Shall I Sell or Shall I Wait: Optimal Liquidation under Partial Information with Price Impact

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Introduction

Context. Liquidation of large amount of a given asset typical problem on financial markets. ⇒ large literature on *optimal portfolio execution.*

Two classes of models (according to [Gatheral and Schied, 2013a])

- *Market impact models.* Here one specifies price process for a given execution strategy. Market impact depends on size of the transaction and on the speed of trading. Fundamental price (price when the trader is inactive) usually a diffusion ([Almgren and Chriss, 2001a].)
- *Order book models.* Here one specifies dynamics of limit order book ⇒ endogenous price impact. Again mostly diffusion models.

Our setup: a point process model

We consider a novel market impact model where the asset price S follows a *pure jump process.*

Interesting features:

- Local characteristics (intensity and jump size distribution) of S depend on the liquidation rate $\Rightarrow$ (permanent) price impact.
- Local characteristics depend on an unobservable Markov chain Y $\Rightarrow$ Liquidity and trend of the market are random and not directly observable; Stochastic filtering is used to estimate state of Y

Setting captures *typical features of high frequency data*:

- In reality the bid price is constant between events
- Introducing (unobservable) Y in the price dynamics helps to reproduce clustering in inter-event durations
- Y can be used to model the feedback effect from the trading activity of the rest of the market.

Our contributions

- Reduction to a complete-information setup by stochastic filtering using the reference probability approach
- Resulting state process is a piecewise deterministic Markov process (PDMP) ([Davis, 1993]). We carry out a detailed analysis of optimization problem via PDMP techniques
  - We identify the optimization problem with a discrete-time Markov Decision Model (MDM)
  - We characterize value function via optimality equation for the MDM
  - We use optimality equation to characterize value function as *viscosity solution* of HJB equation and we derive a novel comparison principle for that equation
- Numerical case study for specific example

Related literature

Optimal liquidation. See eg. [Bertsimas and Lo, 1998], [Almgren and Chriss, 2001b], [He and Mamaysky, 2005], [Schied and Schöneborn, 2009], [Bian et al., 2012], [Ankirchner et al., 2015], [Guo and Zervos, 2015], [Schied, 2013], [Cayé and Muhle-Karbe, 2016];

Surveys [Gökay et al., 2011], [Gatheral and Schied, 2013b] or [Cartea et al., 2015]

Other point process models. [Bäuerle and Rieder, 2010], [Bayraktar and Ludkovski, 2011], [Bayraktar and Ludkovski, 2014]. Differences to our work: trading only at jump times of a Poisson process, no partial information, order book models

Portfolio optimization and hedging for pure jump process [Bäuerle and Rieder, 2009], [Kirch and Runggaldier, 2004]

The Model

Throughout we work on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$, for $\mathbf{P}$ is the historical probability measure.

The trader. She wants to liquidate w_0 units of the stock over given period $[0, T]$. She sells at the nonnegative $\mathbb{F}^S$ adapted rate $\boldsymbol{\nu} = (\nu_t)_{0 \le t \le T}$ so that her inventory is given by

$$W_t = w_0 - \int_0^t \nu_u \mathrm{d}u, \quad t \in [0, T]. \tag{1}$$

Denote by S_t bid price of the stock at t. The *revenue* generated by strategy $\boldsymbol{\nu}$ over $[0, t]$ is

$$\int_0^t \nu_u S_u (1 - f(\nu_u)) \mathrm{d}u,$$

where f models *temporary price impact* (nonnegative, increasing)

Bid price dynamics

Bid price satisfies $\mathrm{d}S_t = S_{t-}\mathrm{d}R_t$. Here the *return process* $R := (R_t)_{t\geq 0}$ is a finite activity pure jump process. Denote random measure associated with R by

$$\mu^R(\mathrm{d}t, \mathrm{d}z) := \sum_{u\geq 0:\ \Delta R_u \neq 0} \delta_{\{u,\Delta R_s\}}(\mathrm{d}t, \mathrm{d}z)\,, \tag{2}$$

Compensator of μ^R. Given a strategy $\boldsymbol{\nu}$, the $\mathbb{F}$-compensator $\eta^{\mathbf{P}}$ of μ^R is absolutely continuous and of the form

$$\eta_t^{\mathbf{P}}(\mathrm{d}t, \mathrm{d}z) = \eta^{\mathbf{P}}(t, Y_{t-}, \nu_{t-}, \mathrm{d}z)\mathrm{d}t\,.$$

Here Y_t is the unobservable Markov chain driving the model; Y has generator matrix $Q = (q^{ij})_{i,j=1,\dots,K}$ and state space $\mathcal{E} = \{e_1, e_2, ..., e_K\}$.

Regularity assumptions on η in the paper.

Examples

Return R follows a bivariate point process, i.e. $\Delta R \in \{-\theta, \theta\}$ and

$$\eta^{\mathbf{P}}(t, e_i, \nu, \mathrm{d}z) = \lambda^+(t, e_i, \nu)\delta_{\{\theta\}}(\mathrm{d}z) + \lambda^-(t, e_i, \nu)\delta_{\{-\theta\}}(\mathrm{d}z)\,.$$

Case 1: $\eta^{\mathbf{P}}$ deterministic. Here we assume $\lambda^+ = c^{\mathrm{up}}$ and $\lambda^-(t, \nu) = c^{\mathrm{down}}(1 + a\nu)$, for constants $c^{\mathrm{up}}, c^{\mathrm{down}}, a > 0$. Strength of market impact is governed by a.

Case 2: $\eta^{\mathbf{P}}$ depends on Y. Here we consider two-state Markov chain Y such that e_1 is a 'good' state and e_2 a 'bad' state, i.e. $\lambda^+(e_1) > \lambda^+(e_2), \lambda^-(e_1) < \lambda^-(e_2)$.
Given $c_1^{\mathrm{up}} > c_2^{\mathrm{up}} > 0$, $c_2^{\mathrm{down}} > c_1^{\mathrm{down}} > 0$, $a > 0$ we let for $i = 1, 2$

$$\lambda^+(t, e_i, \nu) = (c_1^{\mathrm{up}}, c_2^{\mathrm{up}})e_i \;\text{ and }\; \lambda^-(t, e_i, \nu) = (1 + a\nu)(c_1^{\mathrm{down}}, c_2^{\mathrm{down}})e_i,$$

Parameter estimation via the EM algorithm (work in progress)

Optimization problem

Admissible strategies. Recall that chain Y is not observable. Hence admissible strategies are $\mathbb{F}^S$-adapted processes $\boldsymbol{\nu}$ with $\nu_t \in [0, \nu^{\max}]$.

Objective of the trader. Define $\tau := \inf\{t \geq 0 : \; W_t \leq 0\} \wedge T$ and denote discount rate by ρ. Trader wants to maximize expected discounted value of proceeds from liquidation,

$$J(\boldsymbol{\nu}) = \mathbb{E}\Big(\int_0^\tau e^{-\rho u} \nu_u \; S_u^{\boldsymbol{\nu}} (1 - f(\nu_u)) \mathrm{d}u \Big) \, . \tag{3}$$

Comments.

- Liquidation value at T can be added
- Trader is assumed risk neutral
- S can (and will) have nonzero drift (even for $\nu_t \equiv 0$)
- Restricted strategy space: only selling, $\nu_t \leq \nu^{\max}$.

On the upper bound on ν_t

Mathematical reasons for $\nu_t \leq \nu^{\max}$

- facilitates application of control theory for PDMPs and construction of model via change of measure
- Viscosity solution characterization of the value function is in general not valid for $\nu^{\max} = \infty$

Financial justification. Exact value of $\nu^{\max}$ does not matter: Denote by $J^{*,m}$ the optimal value in a model with $\nu^{\max} = m$. $\{J^{*,m}\}$ is obviously increasing. We show that

$$\text{for all } m,\ J^{*,m} \leq S_0 w_0 e^{\bar{\eta} T},$$

where $\bar{\eta}$ is the maximal growth rate of S for $\boldsymbol{\nu} \equiv 0$. Hence the sequence $\{J^{*,m}\}$ is Cauchy.

Filtering

Standard approach for optimization under incomplete information: reduce to full information by including filter distribution in the set of state variables (separated problem)

Here unobserved process is K-state Markov chain $\Rightarrow$ filter distribution is characterized by

$$(\pi_t)_{t\geq 0} = (\pi_t^1, \ldots, \pi_t^K)_{t\geq 0} \text{ with } \pi_t^i := \mathbb{E}\left(\mathbf{1}_{\{Y_t=e_i\}} \mid \mathcal{F}_t^S\right).$$

We derive an SDE system for (π_t) (Kushner Stratonovich equation) using *reference probability approach* by working under equivalent measure $\mathbf{Q}$ such that μ^R is Poissonian random measure under $\mathbf{Q}$; existing literature [Frey and Schmidt, 2012, Ceci and Colaneri, 2012] mostly based on innovations approach.

The Kushner-Stratonovich equation

Proposition. The process $(\pi_t^1, \dots, \pi_t^K)_{t \geq 0}$ solves the following SDE system:

$$\mathrm{d}\pi_t^i = \sum_{j=1}^{K} q^{ji} \pi_t^j \mathrm{d}t + \int_{\mathbb{R}} \pi_{t-}^i u^i(t, \nu_{t-}, \pi_{t-}, z)(\mu^R(\mathrm{d}t, \mathrm{d}z) - \pi_{t-}(\eta^{\mathbf{P}}(\mathrm{d}z))\mathrm{d}t),$$

where $u^i(t, \nu, \pi, z) := \dfrac{\mathrm{d}\eta^{\mathbf{P}}(t, e_i, \nu)/\mathrm{d}\eta_t^{\mathbf{Q}}(z)}{\sum_{j=1}^{K} \pi^j \mathrm{d}\eta^{\mathbf{P}}(t, e_j, \nu)/\mathrm{d}\eta_t^{\mathbf{Q}}(z)} - 1.$

Comments

- K-dimensional SDE system
- Independent of specific choice of reference probability Q
- Deterministic behaviour (ODE) between jumps, updating at jump times T_n of R

KS equation for two-state chain

Consider case where R is a bivariate process and Y a 2-state chain. Define point processes

$$N_t^{\text{up}} = \sum_{T_n \le t} 1_{\{\Delta R_{T_n} = \theta\}}, \quad N_t^{\text{down}} = \sum_{T_n \le t} 1_{\{\Delta R_{T_n} = -\theta\}}$$

Then we get following SDE for π_t^1 (note $\pi_1^2 = 1 - \pi_t^1$)

$$\begin{aligned} d\pi_t^1 &= (q^{11}\pi_t^1 + q^{21}\pi_t^2)dt \\ &+ \pi_{t-}^1 \Big(\frac{c_1^{\text{up}}}{\pi_{t-}^1 c_1^{\text{up}} + \pi_{t-}^2 c_2^{\text{up}}} - 1 \Big) d\Big(N_t^{\text{up}} - (\pi_{t-}^1 c_1^{\text{up}} + \pi_{t-}^2 c_2^{\text{up}})dt \Big) \\ &+ \pi_{t-}^1 \Big(\frac{c_1^{\text{down}}}{\pi_{t-}^1 c_1^{\text{down}} + \pi_{t-}^2 c_2^{\text{down}}} - 1 \Big) d\Big(N_t^{\text{down}} - (\pi_{t-}^1 c_1^{\text{down}} + \pi_{t-}^2 c_2^{\text{down}})(1 + a\nu_t)d \end{aligned}$$

Comments. Deterministic behaviour (ODE) between jumps, updating at jump times T_n of R; general case in the paper.

Optimization: Overview of theoretical results

- State process of the optimization problem is $X := (W, S, \pi)$; state space denoted $\mathcal{X}$. This is a *PDMP* as in [Davis, 1993]
- At each jump time T_n we choose a liquidation strategy to be followed up to $T_{n+1} \wedge \tau$. $\Rightarrow$ optimization problem can be identified with discrete-time Markov decision model (MDM) for $L_n = X_{T_n}$, $n \in \mathbb{N}$ ([Bäuerle and Rieder, 2011].)
- MDM-theory $\Rightarrow$ under regularity conditions value function V is characterized by a fixed-point equation and there is an optimal relaxed control.
- Fixed point equation $\Rightarrow$ V is value function of a deterministic control problem. Can be used to show that V is unique viscosity solution of the 'naive' HJB equation corresponding to the Markov process X ([Davis and Farid, 1999]) and to derive a comparison principle.

State process as a PDMP

A controlled PDMP X is a jump process that follows between jumps an ODE $\frac{d}{dt}X_t = g(X_t, \nu_t)$ and that jumps at random times; jumps governed by $Q_X(\cdot \mid x, \nu)$.

Here: $g^1(\widetilde{x}, \nu) = -\nu, \quad g^2(\widetilde{x}, \nu) = 0$, and for $k = 1, \dots, K$,

$$g^{k+2}(\widetilde{x}, \nu) = \sum_{j=1}^{K} q^{jk}\pi^j - \pi^k \sum_{j=1}^{K} \pi^j \int_{\mathbb{R}} u^k(t, \nu, \pi, z)\eta^{\mathbf{P}}(t, e_j, \nu, \mathrm{d}z);$$

Transition kernel: $Q_X f(x, \nu) := \frac{1}{\lambda(x,\nu)} \bar{Q}_X f(x, \nu)$ with

$$\bar{Q}_X f(x, \nu) = \sum_{j=1}^{K} \pi^j \int_{\mathbb{R}} f\left(t, w, s(1+z), \pi^1(1 + u^1(t, \nu, \pi, z)), \dots, \pi^K(1 + u^K(t, \nu, \pi, z))\right) \eta^{\mathbf{P}}(t, e_j, \nu, \mathrm{d}z).$$

Optimal liquidation as control problem for PDMPs

In control of PDMPs one uses *open-loop controls*: trader chooses at $T_n < \tau$ a liquidation strategy $\nu^n = \nu^n(t, T_n, X_{T_n})$ to be followed up to $T_{n+1} \wedge \tau$.

Strategies. Denote by $\mathcal{A}$ the set of all $\alpha : [0, T] \to [0, \nu^{\max}]$. An *admissible liquidation strategy* is a sequence of functions $\{\nu^n\}_{n\in\mathbb{N}} : \widetilde{\mathcal{X}} \to \mathcal{A}$; the liquidation rate at time t is given by

$$\nu_t = \sum_{n\in\mathbb{N}} \mathbf{1}_{(T_n\wedge\tau, T_{n+1}\wedge\tau]}(t)\nu^n \left(t - T_n; T_n, X_{T_n}\right). \qquad (4)$$

Proposition. Under our regularity assumptions there exists for every admissible $\{\nu^n\}_{n\in\mathbb{N}}$ and every initial value x a unique PDMP with characteristics g, λ, and Q_X.

The PDMP problem ctd

Denote by $\mathbf{P}_{(t,x)}^{\{\nu^n\}}$ law of X provided that $X_t = x \in \mathcal{X}$ and that the strategy $\{\nu^n\}_{n\in\mathbb{N}}$ is used. The associated reward function is

$$V(t,x,\{\nu^n\}) = \mathbb{E}_{(t,x)}^{\{\nu^n\}}\Big(\int_t^\tau e^{-\rho(u-t)}\nu_u S_u(1-f(\nu_u))\mathrm{d}u\Big),$$

and the value function of the liquidation problem under partial information is

$$V(t,x) = \sup\left\{V(t,x,\{\nu^n\}) : \{\nu^n\}_{n\in\mathbb{N}} \text{ admissible liquidation strategy}\right\}.$$

Remark. This optimization problem is discrete: the strategy is chosen at T_n and the system evolves in a deterministic way up to next jump.

Optimality equation

Theorem. 1. Under technical conditions the value function V is continuous and satisfies the optimality equation

$$V(t,x) = \sup_{\alpha \in \mathcal{A}} \Big\{ \int_0^{\tau^\varphi} e^{-\rho u} e^{-\Lambda_u^\alpha(t,x)} \big\{ s\alpha_u(1 - f(\alpha_u)) + \bar{Q}V(\varphi_u^\alpha(t,x), \alpha_u) \big\} \mathrm{d}u \Big\}$$

2. An optimal strategy exists in the space of all *relaxed* or *randomized* controls.

Here φ^α is the flow of the vector field g and τ^φ is the first exit time of the system from the state space (at $t = T$ or at $w = 0$).

The HJB equation

The HJB equation for the optimal liquidation problem is

$$0 = \frac{\partial V'}{\partial t}(t, w, \pi) + \sup\left\{H(\nu, t, w, \pi, V', \nabla V') \colon \nu \in [0, \nu^{\max}]\right\},$$

where H is given by a complicated expression involving the generator of X.

Theorem. V is the unique continuous viscosity solution of HJB equation with appropriate boundary condition. Moreover a comparison principle holds for that equation

Comments

- Proof uses optimality equation and results from [Barles, 1994] on deterministic control problems.
- Different boundary conditions, $V = 0$ only on active boundary
- We have counterexamples that show that V is non-smooth (for $\nu^{\max} < \infty$) and a strict supersolution ifwe consider $\nu^{\max} \to \infty$

Case studies: Overview

We study the example where $\eta^{\mathbf{P}}$ depends on 2-state Markov chain Y and is of the form

$$\eta^{\mathbf{P}}(e_i, \nu, dz) = (1 + a\nu)(c_1^{\text{down}}, c_2^{\text{down}})e_i\delta_{-\theta}(dz) + (c_1^{\text{up}}, c_2^{\text{up}})e_i\delta_{\theta}(dz)\,.$$

Finite differences are used to solve HJB equation numerically and to compute approximately optimal strategy; convergence follows by applying the [Barles and Souganidis, 1991]-approach.

Parameters

w_0	$\nu^{\max}$	T	θ	a	c_f	ς	q^{12}	q^{21}
6000	18000	1 day	0.001	4×10^{-6}	5×10^{-5}	0.6	4	4

Liquidation rate for moderate permanent price impact

Throughout $c_1^{\text{up}} = c_2^{\text{down}} = 1000$, $c_2^{\text{up}} = c_1^{\text{down}} = 950$.

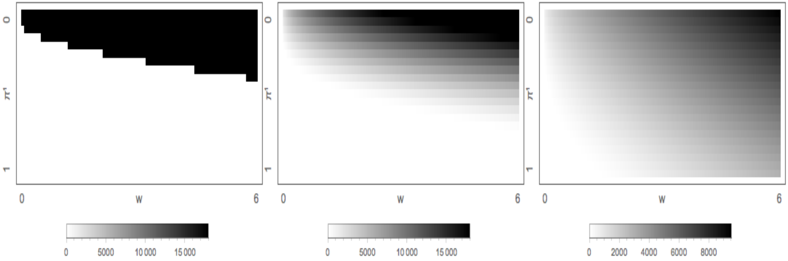

Figure: Liquidation policy for $a = 410^{-6}$, $c_f = 5 \times 10^{-11}$ (left), $c_f = 10^{-5}$ (middle), and $c_f = 5 \times 10^{-5}$ (right) and $t = 0$ Note the different scale in the graphs.

Liquidation rate for large permanent price impact

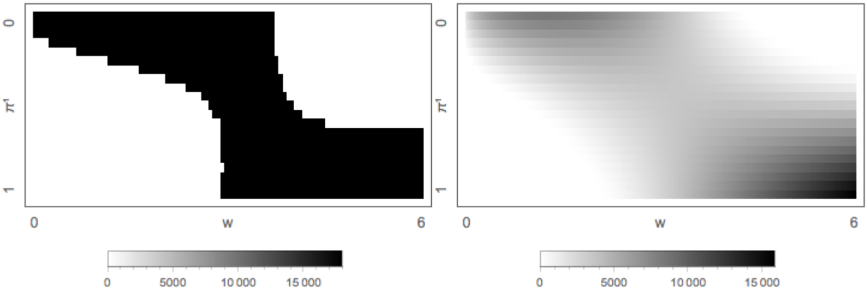

Figure: Liquidation policy for $c_f = 5 \times 10^{-11}$ (left) and $c_f = 10^{-5}$ (right) for $a = 10^{-4}$ and $t = 0$

Interpretation

- Temporary price impact 'smoothes out' trading behavior
- Qualitative behaviour of $\boldsymbol{\nu}$ determined by an interplay of two effects.
  - *price anticipation effect* $\Rightarrow$ 'wait' in the good state where prices are increasing on average and 'sell' in the bad state
  - *liquidity effect*. $\Rightarrow$ sell in the good state and wait in the bad state to reduce the permanent price impact.
- For small a price anticipation effect dominates. For large a and large w liquidity effect dominates
- *Gambling region* for w large and π^1 small

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