Copula Families that Generalise the Archimedean Class

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Contents

1. Copulas

Copulas have found a variety of actuarial/financial applications:

- Life insurance - models for joint (dependent) lives
- Non-life insurance - loss distributions for multi-line insurance losses
- Risk aggregation - models for combining loss distributions in a modular appproach to deriving risk capital
- Capital allocation - models for disaggregating overall capital into contributions
- Market risk - models for asset returns
- Credit risk - multivariate survival models for times-to-default

Some Points in Favour...

- Copulas help in the understanding of dependence at a deeper level;
- They show us potential pitfalls of approaches to dependence that focus only on correlation;
- They allow us to define useful alternative dependence measures;
- They express dependence on a quantile scale, which is natural in QRM;
- They facilitate a bottom-up approach to multivariate model building;
- They are easily simulated and thus lend themselves to Monte Carlo risk studies.

And Some Against...

Copulas are not universally popular among actuarial modellers; some find they have little added value in the bigger picture of multivariate stochastic models.

See [Mikosch, 2006] and [Genest and Rémillard, 2006] for a lively discussion. Main issues are:

- They are often applied very arbitrarily without justification for their appropriateness.
- Too many choices - when do we use Gauss copulas t copulas, Archimedean, or other copulas?
- Static representations of dependence that are not well connected to the theory of multivariate stochastic processes.

What is a copula?

A copula is a multivariate distribution function with standard uniform margins.

Equivalently, a copula if any function $C : [0,1]^d \to [0,1]$ satisfying the following properties:

1. $C(u_1, \dots, u_d) = 0$ whenever $u_i = 0$ for at least one $i = 1, \dots, d$.
2. $C(1, \dots, 1, u_i, 1, \dots, 1) = u_i$ for all $i \in \{1, \dots, d\}$, $u_i \in [0,1]$.
3. For all $(a_1, \dots, a_d), (b_1, \dots, b_d) \in [0,1]^d$ with $a_i \leq b_i$ we have:

$$\sum_{i_1=1}^{2} \cdots \sum_{i_d=1}^{2} (-1)^{i_1+\cdots+i_d} C(u_{1i_1}, \dots, u_{di_d}) \geq 0,$$

   where $u_{j1} = a_j$ and $u_{j2} = b_j$ for all $j \in \{1, \dots, d\}$.

Sklar's Theorem

Let F be a joint distribution function with margins $F_1, \ldots, F_d$. There exists a copula C such that for all $x_1, \ldots, x_d$ in $[-\infty, \infty]$

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

If the margins are continuous then C is unique; otherwise C is uniquely determined on Ran$F_1 \times$ Ran$F_2 \ldots \times$ RanF_d.

And conversely, if C is a copula and $F_1, \ldots, F_d$ are (arbitrary) univariate distribution functions, then

$$C(F_1(x_1), \ldots, F_d(x_d)) \equiv F(x_1, \ldots, x_d)$$

defines a d-dimensional multivariate df with margins $F_1, \ldots, F_d$.

Sklar's Theorem for survival functions

Let $\bar{F}$ be a d-dimensional joint survival function with margins $\bar{F}_1, \ldots, \bar{F}_d$. There exists a survival copula $\bar{C}$ such that for all $x_1, \ldots, x_d$ in $[-\infty, \infty]$

$$\bar{F}(x_1, \ldots, x_d) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)).$$

If the margins are continuous then $\bar{C}$ is unique.

And conversely, if $\bar{C}$ is a copula and $\bar{F}_1, \ldots, \bar{F}_d$ are (arbitrary) univariate marginal survival functions, then

$$\bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)) \equiv \bar{F}(x_1, \ldots, x_d)$$

defines a d-dimensional survival function with survival margins $\bar{F}_1, \ldots, \bar{F}_d$.

The Fréchet-Hoeffding bounds

For every copula $C(u_1, \ldots, u_d)$ we have the important bounds

$$\max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \leq C(\mathbf{u}) \leq \min\left\{u_1, \ldots, u_d\right\}. \qquad (1)$$

The upper bound is the df of $(U, \ldots, U)$. It represents perfect positive dependence or comonotonicity and is often denoted M.

The lower bound is often denoted W but it is only a copula when $d = 2$. It is the df of the vector $(U, 1 - U)$ and represents perfect negative dependence or countermonotonicity.

The copula representing independence is $C(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i$.

2. Archimedean copulas

A copula is called Archimedean if it can be written in the form

$$C(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$$

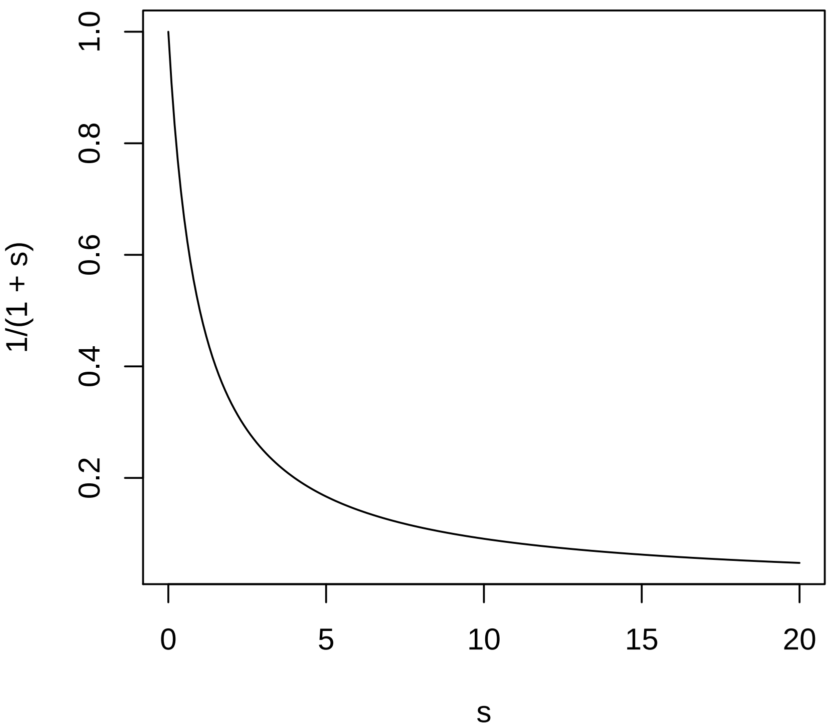
for some generator function ψ and its inverse ψ^{-1}.

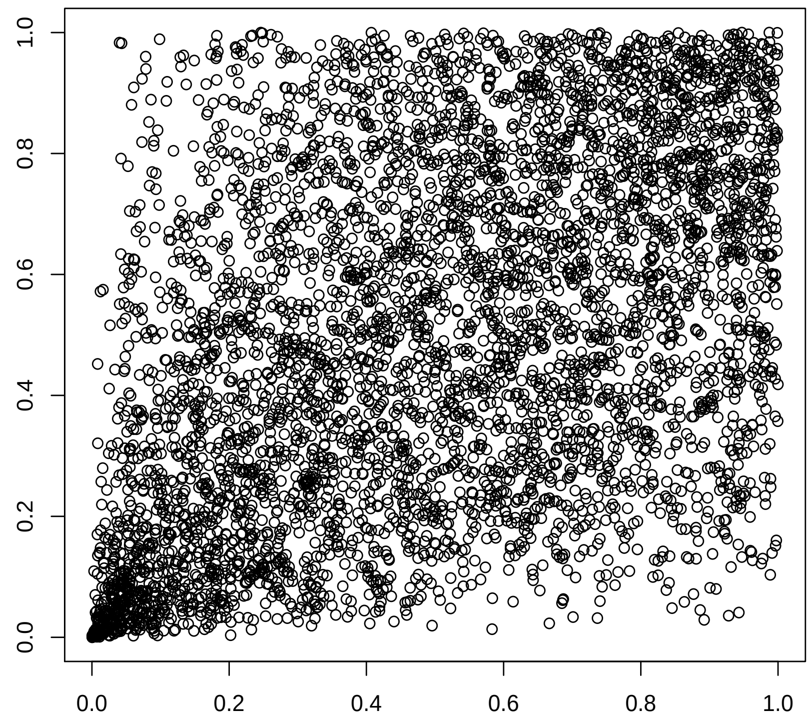
The generator ψ satisfies

- $\psi : [0, \infty) \to [0, 1]$ with $\psi(0) = 1$ and $\lim_{x \to \infty} \psi(x) = 0$
- ψ is continuous
- ψ is strictly decreasing on $[0, \inf\{u : \psi(u) = 0\}]$
- $\psi^{-1}(0) = \inf\{u : \psi(u) = 0\}$

Clayton copula

Take $\psi_\theta(x) = (1+\theta x)_+^{-\frac{1}{\theta}}$ for $\theta \geq -\frac{1}{d-1}$.





Generator and sample in case $\theta = 1$.

Necessary and sufficient conditions

Ling (1965)

A generator ψ induces a bivariate copula if and only if ψ is convex.

[Kimberling, 1974]

A generator ψ induces an Archimedean copula in any dimension if and only if ψ is completely monotone, i.e. $\psi \in C^{\infty}(0, \infty)$ and $(-1)^k \psi^{(k)}(x) \geq 0$ for $k = 1, \ldots$.

[McNeil and Nešlehová, 2009b]

A generator ψ induces an Archimedean copula in dimension d if and only if ψ is d-monotone, i.e. $\psi \in C^{d-2}(0, \infty)$ and $(-1)^k \psi^{(k)}(x) \geq 0$ for any $k = 1, \ldots, d-2$ and $(-1)^{d-2} \psi^{(d-2)}$ is non-negative, non-increasing and convex.

Williamson Transforms and Simplex Distributions

ψ is a d-monotone generator if and only if ψ is the Williamson d-transform of the df F of a non-negative random variable R satisfying $F_R(0) = 0$.

$$\psi(x) = \mathfrak{W}_d F_R(x) = \int_{(x,\infty)} \left(1 - \frac{x}{r}\right)^{d-1} dF_R(r)$$

The distribution of a non-negative random variable is uniquely given by its Williamson d-transform. If $\psi = \mathfrak{W}_d F_R$ then

$$F_R(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k \psi^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} \psi_+^{(d-1)}(x)}{(d-1)!}\,.$$

[Williamson, 1956]

Relationship to Laplace Transform

$$\lim_{d\to\infty} \mathfrak{W}_d F_{dR}(x) = \lim_{d\to\infty} \mathfrak{W}_d F_R(x/d) = \mathcal{L}F_{1/R}(x).$$

Proof.

$$\mathfrak{W}_d F_{dR}(x) = \mathfrak{W}_d F_R\,(x/d) = \int_0^\infty \left(1 - \frac{x}{rd}\right)_+^{d-1} dF_R(r)$$

For fixed $x \geq 0$ and $r > 0$ we have that

$$\lim_{d\to\infty} \left(1 - \frac{x}{rd}\right)_+^{d-1} = \exp\left(-\frac{x}{r}\right) \ ,$$

from which the result follows.

Simplex distributions

Consider a non-negative random variable R with $\mathsf{P}(R = 0) = 0$ and a random vector $\boldsymbol{S}_d$ independent of R and uniformly distributed on

$$\mathcal{S}_d = \left\{\boldsymbol{x} \in \mathbb{R}_+^d : x_1 + \cdots + x_d = 1\right\}$$

Then $\boldsymbol{X} \stackrel{\mathsf{d}}{=} R\boldsymbol{S}_d$ is said to have a simplex distribution.

Interpretation: R is a random amount of resources to be shared out; $\boldsymbol{S}_d$ represents random but equitable sharing; $\boldsymbol{X}$ are amounts obtained by each individual.

Fundamental Theorem

(i) If $\boldsymbol{X}$ has a simplex distribution with radial distribution F_R satisfying $F_R(0) = 0$, then $\mathbf{X}$ has an Archimedean survival copula with generator $\psi = \mathfrak{W}_d F_R$.

(ii) If $\boldsymbol{U}$ is distributed as an Archimedean copula C with generator ψ, then $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))$ has a simplex distribution with radial distribution $F_R = \mathfrak{W}_d^{-1}\psi$.

Proof sketch: (i) By direct calculation, survival function of $\boldsymbol{X}$ is $\bar{H}(\mathbf{x}) = \psi(x_1 + \cdots + x_d)$ where $\psi = \mathfrak{W}_d F_R$ is d-monotone by [Williamson, 1956]. $\boldsymbol{X}$ must have Archimedean survival copula.
(ii) The survival function of $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))$ is also $\bar{H}(\mathbf{x}) = \psi(x_1 + \cdots + x_d)$, the survival function of a simplex distribution. Must have $(\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d)) \stackrel{\mathrm{d}}{=} R\boldsymbol{S}_d$, for some R, and uniqueness of transform means $F_R = \mathfrak{W}_d^{-1}\psi$.

3. Examples

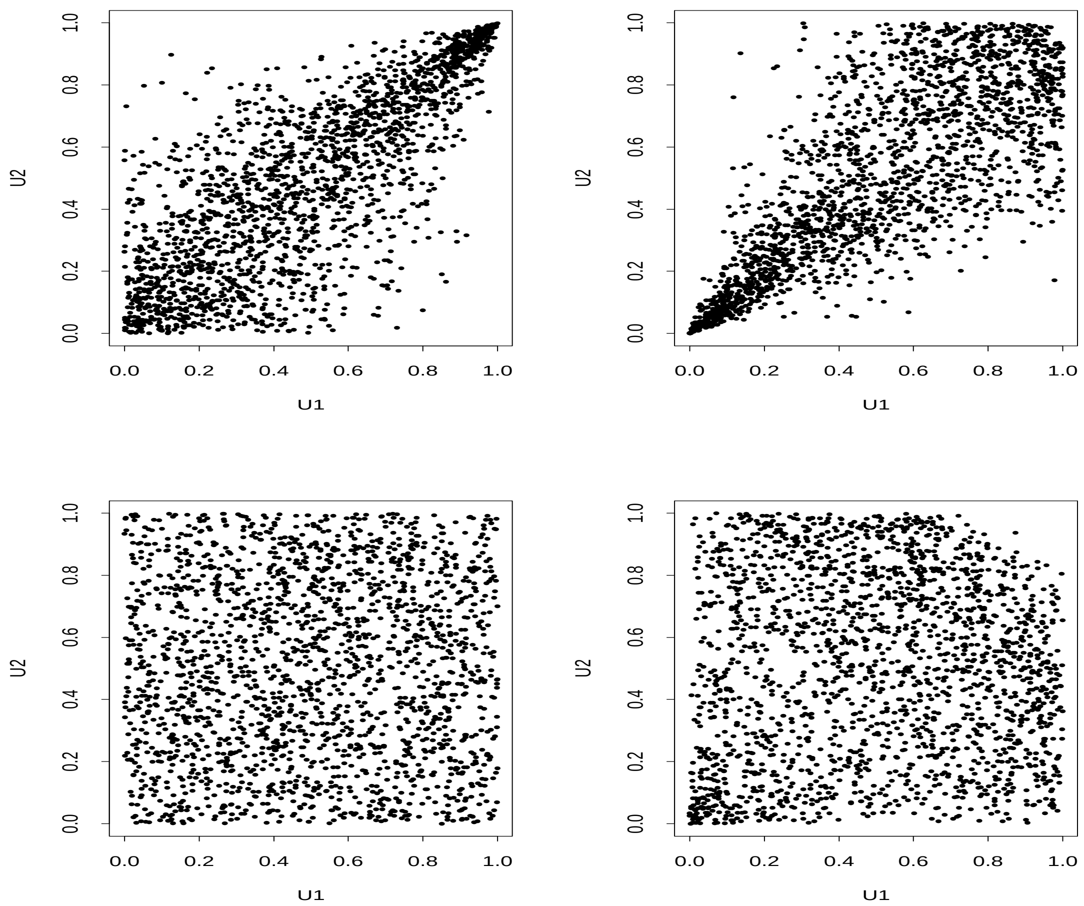
Gamma-simplex copulas.
Let $R \sim \mathrm{Ga}(\theta)$ with density $f_R(r) = r^{\theta-1}\exp(-r)/\Gamma(\theta)$.
This yields a copula family with generators

$$\psi_{\theta,d}(x) = \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-1)^{d-1-k} x^{d-1-k}}{\Gamma(\theta)} \Gamma(k-d+\theta+1, x),$$

where $\Gamma(k,x) = \int_x^\infty t^{k-1} e^{-t}\,dt$ denotes the (upper) incomplete gamma function.

Special case.
When $R \sim \mathrm{Ga}(d)$ (an Erlang distribution) then $\psi_{d,d} = \exp(-x)$, yielding the independence copula in dimension d.

Pictures



Left: gamma-simplex. Right: inverse-gamma-simplex.
Upper copulas have $\theta = 0.3$; lower pictures have $\theta = 2$.

Examples II

Inverse-gamma-simplex copulas.

Suppose $1/R \sim \mathrm{Ga}(\theta)$ for some $\theta > 0$, so that R is inverse-gamma. This yields

$$\psi_{\theta,d}(x) = \sum_{k=0}^{d-1} \binom{d-1}{k} \frac{(-1)^{d-1-k} x^{d-1-k}}{\Gamma(\theta)} \gamma(d+\theta-k-1, 1/x),$$

where $\gamma(k,x) = \int_0^x t^{k-1} e^{-t}\, dt$ denotes the (lower) incomplete gamma function.

In this case $\theta = d$ does not give independence.

Examples III

Pareto-simplex copulas.

If $F_R(r) = 1 - r^{-\kappa}$ for $r \geq 1$ and $\kappa > 0$, we obtain

$$\psi_{\kappa,d}(x) = \kappa x^{-\kappa} B(\min(x,1), \kappa, d) \ ,$$

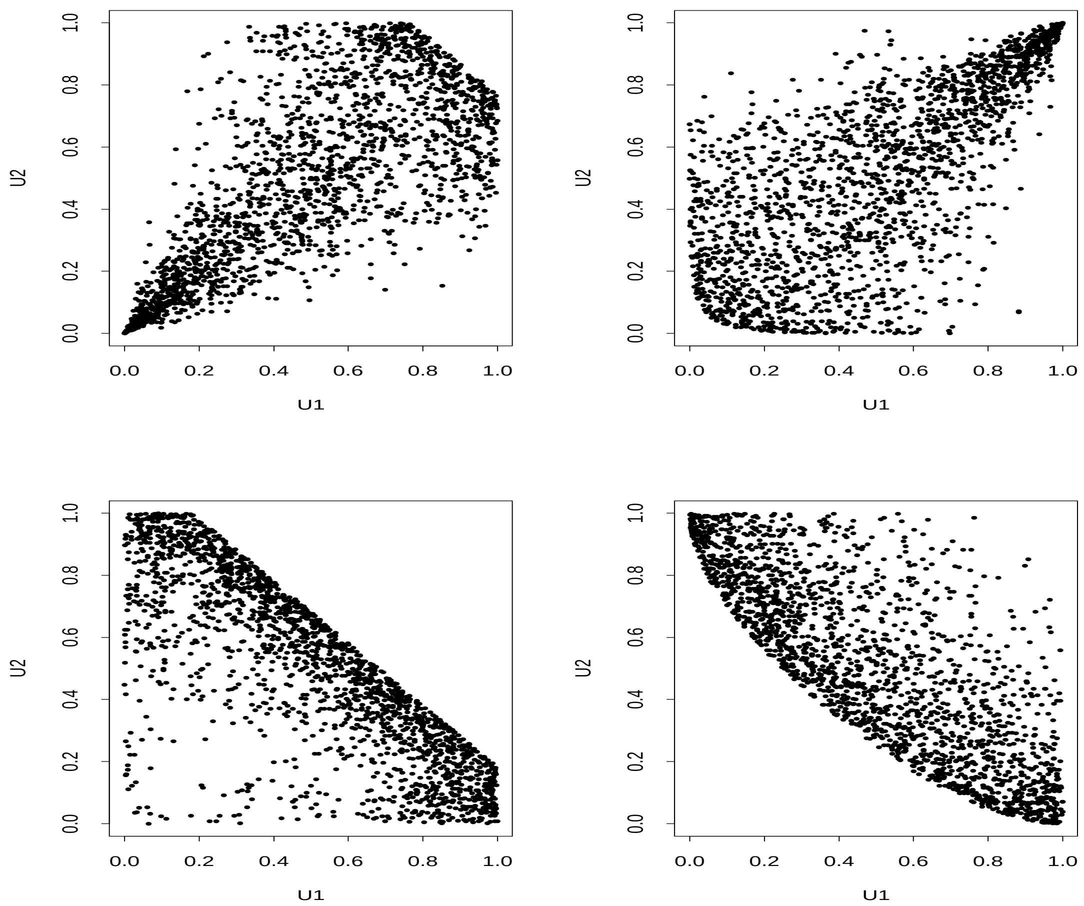
where $B(x, \alpha, \beta)$ denotes the incomplete beta function.

Inverse-Pareto-simplex copulas.

If $f_R(r) = \kappa r^{\kappa-1}$ on the interval $(0,1]$, we obtain

$$\psi_{\kappa,d}(x) = \kappa \sum_{i=0}^{d-1} \binom{d-1}{i} S_i \ , \quad S_i = \begin{cases} (-1)^i \left(\frac{x^\kappa - x^i}{i-\kappa}\right) & i \neq \kappa \\ (-1)^{i+1} x^\kappa \ln(x) & i = \kappa \ . \end{cases}$$

Pictures II



Left: Pareto-simplex. Right: inverse-Pareto-simplex.
Upper copulas have $\theta = 0.3$; lower pictures have $\theta = 4.5$.

The Frailty Subclass

Let $\psi = \mathfrak{W}_d F_R$ for some random variable R satisfying $F_R(0) = 0$ and let Ψ_∞ denote the class of completely monotone generators (which generate copulas in any dimension). We may show that

$$\psi \in \Psi_\infty \iff R \overset{\mathsf{d}}{=} Z_d / W$$

where W is an almost surely positive random variable, independent of $Z_d \sim \mathrm{Erlang}(d)$.

Proof

$\Longleftarrow$ If $R \overset{\mathsf{d}}{=} Z_d / W$ we can show that $\mathfrak{W}_d F_R = \mathcal{L} F_W$ which is completely monotone by Bernstein's theorem.
$\Longrightarrow$ If $\psi \in \Psi_\infty$ then $\psi = \mathcal{L} F_W$ for some W. If $Z_d \sim \mathrm{Erlang}(d)$ independent of W then $\psi = \mathfrak{W}_d F_{Z_d/W}$. Since $\mathfrak{W}_d F_R = \mathfrak{W}_d F_{Z_d/W}$ the uniqueness of the Williamson transform implies $R \overset{\mathsf{d}}{=} Z_d / W$.

The Frailty Subclass II

- A d-dimensional copula with generator $\psi \in \Psi_\infty$ is known as a frailty copula.

- Let $R \sim F_R$ where $F_R = \mathfrak{W}_d^{-1}\psi$. Let $W \sim F_W$ where $F_W = \mathcal{L}^{-1}\psi$. The random vector $\mathbf{X} = R\mathbf{S}_d$ has a simplex distribution with alternative stochastic representation $\mathbf{X} \overset{\text{d}}{=} \mathbf{Y}/W$ where $\mathbf{Y} = (Y_1, \ldots, Y_d)$ is vector of iid unit exponential variables.

- This gives two ways of sampling the copula (using distribution of R or distribution of W).

- The copula is the survival copula of any shared multiplicative frailty model with frailty W.

Shared Frailty Model

Conditional on $W = w$ assume that the lifetimes $T_1, \ldots, T_d$ are independent with the hazard function for the ith individual given by $\lambda_i(t, w) = w\lambda_i(t)$ for some underlying hazard $\lambda_i(t)$. The lifetimes $(T_1, \ldots, T_d)$ are said to follow a multiplicative frailty model with frailty W.

It is easily shown the survival copula of the distribution of $(T_1, \ldots, T_d)$ is Archimedean with generator $\psi(x) = \mathcal{L}F_W(x) = E(\exp(-xW))$.

- Widely used in multivariate survival analysis. [Hougaard, 2000]
- Application to survival of spouses.
- They have been used in CDO pricing models ("lifetimes" of dependent bonds/credit risks).

4. Kendall's tau

Apossible extension of Kendall's tau in dimension $d \geq 2$ is

$$\tau(C) = \frac{2^d}{2^{d-1}-1} \int_{[0,1]^d} C(u_1, \ldots, u_d)\, dC(u_1, \ldots, u_d) - \frac{1}{2^{d-1}-1}.$$

[Joe, 1990]

- Independence. When C is the independence copula, $\int C dC = 2^{-d}$ and $\tau(C) = 0$.

- Comonotonicity. When $C = M$, the Fréchet-Hoeffding upper bound copula, then $\int M dM = 2^{-1}$ and $\tau(M) = 1$.

- Archimedean lower bound. Suppose $C = C_d^{\mathbf{L}}$, the survival copula of $\mathbf{S}_d$ which has generator $\psi_d^{\mathbf{L}}(x) = (1-x)_+^{d-1}$. Then $\int C dC = 0$ and $\tau(C_d^{\mathbf{L}}) = -1/(2^{d-1}-1)$.

New Formulas for Kendall's tau I

Let C be an Archimedean copula with generator ψ and radial part R.

$$\tau(C) = \frac{2^d}{2^{d-1}-1} E\psi(R) - \frac{1}{2^{d-1}-1}$$

Formula follows from observing that

$$\tau(C) = \frac{2^d}{2^{d-1}-1} E(C(\mathbf{U})) - \frac{1}{2^{d-1}-1}.$$

where $\mathbf{U} = (U_1, \ldots, U_d) \sim C$.

$$C(\mathbf{U}) = \psi(\psi^{-1}(U_1) + \cdots \psi^{-1}(U_d)) \stackrel{\text{d}}{=} \psi(R).$$

New Formulas for Kendall's tau II

Let $Y = R/R^*$ where R^* is an independent copy of R.

$$\tau(C) = \frac{2^d}{2^{d-1}-1} E\left\{(1-Y)_+^{d-1}\right\} - \frac{1}{2^{d-1}-1}$$

Follows from

$$E\,(1-Y)_+^{d-1} = \int_0^\infty \int_0^\infty \left(1 - \frac{r}{s}\right)_+^{d-1} dF_R(s)\, dF_R(r)$$
$$= \int_0^\infty \psi(r)\, dF_R(r) = E\psi(R).$$

Kendall's tau: Example

Kendall's tau depends on R through the ratio of radial variables Y: Same formula for the gamma- and inverse-gamma-simplex copulas, or for the Pareto- and inverse-Pareto-simplex copulas.

In latter case, for example, we obtain

$$\tau(C_{\kappa,d}) = \frac{2^{d-1}\kappa B(\kappa, d) - 1}{2^{d-1} - 1}.$$

Both cases turn out to yield comprehensive families, giving all correlations between the lower limit for Archimedean copulas $-1/(2^{d-1} - 1)$ and 1. Moreover they are negatively ordered (in terms of Kendall's tau) by their parameter.

5. Liouville Copulas

Dirichlet distributions

Let $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ be independent random variables such that $Z_i \sim \mathrm{Ga}(\alpha_i)$ for positive parameters $\alpha_1, \ldots, \alpha_d$. Write $\alpha = \sum_{i=1}^d \alpha_i$, $\|\boldsymbol{Z}\| = \sum_{i=1}^d Z_i$ and $D_i = Z_i/\|\boldsymbol{Z}\|$ for $1 \le i \le d$. Then

1. $\|\boldsymbol{Z}\|$ and $(D_1, \ldots, D_{d-1})$ are independent;

2. $\|\boldsymbol{Z}\| \sim \mathrm{Ga}(\alpha)$;

3. the joint density of $(D_1, \ldots, D_{d-1})$ is given by

$$f(x_1, \ldots, x_{d-1}) = \frac{\Gamma(\alpha)}{\prod_{i=1}^d \Gamma(\alpha_i)} \prod_{i=1}^{d-1} x_i^{\alpha_i - 1} \left(1 - \sum_{j=1}^{d-1} x_j\right)^{\alpha_d - 1},$$

   where $\sum_{i=1}^{d-1} x_i \le 1$ and $x_i \ge 0$ for $i = 1, \ldots, d-1$.

Liouville Distribution

Let $\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)} = (D_1, \ldots, D_d)$. The distribution of the random vector $(D_1, \ldots, D_{d-1})$, or equivalently of $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)}$, is known as a Dirichlet distribution on the unit simplex $\mathcal{S}_d$, written $\mathbf{D}_{(\alpha_1,\ldots,\alpha_d)} \sim D(\alpha_1, \ldots, \alpha_d)$.

A random vector $\boldsymbol{X}$ on $\mathbb{R}_+^d = [0, \infty)^d$ is said to follow a Liouville distribution if it permits the stochastic representation

$$\boldsymbol{X} \stackrel{\mathsf{d}}{=} R\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)}$$

where $\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)} \sim D(\alpha_1, \ldots, \alpha_d)$ and R is a positive radial random variable independent of $\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)}$.
[Marshall and Olkin, 1979, Gupta and Richards, 1997, Gupta and Richards, 1987, Song and Gupta, 1997, Fang et al., 1990]

The survival copula of $\boldsymbol{X}$ will be called a Liouville copula.

Liouville Distributions with Integer Parameters

Obviously simplex distributions are special cases of Liouville distributions when $\alpha_1 = \cdots = \alpha_d = 1$. So Archimedean copulas are special cases of Liouville copulas.

If the parameters $\alpha_1, \ldots, \alpha_d$ are positive integers then we can extend the equitable resource sharing analogy to Liouville distributions. We can think of individuals forming coalitions to pool their resources.

For example, suppose that $\mathbf{X} \stackrel{\text{d}}{=} R\mathbf{S}_3$ and agents 1 and 2 form a coalition and pool their resources. In effect we now consider the random vector $\mathbf{Y} = (Y_1, Y_2)$, where $Y_1 = X_1 + X_2$ and $Y_2 = X_3$, which has the stochastic representation $\mathbf{Y} \stackrel{\text{d}}{=} R\mathbf{D}_{(2,1)}$.

Survival functions and Williamson Transforms

Let $\boldsymbol{X}$ be a Liouville distributed random vector with radial part R and parameters $(\alpha_1, \ldots, \alpha_d)$ such that $\alpha_i \in \mathbb{N}$ for $i = 1, \ldots, d$. Furthermore, set $\alpha = \sum_{i=1}^d \alpha_i$ and $\psi(x) = \mathfrak{W}_\alpha F_R(x)$. Then the survival function of $\boldsymbol{X}$ is given on $\boldsymbol{x} \in \mathbb{R}_+^d$ by

$$\bar{H}(\boldsymbol{x}) = \sum_{i_1=0}^{\alpha_1-1} \cdots \sum_{i_d=0}^{\alpha_d-1} (-1)^{i_1+\cdots+i_d} \frac{\psi^{(i_1+\cdots+i_d)}(x_1+\cdots+x_d)}{i_1! \cdots i_d!} \prod_{j=1}^d x_j^{i_j} \, .$$

[McNeil and Nešlehová, 2009a]
If ψ is α-times differentiable then $\boldsymbol{X}$ has density

$$h(\boldsymbol{x}) = (-1)^\alpha \psi^{(\alpha)}(\|\boldsymbol{x}\|) \prod_{i=1}^d \frac{x_i^{\alpha_i-1}}{\Gamma(\alpha_i)}, \quad \boldsymbol{x} \in \mathbb{R}_+^d.$$

Marginal Distributions and Simulation

The marginal distributions are given by

$$H_i(x) = 1 - \sum_{j=0}^{\alpha_i - 1} \frac{(-1)^j x^j \psi^{(j)}(x)}{j!} = \mathfrak{W}_{\alpha_i}^{-1}\psi(x), \quad x \in \mathbb{R}_+ \, .$$

Obviously, Liouville distributions are easy to sample. This means that if we can compute the derivatives of the Williamson α-transfrom of the radial part, we can generate samples from the copula in the usual way:

1. Generate $\boldsymbol{X} = R\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)}$.
2. Return $(H_1(X_1), \ldots, H_d(X_d))$.

6. Examples

Gamma- and inverse-Gamma-Liouville copulas

Take $R \sim \mathrm{Ga}(\theta)$ or $1/R \sim \mathrm{Ga}(\theta)$.

Clayton-Liouville

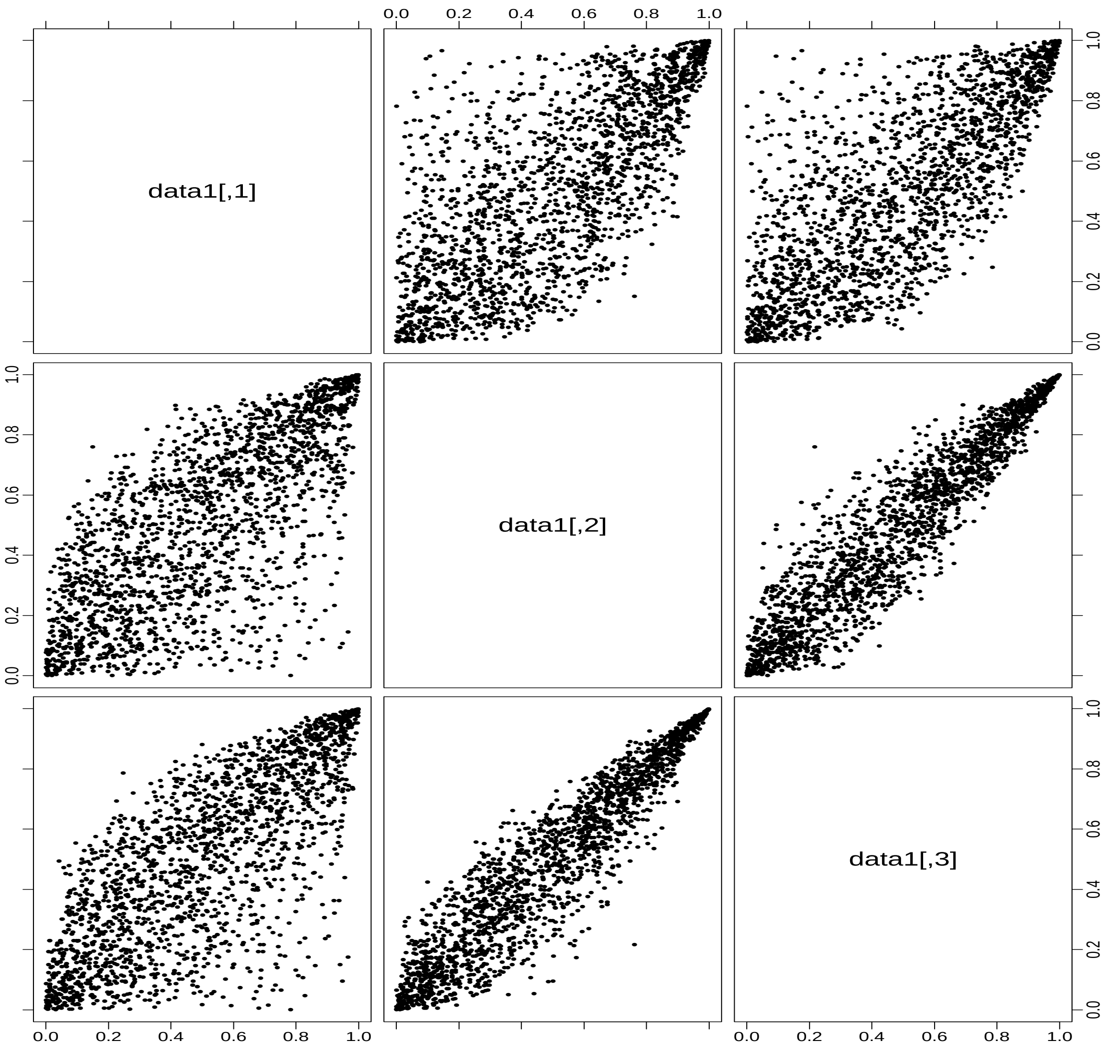
Let $\alpha_1, \ldots, \alpha_d$ be integer and consider a radial part whose Williamson α-transform is given by

$$\psi_\theta(x) = \mathfrak{W}_\alpha F_R(x) = (1+\theta x)_+^{-1/\theta},$$

with $\theta \geq -1/(\alpha - 1)$ and $\alpha = \alpha_1 + \cdots + \alpha_d$.

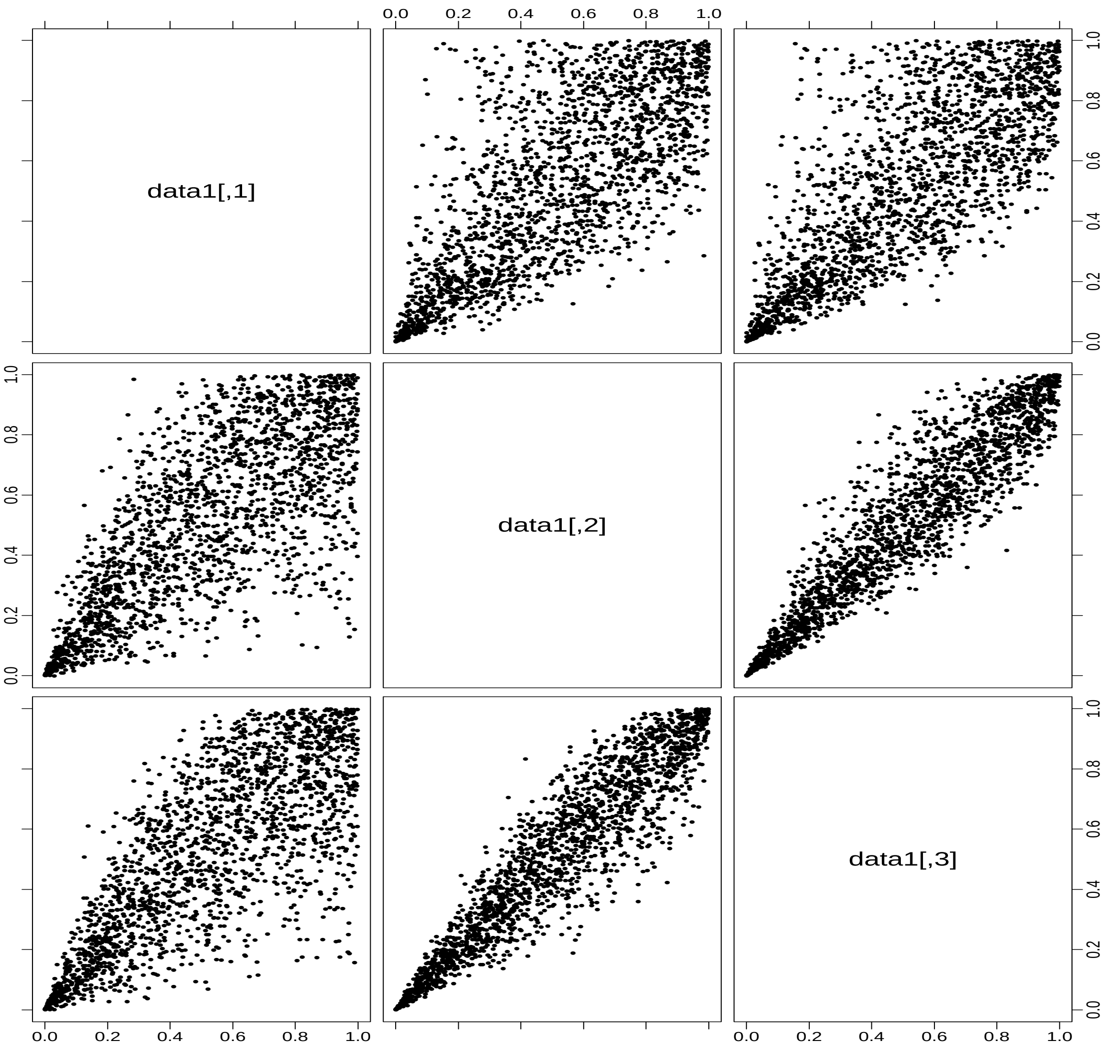
$\tilde{\mathbf{X}} := R\mathbf{S}_\alpha$ has a α-dimensional simplex distribution with a Clayton copula as survival copula and parameter θ. The Liouville random vector $\mathbf{X} = R\boldsymbol{D}_{(\alpha_1,\ldots,\alpha_d)}$ has a survival copula that we call a Clayton–Liouville copula.

Gamma-Liouville



2000 points from a 3-dimensional gamma-Liouville copula with $\theta = 0.6$ and $(\alpha_1, \alpha_2, \alpha_3) = (1, 5, 20)$.

Inverse-Gamma-Liouville



2000 points from a 3-dimensional inverse-gamma-Liouville copula with $\theta = 0.6$ and $(\alpha_1, \alpha_2, \alpha_3) = (1, 5, 20)$.

Bivariate Clayton-Liouville Copula

Let $\alpha_1 = 1$ and $\alpha_2 = 2$ and assume $\theta \geq -1/2$. Let R have distribution function $F_R = \mathfrak{W}_3^{-1}\psi_\theta$ where $\psi_\theta(x) = (1+\theta x)_+^{-1/\theta}$.

The Liouville distribution of $\mathbf{X} = R\mathbf{D}_{(1,2)}$ has survival function

$$\bar{H}(x_1, x_2) = \psi_\theta(x_1 + x_2)\left(1 + \frac{x_2}{1 + \theta(x_1 + x_2)}\right) \quad .$$

The survival margins are

$$\bar{H}_1(x) = \psi_\theta(x)$$

and

$$\bar{H}_2(x) = \psi_\theta(x)\{1 + x/(1 + \theta x)\}.$$

Kendall's tau

Kendall's tau for Liouville copulas can be expressed in terms of the ratio $Y = R/R^*$ between a radial variable R and an independent copy R^*. Consider bivariate case.

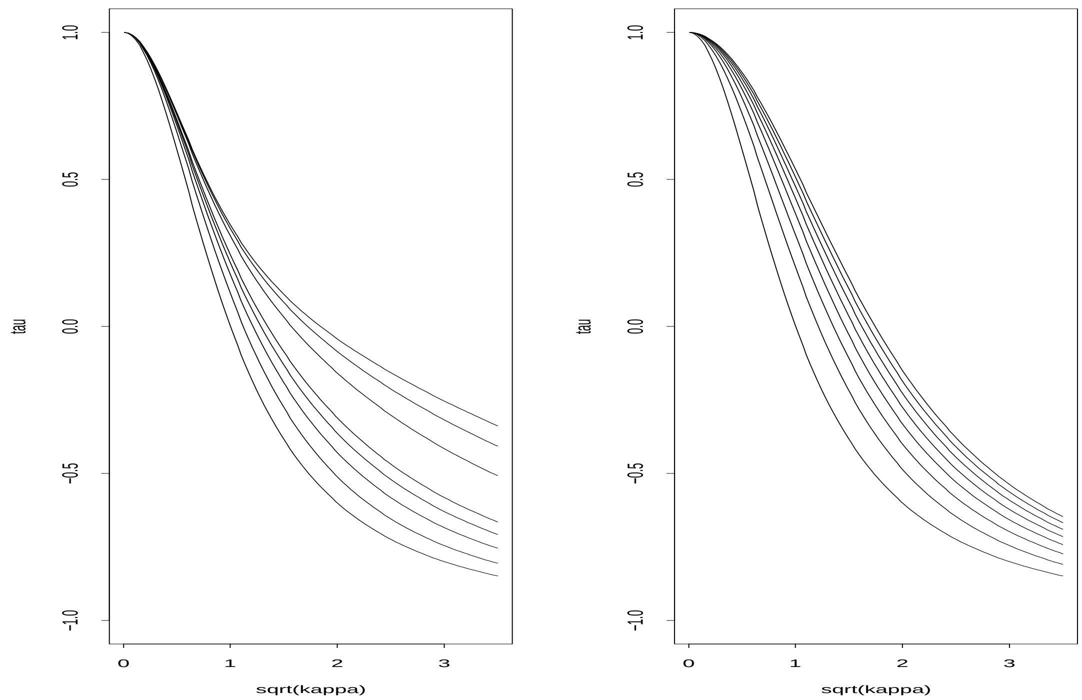
Let C be a bivariate Liouville copula with radial part R and parameters $\alpha_i \in \mathbb{N}$, $i = 1, 2$. Let $\alpha = \alpha_1 + \alpha_2$. $\tau(C)$ is given by

$$4 \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} \frac{\mathrm{B}(\alpha_1 + i, \alpha_2 + j)\Gamma(\alpha)}{\mathrm{B}(\alpha_1, \alpha_2) i! j! \Gamma(\alpha - i - j)} E\left\{ (Y)^{i+j} (1 - Y)_+^{\alpha - i - j - 1} \right\} - 1.$$

Example - Pareto-Liouville copulas. $\tau(C_{\kappa,(\alpha_1,\alpha_2)})$

$$2\kappa \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{\alpha_2 - 1} \frac{\mathrm{B}(\alpha_1 + i, \alpha_2 + j)\Gamma(\alpha)\mathrm{B}(i + j + \kappa, \alpha - i - j)}{\mathrm{B}(\alpha_1, \alpha_2) i! j! \Gamma(\alpha - i - j)} - 1\,.$$

Illustrations



Left plot shows $\tau(C_{\kappa,(1,\alpha)})$ as a function of $\sqrt{\kappa}$ for $\alpha \in \{1, 2, 3, 4, 5, 10, 15, 20\}$; for fixed κ the τ values increase with α.

Right plot shows $\tau(C_{\kappa,(\alpha,\alpha)})$ as a function of $\sqrt{\kappa}$ for $\alpha \in \{1, 2, 3, 4, 5, 6, 7, 8\}$; for fixed κ the τ values increase with α.

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