbb{R}^d \mid w^{\mathsf{T}} z \geq -\alpha^{\mathsf{sys}}(\mathbb{Q}, w)\right\},$$

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- $\mathbb{S} \in \mathcal{M}^e(\mathbb{P})$ equivalent probability measure
- $g(z) = \sup_{x\in\mathbb{R}^d} \left(\Lambda(x) - x^{\mathsf{T}} z\right)$ Legendre-Fenchel conjugate of $x \mapsto -\Lambda(-x)$
- $x \cdot z = (x_1 z_1, \ldots, x_d z_d)^{\mathsf{T}}$

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- $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d) \in \mathcal{M}_d(\mathbb{P})$ vector probability measure with $\mathbb{Q}_i \ll \mathbb{P}$ for each i
- $\mathbb{E}^{\mathbb{Q}}[X] = \left(\mathbb{E}^{\mathbb{Q}_1}[X_1], \ldots, \mathbb{E}^{\mathbb{Q}_d}[X_d] \right)$
- $w \in \mathbb{R}^d_+ \setminus \{0\}$

- Fix an absolutely continuous probability measure $\mathbb{Q}_i$ and a weight w_i for each institution $i \in \{1, \ldots, d\}$.

Dual representation: Interpretation

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- Penalty for using $(\mathbb{Q}, w)$ relative to the society's probability measure $\mathbb{S}$:

$$\mathbb{E}^{\mathbb{S}}\left[g\left(w_1\frac{d\mathbb{Q}_1}{d\mathbb{S}}, \dots, w_d\frac{d\mathbb{Q}_d}{d\mathbb{S}}\right)\right].$$

- "Weighted distance" of the vector probability measure $\mathbb{Q}$ to the society's probability measure $\mathbb{S}$ (multivariate version of the well-known f-divergence).

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$$\alpha^{\mathsf{sys}}(\mathbb{Q}, w) = \inf_{\mathbb{S} \in \mathcal{M}^e(\mathbb{P})} \left(\alpha(\mathbb{S}) + \mathbb{E}^{\mathbb{S}} \left[g \left(w \cdot \frac{d\mathbb{Q}}{d\mathbb{S}} \right) \right] \right).$$

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- A capital allocation vector $z \in \mathbb{R}^d$ is considered feasible with respect to the model $\mathbb{Q} \in \mathcal{M}_d(\mathbb{P})$ and weight vector $w \in \mathbb{R}^d_+ \backslash \{0\}$ if its *weighted sum* exceeds a certain thresold threshold, precisely, if

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- Derived by set-valued convex analysis + a conjugation result for the composition of convex functions (Boţ, Grad, Wanka ('13)).

Dual representation: Example (Entropic model)

- Exponential aggregation: $\Lambda(x) = -\sum_{i=1}^{d} e^{-x_i - 1}$
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- Systemic penalty function becomes

$$\alpha^{\text{sen}}(\mathbb{Q}, w) = \inf_{\mathbb{S} \in \mathcal{M}^e(\mathbb{P})} \left(\mathcal{H}\left(\mathbb{S} \| \mathbb{P}\right) + \sum_{i=1}^{d} w_i \mathcal{H}\left(\mathbb{Q}_i \| \mathbb{S}\right) \right) + c(w).$$

- $\mathcal{H}(\mathbb{Q}_i \| \mathbb{S}) = \mathbb{E}^{\mathbb{S}}\left[\frac{d\mathbb{Q}_i}{d\mathbb{S}} \log\left(\frac{d\mathbb{Q}_i}{d\mathbb{S}}\right)\right]$ (relative entropy)

Dual representation: Example (Eisenberg-Noe model)

- Realized wealth: $x \in \mathbb{R}^d_+$
- Clearing payments: $p(x) = (p_1(x), \ldots, p_d(x))$
- Aggregation function: $\Lambda(x) = \sum_{j=1}^d a_{j0} p_j(x)$
- Multivariate divergence function takes the form

$$\mathbb{E}^{\mathbb{S}}\left[g\left(w \cdot \frac{d\mathbb{Q}}{d\mathbb{S}}\right)\right] = \sum_{i=1}^{d} \mathbb{E}^{\mathbb{S}}\left[\left(\sum_{j=0}^{d} \ell_{ij}\left(w_j \frac{d\mathbb{Q}_j}{d\mathbb{S}} - w_i \frac{d\mathbb{Q}_i}{d\mathbb{S}}\right)\right)^+\right].$$

A model uncertainty interpretation

- Aggregation function Λ is a multivariate utility function.
  - Campi, Owen ('11): same type of utility function for utility maximization
  - A., Hamel, Rudloff ('15): vector-valued versions for shortfall risk measures

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- Use the dual representation of ρ:

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\begin{aligned}
R^{\text{sen}}(X) &= \left\{z \in \mathbb{R}^d \mid \Lambda(X+z) \in \mathcal{A}\right\} \\
&= \left\{z \in \mathbb{R}^d \mid \rho(\Lambda(X+z)) \leq 0\right\} \\
&= \left\{z \in \mathbb{R}^d \mid \sup_{\mathbb{S} \in \mathcal{M}^1(\mathbb{P})} \left(\mathbb{E}^{\mathbb{S}}[-\Lambda(X+z)] - \alpha(\mathbb{S})\right) \leq 0\right\} \\
&= \bigcap_{\mathbb{S} \in \mathcal{M}^1(\mathbb{P})} \left\{z \in \mathbb{R}^d \mid \mathbb{E}^{\mathbb{S}}[-\Lambda(X+z)] \leq \alpha(\mathbb{S})\right\} \\
&= \bigcap_{\mathbb{S} \in \mathcal{M}^1(\mathbb{P})} \left\{z \in \mathbb{R}^d \mid \mathbb{E}^{\mathbb{S}}[\ell(-X-z)] \leq \alpha(\mathbb{S})\right\} \\
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where $R^{\mathbb{S}}$ is a multivariate utility-based shortfall risk measure with threshold value $x^0 = \alpha(\mathbb{S})$ under the model $(\Omega, \mathcal{F}, \mathbb{S})$.

Thank you!

References

- Ç. A., B. Rudloff, *Dual representations for systemic risk measures*, working paper.
- Ç. A., A. H. Hamel and B. Rudloff, *Set-valued shortfall and divergence risk measures*, arXiv e-prints, 1405.4905, 2014.
- Chen, Iyengar, Moallemi (2013), *An Axiomatic Approach to Systemic Risk*, Management Science 59(6), 1373–1388, 2013.
- Feinstein, Rudloff, Weber, *Measures of Systemic Risk*, 2015.
- Kromer, Overbeck, Zilch, *Systemic Risk Measures on General Probability Spaces*, 2014.
- P. Artzner, F. Delbaen, J.-M. Eber and D. Heath, *Coherent measures of risk*, Mathematical Finance, **9**(3): 203-228, 1999.

- R. I. Boţ, S.-M. Grad, G. Wanka, *Generalized Moreau-Rockafellar results for composed convex functions*, Optimization, 2013.
- L. Campi and M. P. Owen, *Multivariate utility maximization with proportional transaction costs*, Finance and Stochastics, **15**(3): 461–499, 2011.
- H. Föllmer and A. Schied, *Stochastic finance: an introduction in discrete time*, De Gruyter Textbook Series, third edition, 2011.
- A. H. Hamel, *A duality theory for set-valued functions I: Fenchel conjugation theory*, Set-Valued and Variational Analysis, **17**(2): 153–182, 2009.
- A. H. Hamel and F. Heyde, *Duality for set-valued measures of risk*, SIAM Journal on Financial Mathematics, **1**(1): 66–95, 2010.
- E. Jouini, M. Meddeb and N. Touzi, *Vector-valued coherent risk measures*, Finance and Stochastics, **8**(4): 531–552, 2004.