Structural Credit Risk Models under Incomplete Information and the Pricing of Contingent Convertibles.

Rüdiger Frey (Research assistence by Lars Rösler and Dan Lu)

WU Vienna

ruediger.frey@wu.ac.at
http://statmath.wu.ac.at/~frey

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Structural credit risk models

Structural models. In this model class default occurs if asset value V of a given firm falls below some threshold K, interpreted as liability. This leads to

$$\tau := \inf\{t \geq 0 \colon V_t \leq K\}.$$

Typically V is modelled as a diffusion $\Rightarrow$ τ is *predictable*.

Problems.

- The model gives unrealistically low short-term credit spreads.
- Asset value V is hard to observe precisely for investors on capital markets

$\Rightarrow$ Consider models where asset value is not perfectly observable.

Notation. We represent information available to the market by the filtration $\mathbb{F}^{\mathbb{M}}$

Structural models with Incomplete information.

[Duffie and Lando, 2001]. V is geometric Brownian motion. Market observes at discrete time points t_n a signal $Z_n = \ln V_{t_n} + \varepsilon_n$ (termed *noisy accounting information*) and moreover default.

Results.

- Under incomplete information, τ admits an *intensity* (an $\mathbb{F}^{\mathbb{M}}$-adapted process λ_t so that $1_{\{\tau \leq t\}} - \int_0^{t \wedge \tau} \lambda_s ds$ is an $\mathbb{F}^{\mathbb{M}}$ martingale).
- Characterization of intensity: $\lambda_t = \frac{1}{2}\sigma^2 K^2 \frac{\partial}{\partial v}\pi(t, K)$, $\pi(t, \cdot)$ the density of $\pi_{V_t|\mathcal{F}_t^{\mathbb{M}}}$.
- Link between structural and firm-value models via information

Structural models with Incomplete information ctd.

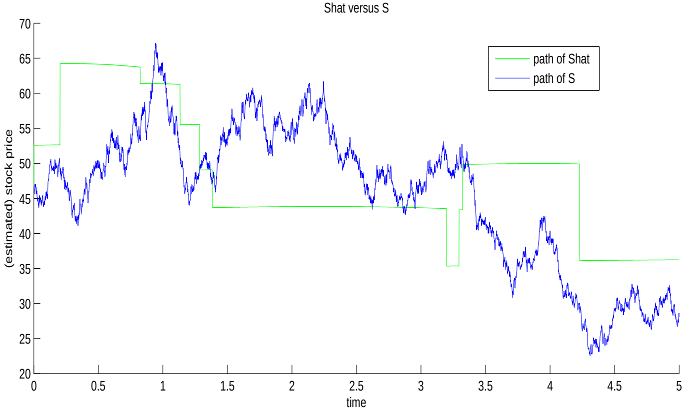
Shortcomings of Duffie Lando

- Noisy accounting information implies unrealistic asset price dynamics (no price volatility)
- No analytic results on asset price dynamics under incomplete information

[Frey and Schmidt, 2009]. Similar setup as in Duffie Lando.

- Dividends are introduced to study pricing of *equity* and debt.
- A systematic link between pricing corporate securities and filtering: Prices are first computed under full information using Markov property, and then projected on market filtration $\mathbb{F}^{\mathbb{M}}$.
- The ensuing filtering problem is studied via a simple Markov-chain approximation

A price path under David-Lando information structure



Our Extensions of Duffie-Lando / Frey-Schmidt

1. Asset-value observation is modelled by a *continuous-time process* $Z_t = \int_0^t a(V_s)ds + W_t$.
   - More in line with continuous-time filtering literature
   - Security price processes follow standard (jump-)diffusion processes.
2. Systematic analysis of the ensuing filtering problem
   - We extend the Duffie-Lando characterization of default intensities to our setting.
   - We derive price dynamics for corporate securities in the market filtration $\mathbb{F}^{\mathbb{M}}$
3. Applications (work in progress)
   - Pricing derivatives and model calibration
   - Analysis of contingent convertibles (Cocos)

Some literature

- Structural credit risk models: [Duffie and Lando, 2001], [Giesecke, 2004], [Jarrow and Protter, 2004], [Coculescu et al., 2008], [Frey and Schmidt, 2009], [Cetin, 2012],. . .
- Reduced-form models: [Collin-Dufresne et al., 2003], [Schönbucher, 2004], [Duffie et al., 2009] (empirical focus), [Frey and Runggaldier, 2010], [Frey and Schmidt, 2012]

Overview

The Model

We use the following setup

- We work on $(\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t \geqslant 0}, Q)$; all processes are $\mathbb{G}$-adapted; Q is the martingale measure used for pricing
- Consider a company with asset value process $V = (V_t)_{t \geqslant 0}$ and default time $\tau = \inf\{t \geqslant 0 \colon V_t \leqslant K\}$.
- Company pays dividend d_n at date T_n, $n \geq 0$. T_n is nth jump time of a Poisson process with intensity λ^D and $d_n = \delta_n V_{T_n}$ for a iid sequence $(\delta_n)_{n=1,2,\dots}$, independent of V, with mean $\bar{\delta}$.
- $D_t = \sum_{\{n \colon T_n \leqslant t\}} d_n$ is the cumulative dividend process and $\varphi(y, V_{T_n})$ denotes conditional density of d_n given V.
- Under Q, V is geometric Brownian motion, $dV_t = (r - \lambda^D \bar{\delta}) V_t dt + \sigma V_t dB_t$. Moreover V_0 has Lebesgue density $\pi_0(v)$ with $\pi_0(K) = 0$.

Market Information

The market uses the following pieces of information to price securities

- *Default information.* Market observes default state $N_t = 1_{\{\tau \leqslant t\}}$ of the firm. We denote the default history by $\mathbb{F}^N = (\mathcal{F}_t^N)_{t \geq 0}$.
- *Dividend information.* Market observes D_t with associated filtration $\mathbb{F}^D = (\mathcal{F}_t^D)_{t \geq 0}$.
- *Noisy asset observation.* Market observes a process Z with $Z_t = \int_0^t a(V_s)ds + W_t$. W is an l-dim $\mathbb{G}$-Brownian motion independent of B, and a is a smooth and bounded with $a(K) = 0$. Z is an abstract process modelling information contained in security prices.
- *Market information* is $\mathbb{F}^{\mathbb{M}} = \mathbb{F}^N \vee \mathbb{F}^Z \vee \mathbb{F}^D$; $\mathbb{F}^Z \vee \mathbb{F}^D$ will be termed *background filtration*.

Pricing basic corporate securities and filtering

Risk-neutral pricing wrt $\mathbb{F}^M \Rightarrow$ price of security with cash flow stream $(H_t)_{0 \leq t \leq T}$ is

$$\Pi_t^H = E^Q\Big(\int_t^T e^{-r(s-t)} dH_s \mid \mathcal{F}_t^M \Big) , \quad t \leq T. \tag{1}$$

Consider a basic corporate securities with $\mathbb{F}^N \vee \mathbb{F}^D$-adapted cash flow such as a bond, a CDS or the equity value of the firm. Iterated conditional expectations gives

$$1_{\{\tau > t\}} \Pi_t^H = E^Q\Big(E^Q\big(1_{\{\tau > t\}} \int_t^T e^{-r(s-t)} dH_s \mid \mathcal{G}_t\big) \mid \mathcal{F}_t^M \Big).$$

Markov property of $V \Rightarrow$ inner conditional expectation is typically of the form $1_{\{\tau > t\}} h(t, V_t)$, (the *full-information value*). Hence

$$1_{\{\tau > t\}} \Pi_t^H = 1_{\{\tau > t\}} E^Q\big(h(t, V_t) \mid \mathcal{F}_t^M\big). \tag{2}$$

Evaluation of this expression is a *nonlinear filtering problem*

Equity Pricing

The *equity value* is defined as value of dividend payments up to default time time τ. Full-information value $S(v)$ satisfies

$$S(v) = E^Q\Big(\int_0^\tau e^{-rs} dD_s \mid V_0 = v\Big) = E^Q\Big(\int_0^\tau e^{-rs}\bar{\delta} V_s \lambda^D \, ds \mid V_0 = v\Big). \tag{3}$$

Hence S is time-independent and solves $\mathcal{L}_V S(v) + \bar{\delta}\lambda^D v = rS(v)$, with boundary condition $S(K) = 0$. For $K = 0$ (and hence $\tau = \infty$) we get $S(v) = v$. For $K > 0$ one has

Proposition. The full information value (3) of the firms equity is given by $S(v) = v - K(\frac{v}{K})^{\alpha^*}$, where α^* is the negative root of the equation $(r - \lambda^D\bar{\delta})\alpha + \frac{1}{2}\sigma^2\alpha(\alpha - 1) - r = 0$.

Remark. Full-information value for *debt securities* can be computed via first passage time of Brownian motion with drift

The filtering problem

Recall that we want to compute recursively the conditional expectation

$$1_{\{\tau>t\}}E^Q\big(f(V_t)\mid\mathcal{F}_t^M\big),\ t\le T,\, f\in L^\infty\big([K,\infty)\big)\,. \qquad (4)$$

This is a *nonstandard filtering problem*, due to inclusion of default history $\mathbb{F}^N$ in the observation filtration:

- Under full information $\mathbb{G}$ τ is a predictable stopping time and does not admit an intensity.
- In standard filtering theory with point process information on the other hand N is assumed to have a $\mathbb{G}$ intensity.

Basic idea. Reduce (4) to a filtering problem wrt background filtration $\mathbb{F}^Z\vee\mathbb{F}^D$ via Dellacherie formula.

Reduction to background filtration

Using the Dellacherie formula, we get

$$1_{\{\tau>t\}} E^Q\big(f(V_t) \mid \mathcal{F}_t^M\big) = 1_{\{\tau>t\}} \frac{E^Q\big(f(V_t) 1_{\{\tau>t\}} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{Q\big(\tau > t \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)} .$$

Denote by V^τ the process $V_t^\tau = V_{t\wedge\tau}$. By definition of τ we have $\{\tau > t\} = \{V_t^\tau > K\}$; moreover, $V_t = V_t^\tau$ for $t \leq \tau$. Hence we get

$$1_{\{\tau>t\}} E^Q\big(f(V_t) \mid \mathcal{F}_t^M\big) = 1_{\{\tau>t\}} \frac{E^Q\big(f(V_t^\tau) 1_{\{V_t^\tau>K\}} \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)}{Q\big(V_t^\tau > K \mid \mathcal{F}_t^Z \vee \mathcal{F}_t^D\big)} . \tag{5}$$

Remark. (5) is a filtering problem with standard diffusion and point process information. On the other hand new signal process $X := V^\tau$ with state space S^X is a *stopped diffusion process.*

Measure transform

Start with independent processes (X, Z) on $(\Omega, \mathcal{G}, Q^*)$ such that X is a stopped geometric Brownian motion and Z is a standard BM. (We largely ignore dividend payments.)
Consider the density martingale $L_t = \frac{dQ}{dQ^*}_{|\mathcal{F}_t}$ with

$$L_t = \exp\Big(\int_0^t a(X_s)^\top dZ_s - \frac{1}{2} \int_0^t |a(X_s)|^2 \, ds \Big) . \qquad (6)$$

Girsanov $\Rightarrow$ the pair (X, Z) has the right law under Q and we have

$$E^Q\big(f(X_t) \mid \mathcal{F}_t^Z\big) = \frac{E^{Q^*}\big(f(X_t)L_t \mid \mathcal{F}_t^Z\big)}{E^{Q^*}\big(L_t \mid \mathcal{F}_t^Z\big)} =: \frac{\Sigma_t f}{\Sigma_t 1} . \qquad (7)$$

SPDE for the density of Σ_t

Suppose that Σ_t is absolutely continuous with density u. u is related to the SPDE

$$du(t) = \mathcal{L}^* u(t)dt + a^\top u(t)dZ_t, \quad u(0) = \pi_0.$$

Formal interpretation. Denote by (f, g) the scalar product on $L^2(S^X)$. Then u is an $\mathbb{F}^Z$ adapted continuous process with values in the Sobolev space $H_1^0(S^X) \cap H^2(S^X)$, and one has for $v \in L^2$

$$\big(u(t), v\big) = \big(u(0), v\big) + \int_0^t (\mathcal{L}^* u(s), v) ds + \int_0^t \big(a^\top u(s), v\big) dZ_s. \quad (8)$$

Theorem. ([Pardoux, 1978]) There is a unique solution u of equation (8). Moreover, for $f \in L^\infty(S^X)$,

$$\Sigma_t f = \big(u(t), f\big) + \nu_K(t) f(K) \text{ where } \nu_K(t) = \int_0^t \frac{1}{2}\sigma^2 K^2 \frac{du}{dx}(s, K) ds\,. \quad (9)$$

Comments and Implications

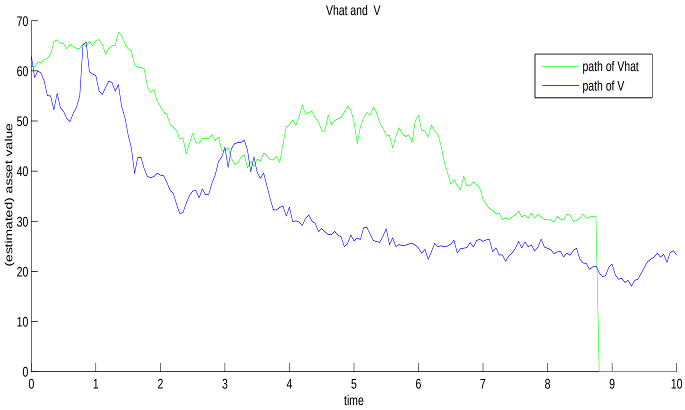
Comments.

- The measure Σ_t consists of two parts: an absolutely continuous part with density $u(t)$ and a point mass $\nu_K(t)$ at the boundary K.
- Boundary term drives form of default intensities.
- We give a simplified presentation here: the result has been shown only for bounded domain $[K, N)$ (see paper).
- Numerical solution via Galerkin approximation of (8)

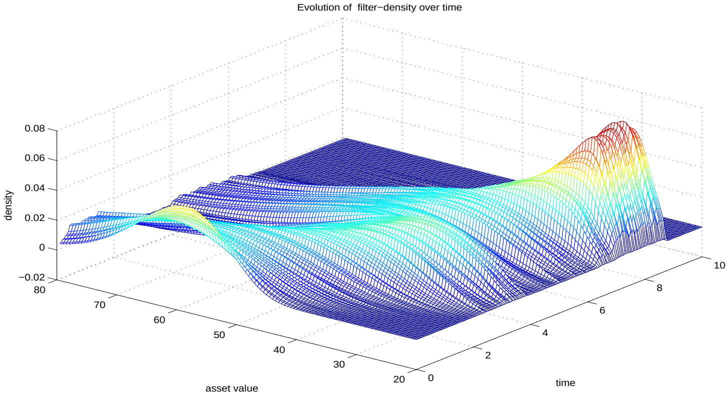
Corollary. We get for the original filtering problem

$$E(f(V_t) \mid \mathcal{F}_t^M) = \big(\tilde{\pi}(t), f\big) \text{ with } \tilde{\pi}(t,x) = \frac{u(t,x)}{\big(u(t),1\big)}. \qquad (10)$$

A Path of V and $\hat{V}$



Corresponding Filter Density



Default Intensity

Theorem. The $\mathbb{F}^M$ compensator of N_t is given by $(\Lambda_{t\wedge\tau})_{t\geq 0}$ where

$$\Lambda_t = \int_0^t \lambda_s ds \quad \text{with } \lambda_t = \frac{1}{2}\sigma^2 K^2 \frac{d\pi}{dx}(t,K)\,. \tag{11}$$

Here $\pi(t,x)$ is conditional density of X_t given $\mathcal{F}_t^M$.

- This extends earlier results of [Duffie and Lando, 2001] and [Frey and Schmidt, 2009] to the case where information arrives continuously.
- An alternative characterization of the compensator of N has recently been given by [Cetin, 2012].

Filter equations

Theorem. For $f \in C^{1,2}([0,T] \times S^X)$ the projection $\widehat{f}_t = E(f(t, X_t) \mid \mathcal{F}_t^M)$ has dynamics

$$\widehat{f}_t = \widehat{f}_0 + \int_0^t (\widehat{\frac{df}{dt}})_s + (1 - N_{s-})(\widehat{\mathcal{L}_X f})_s \, ds + \int_0^{t \wedge \tau} (\widehat{fa^\top})_s - \widehat{f}_s \widehat{a^\top}_s \, dM_s^Z$$
$$+ \int_0^{t \wedge \tau} (f(s, K) - \widehat{f}_{s-}) \, d(N_s - \lambda_s ds)$$
$$+ \int_0^{t \wedge \tau} \int_{\mathbb{R}^+} \frac{(\widehat{f \varphi_d(y)})_{s-} - \widehat{f}_{s-} (\widehat{\varphi_d(y)})_{s-}}{\widehat{f}_{s-} (\widehat{\varphi_d(y)})_{s-}} \, (\mu^D - \gamma^D)(dy, ds) \, .$$

Here $M^Z = Z_t - \int_0^t \widehat{a}_s ds$ is a $\mathbb{F}^M$ Brownian motion and $(\mu^D - \gamma^D)(dy, ds)$ is the $\mathbb{F}^M$-compensated random measure associated with the dividends.

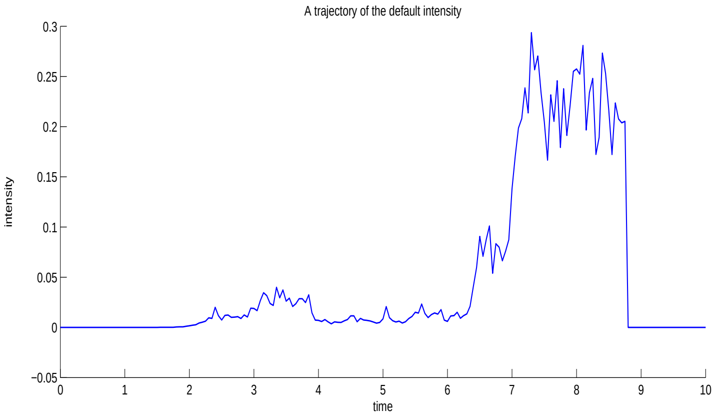
Stock price dynamics

Using the filter equations it is straightforward to compute the semimartingale decomposition of the stock price $\widehat{S}_t$. We get

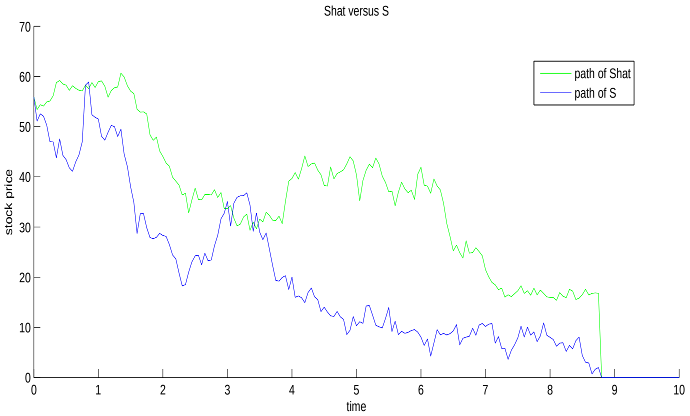
$$d\widehat{S}_t = (1 - N_{t-})(r\widehat{S}_t - \lambda^D \bar{\delta} \widehat{V}_t)dt + (1 - N_{t-})\big((\widehat{Sa^\top})_t - \widehat{S}_t \widehat{a^\top}_t\big)\, dM_t^Z$$
$$- (1 - N_{t-})\widehat{S}_{t-}\, d(N_t - \lambda_t dt) + \text{integral wrt}(\mu^D - \gamma^D)(dy, dt)$$

- Similar formulas can be obtained for debt securities.
- Note that stock-price dynamics can be quite wild even if asset price dynamics follow standard geometric Brownian motion.

A path of the default intensity



A typical stock price trajectory



Derivative Pricing

Basic corporate securities ($\mathbb{F}^D \vee \mathbb{F}^N$-adapted payoff H).
Here $\Pi_t^H = 1_{\{\tau > t\}}(\pi(t), h)$. Note that price is *linear* in $\pi(t)$

Options. Here payoff depends on price path of basic corporate securities; examples include equity options, convertibles, . . . Price is given by some function $C(\cdot)$ of the current density $\pi(t)$, but $C(\cdot)$ needs to be computed with Monte Carlo.

Simulation of price trajectories.

- Crucial point: generate a numerical approximation to the solution $u(t)$ of the Zakai SPDE.
- This is best done via Galerkin approximation: one considers an approximation of the form $u^m(t) = \sum_{i=1}^m \Psi_i(t) e_i$, $e_i \in H_1^0(S^X)$ a suitable sequence of basis functions and one derives an SDE system for $\Psi(t)$ that can be solved numerically. More details in [Frey et al., 2013]

Contingent Convertibles (CoCo

Cocos.

- A CoCo is a corporate bond that is automatically triggered once the issuer enters into financial distress.
- At trigger the CoCo is either converted into equity or into a (low) cash-payment Purpose: strengthen the equity capital of the issuer.

Payoff description. Cocos are characterized by maturity date T; coupons c_j, due at t_j; conversion time $\theta > 0$ and payment at conversion.

- Payoff stream with cash payment $C < 1$ at conversion:

$$\sum_{j=1}^{m} e^{-rt_j} c_j 1_{\{\theta > t_j\}} + e^{-rT} 1_{\{\theta > T\}} + e^{-r\theta} C 1_{\{\theta \le T\}}\,;$$

- Conversion into a fraction γ of equity $\Rightarrow$ last term becomes $e^{-r\theta}\gamma S_\theta 1_{\{\theta \le T\}}$.

Modelling the conversion time θ

CoCos in practice have an accounting trigger based on capital adequacy ratios. Difficult to model directly $\Rightarrow$ use approximations.

Asset trigger. Here $\theta^{\text{ass}} = \inf\{t \geq 0 \colon V_t \leq K^{\text{CoCo}}\}$.

+ conversion always before default $\Rightarrow$ protection of debtholders
– asset value and hence θ^{ass} not publicly observable, hence monitoring difficult

Equity trigger. Here conversion if stock price hits threshold S^{CoCo}, that is $\theta^{\text{eq}} = \inf\{t \geq 0 \colon S_t \leq S^{\text{CoCo}}\}$. A natural choice in our model is $S^{\text{CoCo}} = S(K^{\text{CoCo}})$.

+ Stock price is publicly available.
– It may happen that $\theta^{\text{eq}} = \tau$, that is conversion takes place too late

Comparison of the two trigger mechanisms

Under full information it holds that $\theta^{\text{eq}} = \theta^{\text{ass}}$; under incomplete information the two mechanisms differ.

- asset trigger contains extra information whereas θ^{eq} is $\mathbb{F}^{\mathbb{M}}$ stopping time.
- Late conversion ($\theta^{\text{eq}} = \tau$) most likely if K^{CoCo} close to the default boundary and if asset information is quite noisy
- In our experiments equity trigger led to higher prices for the Coco.

Cocos: numerical experiments

Information	Coco price θ^{eq}	θ^{ass}	$Q(\tau < T)$	$Q(\theta^{\text{eq}} < T)$	$Q(\theta^{\text{eq}} = \tau)$
noisy	1.1	0.828	0.2876	0.2894	0.2861
precise	0.882	0.828	0.2876	0.6099	0.0328

Other parameters. $T = 10$; coupon 5% biannually; $K = 20$; $K^{\text{CoCo}} = 30$; mean initial asset price 40.

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