Structural Credit Risk Models under Incomplete Information and the Pricing of Contingent Convertibles.

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Structural credit risk models

structural models. In this model class default occurs if asset value $V$ of a given firm falls below some threshold $K$, interpreted as liability. This leads to

$$\tau := \inf\{ t \geq 0 : V_t \leq K \}.$$

Typically $V$ is modelled as a diffusion $\Rightarrow \tau$ is predictable.

Problems.

- The model gives unrealistically low short-term credit spreads.
- Asset value $V$ is hard to observe precisely for investors on capital markets

$\Rightarrow$ Consider models where asset value is not perfectly observable.

Notation. We represent information available to the market by the filtration $\mathcal{F}_M^t$. 
Structural models with Incomplete information.

[Duffie and Lando, 2001]. $V$ is geometric Brownian motion. Market observes at discrete time points $t_n$ a signal $Z_n = \ln V_{t_n} + \varepsilon_n$ (termed noisy accounting information) and moreover default.

Results.

- Under incomplete information, $\tau$ admits an intensity (an $\mathbb{F}^M$-adapted process $\lambda_t$ so that $1\{\tau \leq t\} - \int_0^{t \wedge \tau} \lambda_s ds$ is an $\mathbb{F}^M$ martingale).
- Characterization of intensity: $\lambda_t = \frac{1}{2} \sigma^2 K^2 \frac{\partial}{\partial \nu} \pi(t, K), \pi(t, \cdot)$ the density of $\pi_{V_t}|\mathcal{F}^M_t$.
- Link between structural and firm-value models via information
Structural models with Incomplete information ctd.

Shortcomings of Duffie Lando

- Noisy accounting information implies unrealistic asset price dynamics (no price volatility)
- No analytic results on asset price dynamics under incomplete information

[Frey and Schmidt, 2009]. Similar setup as in Duffie Lando.

- Dividends are introduced to study pricing of equity and debt.
- A systematic link between pricing corporate securities and filtering: Prices are first computed under full information using Markov property, and then projected on market filtration $F_M$.
- The ensuing filtering problem is studied via a simple Markov-chain approximation
A price path under David-Lando information structure
Our Extensions of Duffie-Lando / Frey-Schmidt

1. Asset-value observation is modelled by a *continuous-time* process \( Z_t = \int_0^t a(V_s)ds + W_t \).
   - More in line with continuous-time filtering literature
   - Security price processes follow standard (jump-)diffusion processes.

2. Systematic analysis of the ensuing filtering problem
   - We extend the Duffie-Lando characterization of default intensities to our setting.
   - We derive price dynamics for corporate securities in the market filtration \( \mathbb{F}^M \).

3. Applications (work in progress)
   - Pricing derivatives and model calibration
   - Analysis of contingent convertibles (Cocos)
Some literature

- Structural credit risk models: [Duffie and Lando, 2001], [Giesecke, 2004], [Jarrow and Protter, 2004], [Coculescu et al., 2008], [Frey and Schmidt, 2009], [Cetin, 2012],…
- Reduced-form models: [Collin-Dufresne et al., 2003], [Schönbucher, 2004], [Duffie et al., 2009] (empirical focus), [Frey and Runggaldier, 2010], [Frey and Schmidt, 2012]
Overview

Introduction

Corporate Security Prices with Incomplete Information
   The Model
   Pricing corporate securities

Stochastic Filtering and Price Dynamics
   The SPDE for the conditional density
   Default intensity and price dynamics

Applications
   Derivative pricing
   CoCos
The Model

We use the following setup

- We work on \((\Omega, \mathcal{G}, \mathbb{G} = (\mathcal{G}_t)_{t \geq 0}, Q)\); all processes are \(\mathbb{G}\)-adapted; \(Q\) is the martingale measure used for pricing.
- Consider a company with asset value process \(V = (V_t)_{t \geq 0}\) and default time \(\tau = \inf\{t \geq 0 : V_t \leq K\}\).
- Company pays dividend \(d_n\) at date \(T_n\), \(n \geq 0\). \(T_n\) is the \(n^{th}\) jump time of a Poisson process with intensity \(\lambda^D\) and \(d_n = \delta_n V_{T_n}\) for a iid sequence \((\delta_n)_{n=1,2,...}\), independent of \(V\), with mean \(\bar{\delta}\).
- \(D_t = \sum\{n : T_n \leq t\} d_n\) is the cumulative dividend process and \(\varphi(y, V_{T_n})\) denotes conditional density of \(d_n\) given \(V\).
- Under \(Q\), \(V\) is geometric Brownian motion, \(dV_t = (r - \lambda^D\bar{\delta})V_t dt + \sigma V_t dB_t\). Moreover \(V_0\) has Lebesgue density \(\pi_0(v)\) with \(\pi_0(K) = 0\).
Market Information

The market uses the following pieces of information to price securities

- **Default information.** Market observes default state \( N_t = 1_{\{\tau \leq t\}} \) of the firm. We denote the default history by \( \mathbb{F}^N = (\mathcal{F}^N_t)_{t \geq 0} \).

- **Dividend information.** Market observes \( D_t \) with associated filtration \( \mathbb{F}^D = (\mathcal{F}^D_t)_{t \geq 0} \).

- **Noisy asset observation.** Market observes a process \( Z_t = \int_0^t a(V_s)ds + W_t \). \( W \) is an \( l \)-dim \( \mathcal{G} \)-Brownian motion independent of \( B \), and \( a \) is a smooth and bounded with \( a(K) = 0 \). \( Z \) is an abstract process modelling information contained in security prices.

- **Market information** is \( \mathbb{F}^M = \mathbb{F}^N \vee \mathbb{F}^Z \vee \mathbb{F}^D \); \( \mathbb{F}^Z \vee \mathbb{F}^D \) will be termed background filtration.
Pricing basic corporate securities and filtering

Risk-neutral pricing wrt $\mathbb{F}^M \Rightarrow$ price of security with cash flow stream $(H_t)_{0 \leq t \leq T}$ is

$$\Pi^H_t = E^Q \left( \int_t^T e^{-r(s-t)} dH_s \mid \mathcal{F}^M_t \right), \quad t \leq T. \quad (1)$$

Consider a basic corporate securities with $\mathbb{F}^N \lor \mathbb{F}^D$-adapted cash flow such as a bond, a CDS or the equity value of the firm. Iterated conditional expectations gives

$$1_{\{\tau > t\}} \Pi^H_t = E^Q \left( E^Q \left( 1_{\{\tau > t\}} \int_t^T e^{-r(s-t)} dH_s \mid \mathcal{G}_t \right) \mid \mathcal{F}^M_t \right).$$

Markov property of $V \Rightarrow$ inner conditional expectation is typically of the form $1_{\{\tau > t\}} h(t, V_t)$, (the full-information value). Hence

$$1_{\{\tau > t\}} \Pi^H_t = 1_{\{\tau > t\}} E^Q \left( h(t, V_t) \mid \mathcal{F}^M_t \right). \quad (2)$$

Evaluation of this expression is a nonlinear filtering problem.
Equity Pricing

The *equity value* is defined as value of dividend payments up to default time time \( \tau \). Full-information value \( S(v) \) satisfies

\[
S(v) = E^Q \left( \int_0^T e^{-rs} dD_s \mid V_0 = v \right) = E^Q \left( \int_0^T e^{-rs} \delta V_s \lambda^D ds \mid V_0 = v \right).
\]

Hence \( S \) is time-independent and solves \( \mathcal{L}_V S(v) + \delta \lambda^D v = rS(v) \), with boundary condition \( S(K) = 0 \). For \( K = 0 \) (and hence \( \tau = \infty \)) we get \( S(v) = v \). For \( K > 0 \) one has

**Proposition.** The full information value (3) of the firms equity is given by \( S(v) = v - K \left( \frac{v}{K} \right)^{\alpha^*} \), where \( \alpha^* \) is the negative root of the equation \( (r - \lambda^D \delta)\alpha + \frac{1}{2} \sigma^2 \alpha(\alpha - 1) - r = 0 \).

**Remark.** Full-information value for *debt securities* can be computed via first passage time of Brownian motion with drift
The filtering problem

Recall that we want to compute recursively the conditional expectation

\[ 1_{\{\tau > t\}} E^Q \left( f(V_t) \mid \mathcal{F}^M_t \right), \quad t \leq T, \quad f \in L^\infty([K, \infty)) . \]  

This is a nonstandard filtering problem, due to inclusion of default history \( \mathbb{F}^N \) in the observation filtration:

- Under full information \( \mathbb{G} \tau \) is a predictable stopping time and does not admit an intensity.
- In standard filtering theory with point process information on the other hand \( N \) is assumed to have a \( \mathbb{G} \) intensity.

**Basic idea.** Reduce (4) to a filtering problem wrt background filtration \( \mathbb{F}^Z \lor \mathbb{F}^D \) via Dellacherie formula.
Using the Dellacherie formula, we get

\[ 1_{\{\tau > t\}} E^Q \left( f(V_t) \mid \mathcal{F}_t^M \right) = 1_{\{\tau > t\}} \frac{E^Q(f(V_t)1_{\{\tau > t\}} \mid \mathcal{F}_t^Z \lor \mathcal{F}_t^D)}{Q(\tau > t \mid \mathcal{F}_t^Z \lor \mathcal{F}_t^D)}. \]

Denote by \( V^\tau \) the process \( V^\tau_t = V_{t \wedge \tau} \). By definition of \( \tau \) we have \( \{\tau > t\} = \{V^\tau_t > K\} \); moreover, \( V_t = V^\tau_t \) for \( t \leq \tau \). Hence we get

\[ 1_{\{\tau > t\}} E^Q \left( f(V_t) \mid \mathcal{F}_t^M \right) = 1_{\{\tau > t\}} \frac{E^Q(f(V^\tau_t)1_{\{V^\tau_t > K\}} \mid \mathcal{F}_t^Z \lor \mathcal{F}_t^D)}{Q(V^\tau_t > K \mid \mathcal{F}_t^Z \lor \mathcal{F}_t^D)}. \]

**Remark.** (5) is a filtering problem with standard diffusion and point process information. On the other hand new signal process \( X := V^\tau \) with state space \( S^X \) is a *stopped diffusion process.*
Measure transform

Start with independent processes $(X, Z)$ on $(\Omega, \mathcal{G}, Q^*)$ such that $X$ is a stopped geometric Brownian motion and $Z$ is a standard BM. (We largely ignore dividend payments.)

Consider the density martingale $L_t = \frac{dQ}{dQ^*} | \mathcal{F}_t$ with

$$L_t = \exp \left( \int_0^t a(X_s) ^\top dZ_s - \frac{1}{2} \int_0^t |a(X_s)|^2 ds \right). \tag{6}$$

Girsanov $\Rightarrow$ the pair $(X, Z)$ has the right law under $Q$ and we have

$$E^Q \left( f(X_t) \mid \mathcal{F}_t^Z \right) = \frac{E^{Q^*} \left( f(X_t)L_t \mid \mathcal{F}_t^Z \right)}{E^{Q^*} (L_t \mid \mathcal{F}_t^Z)} \equiv \frac{\Sigma_t f}{\Sigma_t 1}. \tag{7}$$
SPDE for the density of $\Sigma_t$

Suppose that $\Sigma_t$ is absolutely continuous with density $u$. $u$ is related to the SPDE

$$du(t) = L^* u(t) dt + a^\top u(t) dZ_t, \quad u(0) = \pi_0.$$  

Formal interpretation. Denote by $(f, g)$ the scalar product on $L^2(S^X)$. Then $u$ is an $\mathbb{F}^Z$ adapted continuous process with values in the Sobolev space $H^0_1(S^X) \cap H^2(S^X)$, and one has for $v \in L^2$

$$(u(t), v) = (u(0), v) + \int_0^t (L^* u(s), v) ds + \int_0^t (a^\top u(s), v) dZ_s. \quad (8)$$

Theorem. ([Pardoux, 1978]) There is a unique solution $u$ of equation (8). Moreover, for $f \in L^\infty(S^X)$,

$$\Sigma_t f = (u(t), f) + \nu_K(t)f(K)$$

where $\nu_K(t) = \int_0^t \frac{1}{2} \sigma^2 K^2 \frac{du}{dx}(s, K) ds$.

$\quad (9)$
Comments and Implications

Comments.

• The measure $\Sigma_t$ consists of two parts: an absolutely continuous part with density $u(t)$ and a point mass $\nu_K(t)$ at the boundary $K$.

• Boundary term drives form of default intensities.

• We give a simplified presentation here: the result has been shown only for bounded domain $[K, N)$ (see paper).

• Numerical solution via Galerkin approximation of (8)

Corollary. We get for the original filtering problem

$$E(f(V_t) \mid \mathcal{F}^M_t) = (\tilde{\pi}(t), f) \text{ with } \tilde{\pi}(t, x) = \frac{u(t, x)}{u(t, 1)}.$$ (10)
A Path of $V$ and $\hat{V}$

The diagram illustrates the path of $V$ and $\hat{V}$ over time. The x-axis represents time, while the y-axis shows the estimated asset value. The green line represents the path of $\hat{V}$, and the blue line represents the path of $V$. The graph demonstrates how both paths fluctuate over time, with $\hat{V}$ generally following $V$ but with some deviations.
Corresponding Filter Density
Default Intensity

Theorem. The $F^M$ compensator of $N_t$ is given by $(\Lambda_t \wedge \tau)_{t \geq 0}$ where

$$\Lambda_t = \int_0^t \lambda_s ds \quad \text{with} \quad \lambda_t = \frac{1}{2} \sigma^2 K^2 \frac{d\pi}{dx}(t, K).$$

Here $\pi(t, x)$ is conditional density of $X_t$ given $F^M_t$.

- This extends earlier results of [Duffie and Lando, 2001] and [Frey and Schmidt, 2009] to the case where information arrives continuously.
- An alternative characterization of the compensator of $N$ has recently been given by [Cetin, 2012].
**Theorem.** For $f \in C^{1,2}([0, T] \times S^X)$ the projection $\hat{f}_t = E(f(t, X_t) \mid \mathcal{F}^M_t)$ has dynamics

$$\hat{f}_t = \hat{f}_0 + \int_0^t \left( \frac{df}{dt} \right)_s + (1 - N_s)(\mathcal{L}_X f)_s ds + \int_0^{t \wedge \tau} (fa^\top)_s - \hat{f}_s a^\top ds \, dM^Z_s$$

$$+ \int_0^{t \wedge \tau} (f(s, K) - \hat{f}_s) \, d(N_s - \lambda_s)$$

$$+ \int_0^{t \wedge \tau} \int_{\mathbb{R}^+} \frac{(f \varphi_d(y))_s - \hat{f}_s(\varphi_d(y))_s}{\hat{f}_s(\varphi_d(y))_s} \left( \mu^D - \gamma^D \right)(dy, ds).$$

Here $M^Z = Z_t - \int_0^t \hat{a}_s ds$ is a $\mathbb{F}^M$ Brownian motion and $(\mu^D - \gamma^D)(dy, ds)$ is the $\mathbb{F}^M$-compensated random measure associated with the dividends.
Stock price dynamics

Using the filter equations it is straightforward to compute the semimartingale decomposition of the stock price $\hat{S}_t$. We get

$$d\hat{S}_t = (1 - N_t)(r\hat{S}_t - \lambda^D \delta \hat{V}_t)dt + (1 - N_t)((\hat{S}a^\top)_t - \hat{S}_t \hat{a}^\top_t) dM^Z_t$$

$$- (1 - N_t)\hat{S}_t d(N_t - \lambda_t dt) + \text{integral wrt}(\mu^D - \gamma^D)(dy, dt)$$

- Similar formulas can be obtained for debt securities.
- Note that stock-price dynamics can be quite wild even if asset price dynamics follow standard geometric Brownian motion.
A path of the default intensity
A typical stock price trajectory
Derivative Pricing

Basic corporate securities \((\mathbb{F}^D \vee \mathbb{F}^N\text{-adapted payoff } H)\).
Here \(\Pi_t^H = 1_{\{\tau > t\}}(\pi(t), h)\). Note that price is \textit{linear} in \(\pi(t)\)

Options. Here payoff depends on price path of basic corporate securities; examples include equity options, convertibles, . . . Price is given by some function \(C(\cdot)\) of the current density \(\pi(t)\), but \(C(\cdot)\) needs to be computed with Monte Carlo.

Simulation of price trajectories.

- Crucial point: generate a numerical approximation to the solution \(u(t)\) of the Zakai SPDE.
- This is best done via Galerkin approximation: one considers an approximation of the form \(u^m(t) = \sum_{i=1}^m \Psi_i(t)e_i\), \(e_i \in H_1^0(S^X)\) a suitable sequence of basis functions and one derives an SDE system for \(\Psi(t)\) that can be solved numerically. More details in [Frey et al., 2013]
Contingent Convertibles (CoCo)

Cocos.

- A CoCo is a corporate bond that is automatically triggered once the issuer enters into financial distress.
- At trigger the CoCo is either converted into equity or into a (low) cash-payment. Purpose: strengthen the equity capital of the issuer.

Payoff description. Cocos are characterized by maturity date $T$; coupons $c_j$, due at $t_j$; conversion time $\theta > 0$ and payment at conversion.

- Payoff stream with cash payment $C < 1$ at conversion:

$$\sum_{j=1}^{m} e^{-r t_j} c_j 1\{\theta > t_j\} + e^{-r T} 1\{\theta > T\} + e^{-r \theta} C 1\{\theta \leq T\};$$

- Conversion into a fraction $\gamma$ of equity $\Rightarrow$ last term becomes $e^{-r \theta} \gamma S_{\theta} 1\{\theta \leq T\}$. 
Modelling the conversion time $\theta$

CoCos in practice have an accounting trigger based on capital adequacy ratios. Difficult to model directly $\Rightarrow$ use approximations.

**Asset trigger.** Here $\theta^{\text{ass}} = \inf\{ t \geq 0 : V_t \leq K^{\text{CoCo}} \}$.

+ conversion always before default $\Rightarrow$ protection of debtholders
- asset value and hence $\theta^{\text{ass}}$ not publicly observable, hence monitoring difficult

**Equity trigger.** Here conversion if stock price hits threshold $S^{\text{CoCo}}$, that is $\theta^{\text{eq}} = \inf\{ t \geq 0 : S_t \leq S^{\text{CoCo}} \}$. A natural choice in our model is $S^{\text{CoCo}} = S(K^{\text{CoCo}})$.

+ Stock price is publicly available.
- It may happen that $\theta^{\text{eq}} = \tau$, that is conversion takes place too late
Comparison of the two trigger mechanisms

Under full information it holds that $\theta^{eq} = \theta^{ass}$; under incomplete information the two mechanisms differ.

- asset trigger contains extra information whereas $\theta^{eq}$ is $\mathbb{F}^M$ stopping time.
- Late conversion ($\theta^{eq} = \tau$) most likely if $K^{CoCo}$ close to the default boundary and if asset information is quite noisy.
- In our experiments equity trigger led to higher prices for the Coco.
Cocos: numerical experiments

<table>
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<th>Information</th>
<th>Coco price $\theta^{eq}$</th>
<th>$\theta^{ass}$</th>
<th>$Q(\tau &lt; T)$</th>
<th>$Q(\theta^{eq} &lt; T)$</th>
<th>$Q(\theta^{eq} = \tau)$</th>
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<tbody>
<tr>
<td>noisy</td>
<td>1.1</td>
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<td>0.2876</td>
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<td>0.2876</td>
<td>0.6099</td>
<td>0.0328</td>
</tr>
</tbody>
</table>

Other parameters. $T = 10$; coupon 5% biannually; $K = 20$; $K^{CoCo} = 30$; mean initial asset price 40.


The term structure of credit risk with incomplete accounting observations.

Pricing credit derivatives under incomplete information: a nonlinear filtering approach.

Pricing corporate securities under noisy asset information.

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