Delta hedging with stochastic volatility in discrete time

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1 Introduction

Since the important contributions by Black and Scholes (1973) and Merton (1973) it is well known that option prices are intimately related to the costs of hedging strategies. The Black/Scholes (BS) option pricing model assumes that the option's payoff can be perfectly duplicated or hedged by an continuously adjusted, self-financing portfolio strategy. The necessary adjustments are based on the option's greeks: delta – the first derivative of the option price with respect to the price of the underlying asset, and theta – the first derivative with respect to time. To exclude arbitrage opportunities the option price must equal the costs necessary to run the duplication strategy. The strategy costs can be calculated as the present value of the accumulated costs until maturity. Since the costs are the same on each possible price path, the present value can be computed by using the risk free rate.

The BS option pricing formula and the related portfolio strategy are based on several assumptions: The underlying of the option is an instantaneously traded and storable asset. Its price follows a geometric Brownian Motion with a constant diffusion. There are no market imperfections such as taxes or transaction costs. The interest rate does not change at different points in time and for different time horizons, i.e., the term structure is flat and constant over time.

The purpose of the present paper is to investigate the consequences of relaxing the assumption of a constant diffusion. Substantial empirical evidence collected in recent years indicates that the temporal behaviour of the variance of stock (or stock index) returns can be conveniently described by generalized autoregressive conditional heteroskedasticity (GARCH) models (see Bollerslev et al., 1992). Duan (1995) has investigated the effect on option pricing when the empirically observed GARCH variance structure is used (GARCH economy) instead of the constant diffusion assumption (BS economy). He proposed a local risk-neutral valuation (LRNV) principle to price options under the GARCH assumption. Our objective is to derive the costs of delta hedging strategies when the variance of the underlying asset returns follows a GARCH process.

In a BS economy the costs of a delta hedging strategy correspond to the option's price calculated from the BS formula. In a GARCH economy it is not straightforward to implement a delta hedge, and it is not clear whether the (average) hedging costs coincide with the price obtained by, for instance, Duan's LRNV approach. It is the purpose of this paper to investigate these issues.

2 Delta Hedging Costs in the Black/Scholes Economy

In this section simulation results for a delta hedging strategy are derived for a BS – or constant variance – economy. The BS economy serves as a benchmark for the GARCH economy, which will be investigated in the next section. We generate 20000 daily return series y_t with standard deviation σ and mean $-0.5\sigma^2$:

$$y_t = -0.5\sigma^2 + \xi_t \qquad \xi_t \sim \mathrm{N}(0, \sigma^2).$$

Prices of the underlying are computed from $S_t = S_{t-1}\exp\{y_t\}$ ($S_0 = 100$). On each price path the delta hedging strategy is pursued using deltas from the BS call price formula. Thus, on each day until maturity the stock position held in the delta hedging portfolio is given by

$$\Delta_t^{BS} = \frac{\partial C_t}{\partial S_t} = \mathrm{N}(d_t) \qquad d_t = \frac{\ln(S_t/X) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \tag{1}$$

Δ_t^{BS} depends only on constant parameters (strike X, risk free interest rate r, variance σ^2 and maturity T) and the current price of the underlying S_t. In the GARCH economy this convenient fact – as will be seen later – is no longer valid.

In continuous time, where adjustments happen instantaneously, the present value of the accumulated costs (a constant amount on all paths) must equal the option price to exclude arbitrage. In the present case the adjustments are solely based on the call's delta and they are made at discrete time instants and for discrete price changes, which introduces (time and price) discretization errors (see Boyle and Emanuel (1980), Leland (1985) and Figlewski (1989) for the effects associated with discrete replication). These errors yield different accumulated delta hedging costs along each simulated price path giving a cost distribution rather than a constant cost amount. The present value of the accumulated costs is calculated as the average of costs across paths discounted by the risk free rate, which is – for simplicity – assumed to be zero.

Table 1 shows the results of a simulated delta hedging strategy for three different call maturities (30, 60 and 90 days) and different moneyness-ratios (S_0/X), assuming $\sigma^2 = 0.3^2/250$. The length of the rebalancing interval is one day. The distribution of the costs is summarized in terms of the mean and standard deviation of 20000 simulated values of hedging costs. As expected, the average hedging costs closely correspond to the prices resulting from the analytical BS call price formula. The deviations from this average can be substantial, however, because of discretization errors.

3 Delta Hedging Costs in the GARCH Economy

In this section the consequences of using stochastic GARCH instead of constant variances on delta hedging costs are investigated. For this purpose we use the GARCH option

Table 1: Results for delta hedging in a discrete time Black/Scholes economy.

S_0/X	T	BS option price	properties of delta hedging costs: average	std.dev.
0.8	30	0.0658	0.0678	0.1898
	60	0.4609	0.4617	0.3782
	90	1.0373	1.0385	0.4815
0.9	30	0.8881	0.8884	0.5023
	60	2.1476	2.1486	0.5953
	90	3.2702	3.2690	0.6313
1.0	30	4.1441	4.1441	0.6550
	60	5.8580	5.8539	0.6418
	90	7.1713	7.1662	0.6476
1.1	30	10.0544	10.0529	0.4634
	60	11.2703	11.2693	0.5334
	90	12.3252	12.3239	0.5624
1.2	30	16.8183	16.8196	0.2208
	60	17.3576	17.3581	0.3597
	90	17.9989	17.9993	0.4212

pricing model proposed by Duan (1995). It is based on the following specification for the return process:

$$\ln \frac{S_t}{S_{t-1}} = r + \lambda\sqrt{h_t} - 0.5h_t + \epsilon_t.$$

λ is a risk aversion parameter. The disturbances ϵ_t are assumed to be conditionally normally distributed:

$$\epsilon_t|\phi_{t-1} \sim \mathrm{N}(0, h_t).$$

ϕ_t denotes the information set at time t and h_t is the conditional variance that follows a GARCH(p, q) model:

$$h_t = a_0 + \sum_{i=1}^{p} a_i\epsilon_{t-i}^2 + \sum_{i=1}^{q} b_i h_{t-i}. \quad (2)$$

The unconditional variance implied by the GARCH parameters is given by:

$$\sigma^2 = \frac{a_0}{1 - \sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i}. \quad (3)$$

If all GARCH parameters a_i, $(i = 1, \ldots, p)$, b_i, $(i = 1, \ldots, q)$ are equal to zero the stochastic return model is identical to the BS model where the conditional variance is a constant and equal to the unconditional variance (i.e., $h_t = a_0 = \sigma^2$) and the mean return is $r + \lambda\sigma - 0.5\sigma^2$ rather than $-0.5\sigma^2$.

3.1 Option Pricing in the GARCH Economy

To derive the delta in the GARCH economy as the derivative of the option price with respect to the price of the underlying, one first needs to know the option price. Duan (1995) shows that the local risk-neutral valuation (LRNV) principle implies the following return process:

$$\ln \frac{S_t^*}{S_{t-1}^*} = r - 0.5 h_t^* + \epsilon_t^*.$$

$\epsilon_t^* | \phi_{t-1} \sim \mathrm{N}(0, h_t^*)$ and the modified conditional variance is given by

$$h_t^* = a_0 + \sum_{i=1}^{p} a_i \left(\epsilon_{t-i}^* - \lambda \sqrt{h_{t-i}^*} \right)^2 + \sum_{i=1}^{q} b_1 h_{t-i}^*.$$

The unconditional variance is given by:

$$(\sigma^*)^2 = \frac{a_0}{1 - (1 + \lambda^2) \sum_{i=1}^{p} a_i - \sum_{i=1}^{q} b_i}$$

and the corresponding terminal asset price is:

$$S_T^* = S_t^* \exp \left\{ (T - t) r - 0.5 \sum_{k=t+1}^{T} h_k^* + \sum_{k=t+1}^{T} \epsilon_k^* \right\}.$$

Based on this asset price the GARCH call option price can be calculated from

$$C_t^G = \exp\{-(T - t) r\} \mathrm{E}[\max(S_T^* - X, 0) | \phi_t]$$

There exists no analytical solution for C_t^G because the conditional distribution of S_T^* cannot be analytically derived. However the GARCH option price can be calculated numerically.

We use a GARCH(1,1) model with parameter values $a_0 = 2.88\text{E}{-5}$, $a_1 = 0.32$ and $b_1 = 0.60$ which are comparable to estimates known from empirical studies and imply a volatility of 30% in annual terms. This is equal to the value used in the BS economy in the previous section. In order to analyse the effect of the parameter λ we simulate two different GARCH economies based on $\lambda = 0.0$ and on $\lambda = 0.4$. The first value implies risk-neutrality, the second value is close to the upper bound $|\lambda| < \sqrt{(1 - a_1 - b_1)/a_1}$ (see Duan, 1995).

In order to compare the delta hedging costs to the GARCH option prices obtained by LRNV we use the empirical martingale simulation (see Duan and Simonato, 1998). The resulting option prices for $\lambda = 0.0$ and $\lambda = 0.4$ are presented in Table 2 together with the prices given by the BS formula. The differences between option prices in the BS and the GARCH economy can be explained on the basis of the differences between the lognormal distribution of the BS economy and the fat-tailed density implied by GARCH returns (see Duan (1995) for a discussion of the differences in prices).

Table 2: Option pricing in a GARCH economy; $T = 30$.

	BS	GARCH option price	
S_0/X	formula	$\lambda = 0.0$	$\lambda = 0.4$
0.8	0.0658	0.1873	0.2180
0.9	0.8881	0.8378	1.0549
1.0	4.1441	3.7505	4.5278
1.1	10.0544	9.9648	10.8168
1.2	16.8183	16.9067	17.4907

3.2 Delta Hedging Strategies in the GARCH Economy

In a discrete time BS economy different hedging costs are obtained for each possible price path. On average, however, the calculated hedging costs correspond to the price from the BS formula (see Table 1). We therefore investigate whether (average) hedging costs in a GARCH economy correspond to the GARCH option prices as derived from the LRNV principle.

For the implementation of a delta hedging strategy in the GARCH economy two aspects are important: First, different price paths are characterized by different realisations of the variance process. Second, GARCH implies a shape of the multiperiod density that deviates from normality. Therefore a different hedge ratio than in the BS economy is required.

The delta that corresponds to the GARCH option pricing model is given by (see Duan, 1995):

$$\Delta_t^G = \exp\{-(T-t)r\}\mathrm{E}\left[\frac{S_T^*}{S_t^*} I(S_T^*, X)|\phi_t\right] \tag{4}$$

$$I(S_T^*, X) = \begin{cases} 1 & \text{if } S_T^* \geq X \\ 0 & \text{if } S_T^* < X \end{cases} .$$

Contrary to the BS economy, no closed form solution for the delta is available in the GARCH economy. Δ_t^G depends on the stochastic evolution of the transformed price process S^*. To calculate the delta along each path requires keeping track (1) of the process S representing the actual price development in the risk-averse economy over time, and at the same time keeping track (2) of the risk-neutralized process S^*.

Since there exists no analytical solution for Δ_t^G it is necessary to compute the GARCH delta numerically, i.e., by simulation. This requires a considerable amount of computations since on each path of the hedging simulation and at each time point the distribution of S_t^* at maturity has to be simulated in order to calculate the GARCH delta from (4). The distribution of S_T^* depends on the time to maturity as well as on the current level of the conditional GARCH variance. To reduce the computational requirements we investigate three cases: the use of (i) a constant variance, (ii) a GARCH variance forecast, and (iii) an approximation of Δ_t^G. The following three different hedging strategies can be derived:

In the constant variance case (i) we calculate the delta from the BS formula by – incorrectly but consciously – assuming a constant variance until maturity on each path and each time instance. The constant variance is the unconditional variance implied by the GARCH parameters (see equation 3). On each of the 20000 simulated paths we calculate Δ_t^{BS} based on equation (1) and hold the corresponding position in the stock.

In the second case (ii) we maintain to calculate the delta from the BS formula. However, we modify equation (1) and replace $\sigma^2(T-t)$ by the variance forecast for the time until maturity:

$$\sum_{k=t+1}^{T} \mathrm{E}\left[h_k|\phi_t\right].$$

To compute this sum we use the current (time t) values of h_t and ϵ_t, plug them into equation (2), and calculate h_k iteratively for $k = t+1, \ldots, T$ with $\mathrm{E}[\epsilon_k^2|\phi_t] = h_k$. Note that the sum is different on each path for each time instant. This takes into account the conditional nature of the variance implied by the GARCH model. However, the fact that the multiperiod return distribution implied by GARCH is not accounted for. Nevertheless we expect the distribution of hedging costs to be narrower than in the constant variance strategy.

In the third case (iii) we derive a hedging strategy by approximating the GARCH delta on the basis of a two-step procedure: In the first step we simulate GARCH deltas according to Duan and Simonato (1998) for a range of different parameter values that determine the shape of S_T^* and consequently Δ_t^G. These are the time to maturity T, the moneyness-ratio S_t/X and the conditional variance h_t which is expressed relative to the unconditional variance. S_t/X is varied in the range of 0.5 to 5.0 in steps of 0.025 (from 0.725 to 1.475), 0.05 (from .05 to 0.7 and 1.5 to 3.0) and 0.1 (from 3.1 to 5.0). $\sqrt{h_t}/\sigma$ is varied from 0.3 to 3.0 in steps of 0.1. We simulate GARCH deltas for this grid of parameters for every maturity in the range of $T = 1$ to $T = 30$ using $\lambda = 0.0$ and the GARCH parameters estimated from returns. We repeat this procedure for $\lambda = 0.4$.

In order to calculate GARCH deltas in the hedging simulations for every possible value of S_t/X and $\sqrt{h_t}/\sigma$ in the second step we fit the nonlinear function $f(x,\beta) = (1 + \exp\{-x\beta\})^{-1}$ by least-squares to the simulated GARCH deltas from the first step. x denotes a vector of 'explanatory variables' that consists of a constant term, S_t/X, $\sqrt{h_t}/\sigma$, and square-roots, squares and cross-products of these variables. The parameter vector β of this function is determined for every T and the two cases $\lambda = 0.0$ and $\lambda = 0.4$. As it turns out the nonlinear least-squares fitting procedure provides highly accurate approximations to the simulated GARCH deltas. Although this procedure reduces the amount of computations in the course of hedging simulations it still involves rather heavy preparatory computations. We have therefore restricted the analysis to considering only the maturity $T = 30$.

For obvious reasons the three hedging strategies will be termed as follows: (i) constant variance strategy, (ii) conditional variance strategy, and (iii) approximate delta strategy.

Table 3: Hedging costs in a GARCH economy; $\lambda = 0.0$, $T = 30$.

strategy	S_0/X	GARCH option price	properties of delta hedging costs: average	std.dev.
constant			0.1923	1.3481
conditional	0.8	0.1873	0.1897	1.1107
approximate delta			0.1907	1.2194
constant			0.8382	1.8912
conditional	0.9	0.8378	0.8388	1.6308
approximate delta			0.8394	1.7533
constant			3.7436	2.1245
conditional	1.0	3.7505	3.7435	1.9399
approximate delta			3.7465	2.0515
constant			9.9567	1.7038
conditional	1.1	9.9648	9.9577	1.4586
approximate delta			9.9590	1.5959
constant			16.9118	1.2527
conditional	1.2	16.9067	16.9065	0.9706
approximate delta			16.9090	1.1203

3.3 Delta Hedging Results and Interpretation

The numerical results presented in this section are based on the same 20000 series of standard normal random numbers ξ_t used in the BS economy. The return series are not identical, however, because of the different assumptions about the process. In order to eliminate effects from the initial level of conditional variances at $t = 0$ we have started to simulate the return process paths at time $t = -20$. Hedging activities started at $t = 0$.

We first consider the case $\lambda = 0.0$ in Table 3 and find that there are hardly any differences between the average costs implied by the three different delta hedges. Deltas based on the conditional GARCH variance forecasts yield a narrower spread of hedging costs than the other deltas if the moneyness-ratio is less than 1.0. For in-the-money calls the approximate GARCH delta provide the smallest spread. However, we find a discrepancy between the average hedging costs (from any strategy) and the GARCH option prices implied by LRNV, in particular for out-of-the-money calls. The bias is surprisingly small, however, given that the constant and conditional delta hedging strategy is based on using the (inappropriate) deltas from the BS formula. This result suggests that the BS deltas may be quite useful for hedging options even if returns follow a GARCH process.

The picture changes strongly if we consider the case of $\lambda = 0.4$ (see Table 4). The average hedging costs from the three strategies differ considerably. Moreover, other than for $\lambda = 0.0$, average hedging costs and GARCH option prices deviate strongly for out-of-the-money calls. We find – without presenting details – that these deviations increase with λ. We obtain similar results for other choices of the GARCH parameters a_1 and b_1. Several attempts to obtain 'better' approximations of the GARCH deltas did not change these results. Large discrepancies between costs and prices prevailed, for instance, when the simulated grid of GARCH deltas was refined and/or was approximated by neural nets.

Table 4: Hedging costs in a GARCH economy; $\lambda = 0.4$, $T = 30$.

strategy	S_0/X	GARCH option price	properties of delta hedging costs: average	std.dev.
constant			0.3039	2.2876
conditional	0.8	0.2180	0.2877	1.8204
approximate delta			0.3065	2.0417
constant			0.8065	2.3040
conditional	0.9	1.0549	0.9113	2.1103
approximate delta			0.6852	2.1512
constant			4.1406	1.7392
conditional	1.0	4.5278	4.1119	1.8372
approximate delta			3.7509	1.7539
constant			10.1150	1.1029
conditional	1.1	10.8168	10.1514	1.2603
approximate delta			10.1056	1.1156
constant			16.8608	0.6786
conditional	1.2	17.4907	16.9336	0.8385
approximate delta			16.9956	0.6973

The results do not seem to depend on the simulation design as some experiments with more or less simulated paths and different random samples show.

4 Summary

The purpose of this study was to investigate issues involved in delta based hedging in discrete time, if the underlying returns follow a GARCH process. The results can be summarized as follows: Hedging strategies based on simple approximate deltas can yield average hedging costs that are close to option prices implied by Duan's (1995) GARCH option pricing model. The strategies mainly differ with respect to the variance of hedging costs across the possible time paths. However, if the GARCH return process underlying Duan's pricing model is based on a value of the risk parameter λ different from zero, we find strong discrepancies between prices and average hedging costs, in particular for far out-of-the-money calls.

This raises the question whether the local risk-neutral valuation principle is not generally applicable, or the approximate deltas used in this study are inappropriate. At this point it is unclear what the reasons for the discrepancies between prices and average hedging costs are. Several attempts to improve the delta approximations did not succeed. The derivation of other hedging strategies – not only based on the option's delta – are left for further research. For $\lambda > 0$ we conclude that GARCH option prices for out-of-the-money calls may be a biased reference for average hedging cost obtained by discrete time delta strategies.

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