Contagion Effects and Collateralized Credit Value Adjustments for Credit Default Swaps

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Risk Management of Counterparty Credit Risk

Counterparty Credit Risk is concerned with the risk that one of the parties in a OTC derivative transaction defaults before the final settlement of the transaction's cash flows.

These days counterparty risk is a big concern in financial risk management

- Increased awareness of counterparty risk after financial crisis and default of Lehman Brothers
- Value adjustments for counterparty risk are nowadays standard in derivative trades
- Appears in new Basel III market risk regulation

Managing counterparty risk

Possible approaches for mitigation of counterparty risk

- Exposure limits
- Netting agreements
- Value adjustments for swaps etc
- Economic capital for counterparty risk
- Hedging, e.g. using CDSs
- Collateralization (posting of securities that serve as a pledge for the collateral taker.)

Economic capital computation, hedging and collateral management should be based on dynamic models for the price of the underlying asset and for the default times of contracting parties but in practice many heuristics and approximations are used.

In this talk . . .

we are concerned with a detailed analysis of counterparty risk and collateralization strategies for a CDS

- We compute value adjustments in 2 versions of the dynamic credit risk model of [Frey and Schmidt, 2012]
- We study the impact of different credit spread dynamics on the performance of collateralization strategies. We are particularly interested in contagion (upward jump in credit spread of surviving firms at the default of one of the contracting parties)
- We derive optimal collateralization strategies that take credit-spread dynamics into account
- We illustrate our results by numerical experiments

The talk is based on [Frey and Rösler, 2013]

Some literature

- General literature on counterparty risk
  - books: [Gregory, 2012], [Cesari et al., 2009] and [Brigo et al., 2013].
  - papers [Hull and White, 2012]
- Structural models: [Lipton and Sepp, 2009]
- Credit Value Adjustments and applications to credit risk [Crepey, 2013a, Crepey, 2013b] [Brigo and Chourdakis, 2009] [Brigo et al., 2014] (Last two papers are close to our analysis)

Overview

1. Introduction
2. Collateralized value adjustments
3. The credit risk model
4. Computation of value adjustments
5. Results and improved collateralization strategies

Basic Notation

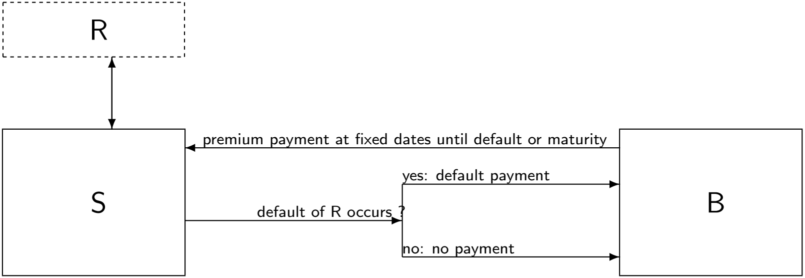
$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ denotes the filtered probability space under risk-neutrl measure $\mathbb{Q}$. We use the abbreviations B, R and S to denote the protection buyer, the reference entity respectively the protection seller. Moreover:

- τ_i default time of entity $i \in \{B, R, S\}$,
- $\tau = \min(\tau_B, \tau_R, \tau_S)$,
- Default indicator processes $H_t^i := 1_{\{\tau_i \le t\}}$ for $i \in \{B, R, S\}$ and we put $H := (H^B, H^R, H^S)$,
- ξ entity which defaults first,
- LGD_i loss given default of entity i,
- T maturity of the CDS.
- $D(t, s) = \exp(-r(s-t))$, $t < s$ is the discount factor

Risk-free Credit Default Swap

Payoff description. Three parties involved:

- R (reference entity); default triggers default payment.
- B (protection buyer); makes periodic premium payments to S
- S (protection seller); makes default payment to B if $\tau_R < T$.



CDS payoff: formal description

P_t denotes the time t price of a riskfree CDS, i.e. a CDS without counterparty risk and without collateralization. We assume that the cashflows arising from a risk free CDS from time t to time s are given by:

$$\Pi(t,s) \quad := \quad 1_{\{t<\tau_R\leq s\}}\,\mathrm{LGD}_R\,D(t,\tau_R) - \int_t^s S_R D(t,u) 1_{\{\tau_R>u\}} du,$$

where S_R represents the spread of the CDS.

Risk-neutral pricing $\Rightarrow$ price of a risk free CDS is given by $P_t := \mathbb{E}\left(\Pi(t,T)|\mathcal{F}_t\right)$.

Collateralization

Collateralization refers to the practice of posting securities (the so-called *collateral*) that serve as a pledge for the collateral taker.

Mathematical description of collateralization

A collateralization strategy is an $\mathcal{F}$-adapted RCLL process $(C_t)_{t\in[0,T]}$ with $C_t : \Omega \to \mathbb{R}$. C_t denotes the amount of collateral which is available to the protection buyer ($C_t > 0 \Leftrightarrow B$ is the collateral taker).

Note that this definition implies the following properties:

- The amount of collateral can be updated continuously.
- C is updated only wrt current information, for example current price changes of the CDS.

Examples of collateralization strategies

We will only consider collateralization strategies of the form $C_t = g(t, P_t)$ with a deterministic function. Among others the following strategies belong to this class:

- No collateralization: $C \equiv 0$.
- Threshold collateralization: for an initial margin γ and thresholds M_1, $M_2 \geq 0$:

$$C_t^{\gamma, M_1, M_2} := \gamma + (P_t - M_1)1_{\{P_t > M_1\}} + (P_t + M_2)1_{\{P_t < -M_2\}}$$

Credit Value Adjustments

Credit value adjustments are used to take the credit worthiness of the contracting parties into account (in pricing). The price of the contingent claim is decomposed in the following way:

$$\begin{aligned} \textit{price} = &\ \text{(counterparty) risk-free price} \\ &- \text{adjustment for default of seller} \\ &+ \text{adjustment for default of buyer}\ , \end{aligned}$$

where adjustment for the seller = Credit Value Adjustment (CVA) and adjustment for the buyer = Debt Value Adjustment (DVA).

Impact of collateralization on value adjustments

- Collateral can be used to (partially) cover replacement costs $\Rightarrow$ loss reduction
- Counterparty might be unable to return posted collateral (*re-hypothecation*).

Bilateral Collateralized Value Adjustment (BCCVA)

Definition. Denote by $\Pi(t,T)$ the discounted cash-flow stream of the risk free CDS and by $\Pi^D(t,T,C)$ the actual cash-flow stream for a given C. Then

$$\mathrm{BCCVA}(t,T,C) := \mathbb{E}\left(\Pi(t,T)|\mathcal{F}_t\right) - \mathbb{E}\left(\Pi^D(t,T,C)\middle|\mathcal{F}_t\right). \quad (1)$$

[Brigo et al., 2014] give the following decomposition of the BCCVA

$$\mathrm{BCCVA}(t,T,C) = \mathrm{CCVA}(t,T,C) - \mathrm{CDVA}(t,T,C), \quad (2)$$

where the *collateralized credit value adjustment* (CCVA) and the *collateralized debt value adjustment* (CDVA) are given by:

$$\begin{aligned}\mathrm{CCVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\ &\qquad + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)|\mathcal{F}_t\big) \\ \mathrm{CDVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(t,\tau)(\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\ &\qquad + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)|\mathcal{F}_t\big).\end{aligned}$$

Comments

The CCVA reflects the possible loss for B due to default of S; the CDVA reflects the loss of S due to default of B.

Without collateralization value adjustments take the form of options on P_t with $K = 0$.

Possible measure of effectiveness of a given strategy C is $m(C) := \text{CCVA}(T, C) + \text{CDVA}(T, C)$

No updating of collateral possible after default of B or S. Hence $C_{\tau-}$, the collateral position immediately prior to the first default, enters the formula.

Simplified BCCVA formula

Markets often use a simplified formula which implicitly assumes that H^B, H^S and P are independent (*no wrong-way risk*). For $C_t \equiv 0$ the simplified bilateral credit value adjustment at $t = 0$ is given by

$$\begin{aligned} \mathrm{BCVA}^{\mathrm{indep}} = \mathrm{LGD}_S & \int_0^T \bar{F}_B(s) D(0,s) \mathbb{E}(P_s^+) f_S(s)\, ds \\ - \mathrm{LGD}_B & \int_0^T \bar{F}_S(s) D(0,s) \mathbb{E}(P_s^-) f_B(s)\, ds \,. \end{aligned} \quad (3)$$

where $\bar{F}_S(s) = \mathbb{Q}(\tau_S > s)$ and $f_S(s) = -\bar{F}'(s)$

Credit Risk Model

We use the framework from [Frey and Schmidt, 2012]:

- X is a finite state Markov chain with state space $S^X = \{1, \dots, K\}$ and generator matrix given by $W = (w_{ij})_{1 \le i,j \le K}$.
- τ_B, τ_R and τ_S are conditionally independent, doubly stochastic random times whose default intensities are increasing functions of X. Therefore

$$\mathbb{Q}(\tau_R > t_1, \tau_B > t_2, \tau_S > t_3 \mid \mathcal{F}^X_\infty) = \prod_{i \in \{B,R,S\}} \exp\Big(- \int_0^{t_i} \lambda_i(X_s) ds \Big)$$

where $\lambda_i : S^X \to \mathbb{R}$.

Credit Risk Model II

We consider two versions of the model, which differ with respect to the available investor information:

- Complete information with filtration $\mathbb{F}^O$: Default times and X are observable.
- Incomplete information with filtration $\mathbb{F}^U$: Investors observe the time points of defaults and noisy observations of X, namely:

$$Z_t = \int_0^t a(X_s)ds + B_s,$$

where B is a standard d-dimensional Brownian motion, independent of all other processes and $a : S^X \to \mathbb{R}^d$.

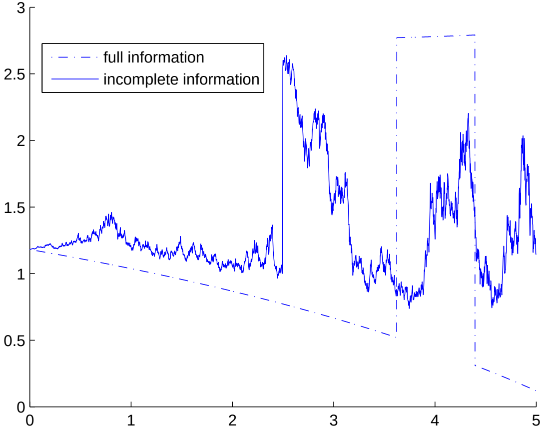
Moreover, we set $\mathbb{F}^U = \mathbb{F}^{H,Z}$ and $\mathbb{F}^O = \mathbb{F}^{H,X,B}$ to obtain $\mathbb{F}^U \subset \mathbb{F}^O$. We will use the superscripts U and O to distinguish the models.

Model variants compared

Note that joint law of τ_B, τ_R and τ_S is identical in both versions of the model. However, model differs wrt *credit-spread dynamics*

- In full-information model credit spreads are piecewise deterministic, jumping only at jumps of X.
- Under partial information there is *spread volatility* and *information-based* default contagion.

Trajectory of the fair CDS spread of R



Complete Information Model

The complete information model exhibits a lot of explicit pricing formulas involving the matrix exponential of the generator matrix.

Default probability of R is given by

$$\mathbb{Q}\left(\tau_R \leq s \middle| X_t = k, H_t^R = 0\right) \;=\; 1 - e_k^\top e^{Q_R(s-t)} \mathbf{1}_K,$$

where $Q_R := W - \Lambda_R$ with $\Lambda_R = \text{diag}(\lambda_R(1), \ldots, \lambda_R(K))$, $\mathbf{1}_K = (1, \ldots, 1)^\top$ e_l denotes the l-th unit vector in $\mathbb{R}^K$.

Pricing formula for risk-free CDS: P_t is a function p^O of t and X_t. For $t \in [0, T]$ and $l \in \{1, \ldots, K\}$ we have

$$p^O(t, l) \;:=\; \left(-\,\text{LGD}_R\, e_l^\top Q_R - S e_l^\top\right)\left(Q_R - rI\right)^{-1}\left(e^{(Q_R - rI)(T-t)} - I\right)\mathbf{1}_K.$$

CCVA formula under full information

Suppose that the collateral process satisfies $C_t = g(t, X_t)$ with a deterministic function g. Then, given $X_0 = j$,

$$\text{CCVA} = \sum_{k \in \{1,\dots,K\}} \int_0^T D(0,s) \left(\text{LGD}_S \left(p^O(s,k)^+ - g(s,k)^+ \right)^+ \right.$$
$$\left. + \, \text{LGD}'_S \left(g(s,k)^- - p^O(s,k)^- \right)^+ \right) f^S_{j,k}(s) ds.$$

Note that the function $f^S_{j,k}$ is given by

$$\begin{aligned} f^S_{j,k}(s) &:= \frac{d}{ds} \mathbb{Q}(\tau \le s, \xi = S, X_\tau = k \mid X_0 = j) \\ &= e_j^\top e^{Q_{(1)}(s-t)} \Lambda_S e_k . \end{aligned}$$

A similar formula can be given for the CDVA

Incomplete Information Model - Pricing

The following process will play an important role:

$$\pi_t^k := \mathbb{Q}(X_t = k \mid \mathcal{F}_t^U) \text{ for } 1 \leq k \leq K, \quad \text{and } \pi_t := (\pi_t^1, \dots, \pi_t^K)^\top.$$

It is possible to recover pricing formulas for contingent claims with $\mathbb{F}^H$-adapted cashflows such as a risk-free CDS. For a contingent claim with $\mathbb{F}^H$-adapted cashflow $\Pi(t,s)$ we introduce:

- p_t^U - incomplete information model price,
- $p^O(t, X_t)$ - complete information model price given X_t.

We obtain the following relationship:

$$\begin{aligned} p_t^U &= \mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^U\right) = \mathbb{E}\left(\mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^O\right) \middle| \mathcal{F}_t^U\right) = \mathbb{E}\left(p^O(t,X_t) \middle| \mathcal{F}_t^U\right) \\ &= \sum_{k=1}^K p^O(t,k)\pi_t^k. \end{aligned}$$

Computation of BCCVA

The BCCVA is essentially an option on the price process P^U of the risk-free CDS. Hence the above projection-approach is insufficient for computing the BCCVA and we need the dynamics of π_t.

Notation. Denote by by $\widehat{G}$ the optional projection of a process $G = (G_t)_{t\in[0,T]}$ with respect to $\mathbb{F}^U$

$$\widehat{\lambda_{t,i}} := \widehat{\lambda_i(X_t)} = \mathbb{E}\left(\lambda_i(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{j=1}^{K} \lambda_i(j)\pi_t^j.$$

$$\widehat{a(X_t)} = \mathbb{E}\left(a(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{j=1}^{K} a(j)\pi_t^j$$

Moreover, we introduce the $\mathbb{F}^U$-Brownian motion μ defined by

$$\mu_t = (\mu_t^1, \ldots, \mu_t^d) \text{ with } \mu_t^i = Z_t^i - \int_0^t \widehat{a(X_s)}ds.$$

Kushner-Stratonovich-Equation for Π_t

[Frey and Schmidt, 2012] show that for $k = 1, \dots, K$

$$d\pi_t^k = \sum_{i=1}^{K} w_{ik}\pi_t^i dt + \sum_{j\in\{R,B,S\}} \Big(\gamma_j^k(\pi_{t-})\Big)^\top d\big(H_t^j + (1 - H_t^j)\widehat{\lambda}_{t,j} dt\big) + \Big(\alpha^k(\pi_t)\Big)^\top d\mu_t\,, \text{ with}$$

$$\gamma_j^k(\pi_t) = \pi_t^k \left(\frac{\lambda_j(k)}{\sum_{i=1}^K \lambda_j(i)\pi_t^i} - 1 \right) \text{ for } 1 \le j \le K \text{ and}$$

$$\alpha^k(\pi_t) = \pi_t^k \left(a(k) - \sum_{i=1}^K \pi_t^i a(i) \right).$$

The KS equations are a K-dim SDE system that can be used to generate price trajectories under incomplete information.

Contagion effects

It follows from KS that

$$\widehat{\lambda}_{\tau_j,i} - \widehat{\lambda}_{\tau_j-,i} \quad = \quad \sum_{k=1}^{K} \lambda_i(k)\pi^k_{\tau_j-}\left(\frac{\lambda_j(k)}{\sum_{l=1}^{K}\lambda_j(l)\pi^l_{\tau_j-}} - 1\right) = \frac{\text{cov}^{\pi_{\tau_j-}}(\lambda_i,\lambda_j)}{E^{\pi_{\tau_j-}}(\lambda_j)}.$$

An inspection of the formula shows that:

- Contagion effects are inversely proportional to the instantaneous default risk of the defaulting entity (firm j): a default of an entity with a better credit quality comes as a bigger surprise and the market impact is larger.
- Contagion effects are proportional to the covariance of the default intensities $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under the 'a-priori distribution' π_{τ_j-}.

Relationship between CCVA in the 2 models

By using Jensen's inequality one can show that for the collateralization strategy $C \equiv 0$ the following is true:

$$\mathrm{CCVA}^{O} \geq \mathrm{CCVA}^{U} \text{ and } \mathrm{CDVA}^{O} \geq \mathrm{CDVA}^{U} .$$

Simulation study

We calibrate the model to the following credit spreads in base points and default correlations in percentage points:

	B	R	S
Spread	50	1000	500

	ρ_{BR}	ρ_{BS}	ρ_{RS}
Correlation	2.0	1.5	5.0

Findings from simulation study

- Threshold-collateralization with initial margin $\gamma = 0$ works fine in the complete information model; performance less satisfactory under incomplete information due to contagion.
- Simplified value adjustment formula (3) leads to a substantially lower value adjustment becaus of default correlation (wrong way risk)

Value adjustments for threshold collateralization

	full information			incomplete information		
threshold	CCVA	CDVA	BCCVA	CCVA	CDVA	BCCVA
$M_1 = M_2 = 0$	0	0	0	35	0	35
$M_1 = M_2 = 0.02$	16	0	15	45	0	45
$M_1 = M_2 = 0.05$	38	1	37	60	0	60
no collateralization						
(i) correct formula	93	1	92	83	1	82
(ii) simplified formula	68	6	62	54	4	49

Table: Value adjustments in the complete-information model (left) and in the incomplete-information model (right) with threshold-collateralization and market value collateralization ($M_1 = M_2 = 0$) for $\gamma = 0$. In the last row we report the value adjustment corresponding to the simplified value adjustment formula (3).

Improving Collateralization Strategies

Goal: find a strategy C which minimizes $m(C) := CCVA(C) + CDVA(C)$. Recall the definitions:

$$
\begin{aligned}
\mathrm{CCVA}(T, C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(0,\tau) \quad (\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\
&\qquad + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)\big) \\
\mathrm{CDVA}(T, C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(0,\tau) \quad (\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\
&\qquad + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)\big).
\end{aligned}
$$

We want to find an $\mathcal{F}$-adapted RCLL process C which minimizes $m(C)$.

Complete Information model

Consider the collateralization strategy given by $C_t = P_t$ (market value collateralization):

- C coincides with the threshold strategy given by $\gamma = 0$ and $M_1 = M_2 = 0$,
- $C_{\tau-} = P_{\tau-} = P_\tau$ Q-a.s. and therefore $\mathrm{CCVA}(C) = 0$ and $\mathrm{CDVA}(C) = 0$.

Therefore C is a collateralization strategy which minimizes m. (holds in any model where P does not jump at τ_B or τ_S

Incomplete Information model

- Contagion effects $\Rightarrow P_{\tau-} < P_\tau$ $\mathbb{Q}$-a.s.
- Threshold collateralization with $\gamma = 0$ systematically underestimates the price of the CDS.

Is it possible to find an improvement?
In [Frey and Rösler, 2013] a minimizer of $C \mapsto m(C)$ in the incomplete information model is found. It depends only on the quantities:

$$d_j(\pi_{\tau-}) := \mathbb{Q}\left(\xi = j | \mathcal{F}_{\tau-}\right) = \frac{(\widehat{\lambda}_j)_{\tau-}}{\sum_{i \in \{B,R,S\}} (\widehat{\lambda}_i)_{\tau-}},$$

$$x_j(\tau, \pi_{\tau-}) := p^U\left(\tau, \pi_{\tau-} + \operatorname{diag}\left(\gamma_j^1, \dots, \gamma_j^K\right)\pi_{\tau-}\right)$$

Simulation analysis shows that improved strategy performs much better, at least within the model framework.

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