Optimal Endowment Investing

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Abstract

Why doesn't the Modigliani-Miller theorem apply to the relationship between endowments ("firms") and donors ("shareholders"), rendering an endowment's investment policy neutral? We show that donor altruism (even if impure) breaks neutrality. Under a condition—which quickly converges to standard DARA preferences in the number of donors—endowment risk-taking reduces donor free-riding, is Pareto improving, and required by endowment competition. Contrary to asset-liability matching, large endowments optimally take substantial risk even if donors are very averse to changes in endowment-supported spending, reversing the standard moral hazard interpretation. Total giving grows unbounded in the number of donors even with pure altruism.

Keywords: Endowments, risk taking, charitable giving, free riding
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1 Introduction

The endowment investing literature is quite large.[1] The conventional wisdom—tracing back to Litvack et al. (1974), Tobin (1974), Black (1976) and Swensen (2009)—is that an endowment can afford to take on substantial investment risk since it has a long time horizon, as much of the supported spending occurs in the future. Consistently, endowments take on substantial risk, including investments in equities, alternatives, and illiquid assets (e.g., timber).[2] The main funds of university endowments are commonly labeled "long-term investment pools," or similar names. The so-called "endowment model" or "Yale model" of investment (Swensen, 2009)) has become standardized, creating a blueprint for systematic endowment-like investing throughout the world (Leibowitz et al., 2010).

At the same time, Black (1976) strongly argues that the standard argument of a long horizon is not well grounded because it considers an endowment in isolation of all of its stakeholders.[3] Neither his paper nor the subsequent literature, though, explores this point with sufficient micro-based foundations. This paper addresses his concern.

The current optimal endowment investment literature, in particular, does not incorporate explicit value maximization of an endowment's most important stakeholders—its donors. Instead, papers in this literature model the endowment's objective function as an isolated decision sans donors or using some reduced-form proxy for a range of stakeholders.[4] Maximizing an assumed endowment objective function without considering the donor problem raises an important question: How do you know that donors won't walk away? Donor utility maximization would be *required* with perfect competition across endowments for donor funds. Donor utility maximization is even required with *imperfect competition* but *complete information*, where donors oversee the endowment by, for example, serving on investment committees, as is common practice (Brown et al., 2011). To be sure, deviations from complete information are entirely plausible. But, even if the endowment environment features asymmetric information, any claim of an agency issue must be made relative to *donors* as distinct principals.[5]

[1]See, for example, Litvack et al. (1974); Tobin (1974); Black (1976); Merton (1992, 1993); Dybvig (1999); Fisman and Hubbard (2005); Swensen (2009); Constantinides (1993); Gilbert and Hrdlicka (2012); Cejnek et al. (2014a); Cejnek et al. (2014b); Brown and Tiu (2015), and other papers cited below.

[2] See, for example, Lerner et al. (2008); Dimmock (2012); Cejnek et al. (2014a); and Ang et al. (2018).

[3] His argument is best understood using the more modern language of missing markets. Whereas the government can potentially take on more risk to complete missing markets between generations, an endowment lacks the necessary taxation authority. In particular, an endowment can't pre-commit future generations to donate during hard economic times.

[4] Some papers cited above incorporate an exogenous donation stream independent of the endowment's actions. The endowment does not maximize donor utility in these papers.

[5] Put differently, if the endowment's problem were taken in isolation of the donor problem, there is no agency issue since the endowment is both principal and agent.

The current optimal endowment literature, therefore, is a bit at odds with the field of corporate finance. Commencing with the seminal 1958 theorem by Modigliani and Miller (M-M), a literature in corporate finance began analyzing a firm's optimal capital structure by including its key stakeholders—the firm's shareholders—into the model. Barring financial market frictions (e.g., asymmetric information), the M-M theorem shows that a firm's capital structure is irrelevant when shareholders are considered. Shareholders neutralize the firm's capital structure decision within their own private portfolios.[6]

Similarly, an endowment does not operate independently of its donors. Like corporate shareholders, donors give money to the endowment's sponsor (e.g., university) to receive some form of consumption, such as a private benefit (e.g., naming rights of a building) or altruism (contributing to a university's common mission). So, it would seem that the endowment portfolio problem could be relabeled as a corporate finance problem, producing a M-M type of theorem where the endowment's investment policy is irrelevant.

Indeed, suppose that donors make contributions purely for private gain (i.e., "warm glow" (Andreoni, 1998)) and this form of consumption is perfectly substitutable with the donor's other consumption. (Perfect substitution is analogous to "value additivity" in the corporate finance literature.[7]) Then, we get a striking result (Section 2): an endowment's optimal portfolio is indeterminant (neutral). Analogous to the M-M theorem, warm glow donors offset, within their own portfolios, an endowment's investment choice. This result holds even in complex settings where the endowment faces random costs and donations are impacted by economic shocks, features considered by some previous papers.

Of course, donating for a pure private return is fairly unrealistic, at least on average.[8] The endowment problem is then different from the standard firm problem in one key way: the endowment's money comes from donors making voluntary gifts to a "public good" that supports a common mission among its donors. This public good might include a passion for a university's sports teams, the shared joy of knowing that the university offers subsidies to low-income matriculates, the shared prestige of a university's contribution to basic research, or even a university's prestige as measured by the endowment size itself

[6] Stiglitz (1969) shows that the M-M theorem holds even upon relaxing many of the assumptions in the original 1958 paper.

[7] Suppose C_1 and C_2 represent streams of cash flows. Then, value additivity implies that the value of a firm (or firms), $V()$, satisfies $V(C_1 + C_2) = V(C_1) + V(C_2)$.

[8] A large literature has investigated actual giving motives to charities in general (Becker, 1976; Andreoni, 1998; Fama and Jensen, 1985; Rose-Ackerman, 1996; Fisman and Hubbard, 2005) and to education institutions, specifically (Baade and Sundberg, 1996; Clotfelter, 2003; Ehrenberg and Smith, 2003; Meer and Rosen, 2009; Butcher et al., 2013; Brown et al., 2015). In aggregate, the evidence suggests some form of "impure altruism" that is partly private consumption ("warm glow") such as increasing the chances of a child's admission to a college and public consumption ("altruism") that produces a shared benefit. For our purposes, however, we just need *some* altruism to motivate our main findings.

(James, 1990; Hansmann, 1990; Conti-Brown, 2011; Brown et al., 2014; Goetzmann and Oster, 2015; Chambers et al., 2015; Rosen and Sappington, 2016).

In 1954, Paul Samuelson published his seminal (and second-most cited) article showing that the private sector will under-provide a public good relative to the "socially optimal" level, resulting in market failure.[9] The subsequent literature interpreted Samuelson's setting as "altruistic" agents donating to a public good, resulting in "free-riding" (under-provision) in Nash equilibrium (e.g., Steinberg 1987; Andreoni 1988b). Unlike a private good (e.g., a football ticket), each dollar gifted to a public good (e.g., basic research) produces at least *some non-rivalrous* consumption simultaneously enjoyed by all donors. Free-riding in giving emerges because each donor receives utility value from the contributions of other donors even if she contributes little herself. In other words, each donor's contribution produces a *positive non-pecuniary "externality"* to other donors, resulting in a Prisoner's Dilemma in giving.

If each donor internalized this externality, she would make a larger ("socially optimal") level of gift where all donors are better off, thereby achieving a Pareto improvement. This outcome could be achieved using a Coasian contracting mechanism if there were few transaction costs to *centralizing* donor activity. It could also be achieved by force if a social planner imposed a head tax on each donor. This outcome would then be "first best" because it would *directly* achieve the socially optimal level of gifts *without distortion*.

First-best mechanisms, of course, are rarely feasible. An endowment sponsor might engage in a capital campaign to inform donors of a desired target and reduce free-riding. But the target itself is an endogenous outcome of the Nash game, reflecting free-riding. As with many public goods problems, both organizational costs and enforcement of commitments make the first-best solution challenging.[10] Instead, the endowment's sponsor must rely on an *indirect*, "second-best," *decentralized* mechanism where some distortion is required. This paper shows that the endowment's risk policy is such a mechanism, and a very powerful one, creating a new theoretical basis for endowment risk taking.

[9] Standard examples of public goods include the shared benefit of non-congested roads and the military, the shared joy of knowing that low-income people are fed, and basic scientific research that produces non-pecuniary externalities.

[10] Similarly, economists generally believe that the *private* provision of a military would lead to substantial free-riding, thereby requiring the government to use second-best taxes to fund it. In theory, without government, a private firm could refuse to provide any military services unless *everyone* in the nation participated, thereby eliminating free-riding. But such a plan would be costly to enforce. It would also lead to bargaining problems if the credibility of some people to commit to not donate (e.g., fervent religious objections) were very strong; it is not sub-game perfect for the private firm to not proceed even with some free-riding. The endowment's sponsor might even face larger challenges. Virtually everyone in a nation gains from military protection (the extensive margin), albeit to differing degrees (the intensive margin). However, identifying even the extensive margin of potential donors that benefit from, for example, enhancing a university's reputation is trickier since, for example, many alumni may not care.

Until now, a concern of free-riding has not been incorporated into the optimal endowment investment literature since, as noted earlier, this literature has not considered explicit donor utility maximization. This paper, therefore, presents a micro-based model with donor utility maximization for determining an endowment's optimal capital structure. Our model combines the key insights of Modigliani and Miller's famous theorem with Samuelson's groundbreaking insights regarding free-riding. As we show, the presence of free-riding makes the endowment's optimal investment policy very *non-neutral*.

In particular, we first show that endowment risk taking reduces free-riding by generating equilibrium "precautionary donations" of prudent donors. (All reasonable preferences exhibit prudence.) This reduction in free-riding, though, comes at a cost of the endowment taking on risk (a second-best distortion) that it would not have taken if it were not for donor free-riding. We then derive a condition—in which the absolute coefficient of prudence is sufficiently larger than the absolute coefficient of risk aversion—where it is, indeed, *optimal* for an endowment to take on this risk, that is, where the value of reducing free-riding exceeds the cost of additional risk. As the size of the donor base grows, this optimality condition quickly converges to standard DARA preferences. Because our model maximizes a donor's expected utility, this additional endowment risk taking is Pareto improving and *required* by *competition* among endowments for donations, or, more generally, required by *complete information*.[11] In fact, contrary to "asset-liability matching," it is optimal for a large endowment to take on substantial risk even if donors are very risk averse toward variations in the endowment's value.[12] This result holds even if the endowment uses high-fee investments that pay no more than cash on average.[13]

At first glance, these results seem far-fetched. However, our model only makes two key assumptions: (i) donor maximization, consistent with competition, or, more generally, complete information; and, (ii) donors are making investments into a public good that, *in part*, produces non-rivalrous consumption for other donors. In fact, with additional standard utility curvature assumptions, the optimal capital structure produced by our model is the *unique solution* compatible with *complete information* and *altruistic donors*.

Of course, the real-world *is* characterized by asymmetric information. Indeed, it is well known that the Modgliani-Miller theorem fails in its presence. Yet, the M-M theorem continues to be taught in introductory corporate finance and is one of the most cited

[11] In particular, this result is *not* driven by a "Samaritan's Dilemma" (Buchanan, 1975) moral hazard problem where donors effectively provide the endowment with put options; such an attempt to exploit donors would not be competitive.

[12] We directly compare our findings against asset-liability matching in Section 8.

[13] Of course, we are not advocating using expensive investment products. Rather, our point is simply that failing to take risk can be more expensive than existing investment products.

papers in finance precisely because it tells us where to look for theoretically valid determinants of a firm's optimal capital structure. Similarly, we recognize that endowments face agency problems (Ehrenberg and Epifantseva, 2001; Core et al., 2006; Dimmock, 2012; Gilbert and Hrdlicka, 2013; Hoxby, 2015). Brown et al. (2014), for example, shows that the length of a university president's tenure is predictive of risk taking. Still, any agency concern must be measured against the equilibrium with complete information, that is, with donor maximization. Indeed, without donor maximization, the endowment would be both principal and agent, and so any choice it makes could not produce agency concerns.

Similar to the Modigliani-Miller theorem, by focusing on the mostly friction-less case, this paper creates a comprehensive theoretical framework for the endowment investing problem inclusive of the requirements of complete information. A key difference between our contribution and the M-M theorem is that we also incorporate the free-riding problem that makes the endowment problem fundamentally different from corporate finance. Of course, as with the original M-M theorem, the real world is more complicated. But interpreting these complications is challenging without a comprehensive baseline framework.

In fact, with the donor viewed as the principal, our results reverse the conventional thinking about the relationship between endowment risk taking and principal-agent problems. A *low* level of risk taking by a large endowment indicates a principal-agent conflict in our model since this investment fails to maximize the expected utility of rational, fully-informed donors.[14] Conversely, the backlash in recent years against university endowment managers receiving large performance-based payouts, often in excess of endowment contributions toward student financial aid (e.g., Fleischer 2015),[15] could, therefore, actually *increase* this principal-agent problem.

More generally, our framework squarely puts the interpretation of an endowment's capital structure within the unifying "transactions cost" framework popularized by Merton Miller. Without transaction costs (including taxes), even information asymmetries could be addressed, rendering a firm's capital structure irrelevant. For endowments, if there were no transaction costs, both moral hazard (by endowment managers) and free-riding (by donors) could be eliminated. For example, capital campaigns are designed to

[14] A salaried endowment investment manager, for example, might take on a low level of risk for the same reason as a salaried CEO of a for-profit company: because his or her job represents a personalized, non-diversified risk. We are not, of course, ascribing intent, as a salaried endowment manager might unknowingly exploit the asymmetric information or bounded rationality of its donors. Performance-based executive stock options were created to increase CEO risk taking inside for-profit companies (Hall and Murphy, 2002; Dittmann et al., 2017).

[15] The Yale endowment, for example, pays more to its asset managers than it distributes toward supporting financial aid, which has been cited as one of the motivations for the new "endowment tax" inside of the 2017 Tax Cuts and Jobs Act. Almost three-quarters of its management fees are paid based on performance (*Ibid*).

reduce free-riding but the capital targets themselves reflect the presence of free-riding.

Section 2 presents the model. Before diving into the model's details, Section 3 considers a simple example that helps to set the stage and drive intuition for the rest of the paper. Section 4 derives the socially optimal ("first best") endowment investment policy, consistent with a hypothetical social planner who directly resolves the non-rivalrous free-rider problem. Section 5 then presents the more general and formal setting, including deriving the optimality condition noted above, where it is optimal for an endowment to take on more risk in Nash equilibrium relative to first-best. That section also shows that the second-best solution can never deliver the first-best level of expected utility. Section 6 provides additional examples. Section 7 considers the case of "impure altruism" where the motivation for giving is a mixture of warm glow and pure altruism. Section 8 presents additional discussion. It explains why our results differ from the asset-liability matching convention that is more appropriate for pension plan investments. It also explains why our results differ from a key maxim from the charitable giving literature that argues that pure altruism cannot explain large amounts of giving. This section also shows how to interpret other real-world elements within our framework, including donations being tax deductible, capital campaigns, mixed investment accounts, committed spending needs and private foundations. Section 9 concludes. Appendix B contains proofs.

The focus of this paper, like much of the past literature, is normative, i.e., about deriving the *optimal* investment policy. Nonetheless, Section 7 shows that our model naturally generates a key stylized fact, namely, the cross-sectional "size effect" where the *share* of an endowment's investment in risky assets increases in the endowment's size (NCSE (2017), Figure 3.2). This correspondence, therefore, provides an *indirect* test of the model. Appendix A provides a more *direct* test of the model's mechanism. Using confidential data of university and college endowment asset information, it shows that gifts per potential donor (proxied by the number of full-time equivalent students) *decrease* in the number of potential donors, consistent with standard Nash equilibrium giving. Moreover, gifts per donor *increase* in the share of an endowment's assets invested into risky assets, consistent with the prudence effect. The results are strongly statistically significant and robust to various controls and data slices.

2 Three-Stage Game

For exposition purposes, we present a very simple model. An endowment is funded by N identical donors, each endowed with wealth $w = 1$. Both the endowment and donors have access to the same risky and risk-free assets. The risk-free asset pays a guaranteed zero real return, $r = 0$. The risky asset with random net return $\tilde{x}$ also has a zero expected value, $E[\tilde{x}] = 0$. The risk-free asset, therefore, second-order stochastically dominates the risky asset, and so risk-averse donors should never take on risk. We can interpret the risky asset as an expensive investment that does not over-perform cash on average. Nonetheless, we show that the endowment, which competitively maximizes ex-ante donor expected utility, should optimally hold the risky asset under a new condition we derive.[16]

The timing of the model (with commitment[17]) is as follows:

Stage 1: The endowment announces its optimal investment policy in the risky asset λ that maximizes each donor's identical expected utility.

Stage 2: Each donor i picks her own optimal gift g_i. She simultaneously picks her own investment in risky assets α_i for the remainder of wealth not gifted, $1 - g_i$. A Nash (non-cooperative) game is played with other donors.

Stage 3: The risky return $\tilde{x}$ is realized.

The endowment is forward looking and so the game is solved backward, starting with the decision-making at Stage 2.

2.1 Stage 2: Donors

Each donor i donates gift g_i to the endowment and also chooses the amount α_i to invest in risky assets from the remainder of her assets, $1 - g_i$. Donor i makes these choices, conditional on the value of λ announced by the endowment at Stage 1 as well as the donation decisions of other donors, to maximize donor i's expected utility:[18]

$$EU_i\left(g_i, \alpha_i | \lambda, \overrightarrow{g_{-i}}\right) = E\left[u\left(1 + \alpha_i \tilde{x} - g_i\right)\right] + E\left[v\left(g_i + \sum_{j=1, j\neq i}^{N} g_j + \lambda \tilde{x}\right)\right]. \tag{1}$$

Here, $\overrightarrow{g_{-i}} = (g_1, ..., g_{i-1}, g_{i+1}..., g_N)$ is the vector of donations made by donors other than donor i. (Consistently, we will use the notation $\vec{g} = (g_1, ..., g_N)$ to be the vector of all donations, including donor i.) The increasing and concave functions $u(\cdot)$ and $v(\cdot)$ provide felicity over personal consumption and gifts contained in the endowment, respectively.

[16] Numerical results are discussed below for the case $E[\tilde{x}] > r = 0$, consistent with a positive equity premium. The key results remain qualitatively unchanged but become quantitatively stronger.

[17] Complete information (or competition) requires renegotiation-proof contracts in our setting, ruling out an endowment announcing a risky portfolio in Stage 1 and investing in no risk in Stage 3.

[18] In particular, donor i invests α_i into the risky asset and the remainder, $(w - g_i - \alpha_i)$, into the risk-free asset, for a gross return at Stage 3 of $\alpha_i(1 + \tilde{x}) + (w - g_i - \alpha_i)(1 + r)$, which reduces to $1 + \alpha_i \tilde{x} - g_i$ with $w = 1$ and $r = 0$, as shown in the first term on the right-hand side of equation (1). Similarly, the endowment invests λ of total gifts, $G = \sum_{j=1}^{N} g_j$, into the risky asset and the remainder into the risk-free asset, to receive $\lambda(1 + \tilde{x}) + (G - \lambda)(1 + r) = G + \lambda \tilde{x}$, which is shown in the second term.

The expression in function $v(\cdot)$ assumes that each donor only cares about the *sum* of gifts, consistent with pure altruism, and so the donor does not distinctively weigh her own gift. The endowment, therefore, provides a "public good" to donors in the tradition of Samuelson (1954), which can lead to free-riding.

2.2 Remark: A Neutrality Result with No Altruism

Instead of altruism, suppose for a moment that each donor *only* receives utility from her own gift, a "warm glow" (Andreoni (1998)) without any shared benefit to other donors. Moreover, suppose donor's gift is perfectly substitutable with her own private consumption. Then, the donor's problem (1) becomes

$$\begin{aligned} EU_i\left(g_i, \alpha_i; \lambda, \overrightarrow{g_{-i}}\right) &= E\left[u\left(1 + \alpha_i \tilde{x} - g_i + (g_i - \lambda)(1 + r) + \lambda(1 + \tilde{x})\right)\right] \\ &= E\left[u\left(1 + (\alpha + \lambda)\tilde{x}\right)\right], \end{aligned} \tag{2}$$

where, recall, $r = 0$. We drop the i subscript on α in the second equality since donors are still ex-ante identical and there is no public good. The free-riding problem, of course, vanishes. But, notice that each donor now only cares about the *total* risk, $(\alpha + \lambda)$, rather than its decomposition. As a result, the actual endowment risk policy λ, that competitively maximizes donor ex-ante utility, is now irrelevant since each donor simply neutralizes any choice of λ with an offsetting choice of α, similar to the M-M theorem. This neutrality result easily generalizes to richer economic settings, including model settings cited in Section 1 that consider shocks to the economy or endowment sponsor costs.

To be sure, program (2) is not realistic. Program (2) models giving as a pure private good, in contrast to the evidence of "impure altruism" (Section 1). Program (2) also assumes that the private return for the charitable donation is fully substitutable with other private consumption. This "value additivity" type of assumption is natural in the corporate finance setting where investors only care about cash flows and not the identity of each cash flow. It is a much stronger assumption for donations.

Still, this neutrality result emphasizes the power of allowing for donor maximization, a result that easily generalizes to even richer settings with economic or cost shocks. Models with impure altruism fall in-between and produce free-riding, albeit less free-riding, than full altruism (Andreoni, 1989). Our subsequent key theoretical results, however, only require the presence of *some* free-riding. Hence, most of the remainder of this paper considers the case of pure altruism giving shown in equation (1). But, we return to the case of impure altruism in Section 7.

2.3 Stage 1: The Endowment

The endowment fund picks its investment policy λ to maximize the sum of donor utilities,

$$\sum_{i=1}^{N} E\left[u\left(1+\alpha_i^*(\lambda)\tilde{x}-g_i^*(\lambda)\right)\right] + N\cdot E\left[v\left(\sum_{i=1}^{N} g_i^*(\lambda)+\lambda\tilde{x}\right)\right],$$

where $g_i^*(\lambda)$ and $\alpha_i^*(\lambda)$ are the equilibrium policy functions that solve the Stage-2 problem.[19] However, since donors are ex-ante identical, maximizing their sum of utilities is identical to choosing λ to maximize the ex-ante utility of a single donor, consistent with a competitive equilibrium, where endowments compete for donations:

$$\Omega\left(\lambda\right) = E\left[u\left(1+\alpha^*(\lambda)\tilde{x}-g^*(\lambda)\right)\right] + E\left[v\left(Ng^*(\lambda)+\lambda\tilde{x}\right)\right]. \tag{3}$$

2.4 Nash (Second-Best) vs. Social (First-Best) Values of λ

Denote λ^* as the value of λ that maximizes equation (3). We say that λ^* is the *second-best* solution for the endowment problem because λ^* is conditional on the equilibrium policy functions produced by the non-cooperative Nash game in Stage 2. As shown below, the second-best solution is not Pareto efficient because each donor fails to internalize the value of her contribution on the utility of the other $N-1$ donors in Stage 2.

The endowment problem outlined above looks similar to the "social planner" problem considered by Samuelson (1954) and the large subsequent public goods literature. But, there is a subtle yet important difference. While picking λ, the hypothetical social planner also gets to pick the gifts vector $\vec{g}$ when maximizing equation (3). The social planner, therefore, directly solves the gifts free-riding problem, thereby producing the first-best expected donor utility, which is Pareto efficient. Let hatted variables correspond to the social planner solution, and so $\widehat{\lambda}$ is the *first-best* solution for the endowment problem (3).

3 Setting the Stage: An Example

We present general results starting in the next section. For now, let's set the stage by considering a simple example to help drive subsequent intuition.

Suppose that each donor has separable log-log preferences: $u(\cdot) = v(\cdot) = log(\cdot)$. Moreover, suppose that the net risky return $\tilde{x}$ follows a two-point distribution that takes the values $+1$ and -1 with equal probability, $\frac{1}{2}$. The following results will be proven later.

[19] Section 5 defines the non-cooperative Nash equilibrium in more detail, but it is standard.

First, consider the centralized, socially optimal solution. Regardless of the total number of donors N, the social planner would require the endowment take no risk, $\widehat{\lambda} = 0$, since the risky asset is stochastically dominated. Instead, the social planner directly solves the free-riding problem by commanding each donor to give (or taxing them) sufficiently, producing a first-best total level of giving of $\frac{N}{2}$, which increases unbounded in N.

Now, consider the decentralized setting. Suppose that the endowment arbitrarily adopts a "no-risk rule," i.e., $\lambda = 0$. However, there is no social planner to address free-riding. Instead, donors give voluntarily in a non-cooperative setting. With $\lambda = 0$, this Nash game produces $\frac{N}{N+1}$ in total giving, which converges to just unity in N.

This "no risk" rule, however, is not second-best optimal if there are more than three donors, $N > 3$.[20] The second-best solution requires taking on some risk, $|\lambda^*| > 0$.[21] Specifically, the endowment can increase donor utility by growing the second-best $|\lambda^*|$ in the number of donors N, as plotted in Figure 1.[22] (For comparison, the first-best solution, $\widehat{\lambda} = 0$, and "no-risk rule," $\lambda = 0$, are also shown.) Growing $|\lambda^*|$ in N according to this plotted function produces $\frac{N}{4}$ in total giving. Notice that with $N > 3$:

$$\frac{N}{N+1} \text{ (i.e., "no-risk rule" total giving with } \lambda = 0)$$
$$< \frac{N}{4} \text{ (i.e., second-best equilibrium total giving, } |\lambda^*| > 0)$$
$$< \frac{N}{2} \text{ (i.e., first-best giving, } \widehat{\lambda} = 0)$$

Figure 2 shows total giving in N under the first-best, second-best and "no-risk rule" solutions. Each value of total giving for a given value of N is consistent with its respective value of λ shown in Figure 1.

Consider the first inequality above. It reflects a reduction in free-riding when risk taking $|\lambda|$ increases in N. Since donor expected utility is being maximized, this change is also Pareto improving. Of course, at first glance, this result seems absurd since the risky return $\tilde{x}$ produces no commensurate risk premium. Put differently, this endowment is simply introducing *mean-preserving* risk. How could "junk" variance be optimal since log-utility donors are risk averse to changes in the endowment's value? The answer is that with positive *prudence*, even a mean-preserving increase in risk *effectively pre-commits* each donor to

[20] We prove in Section 6 that $N > 3$ is required for the general "optimality condition," derived in Section 5, to hold for log utility. Intuitively, endowment risk taking is costly since the risky asset is dominated. But, the benefit of taking on some risk increases in the size of the free-riding problem which, itself, increases in N. For log utility, only $N > 3$ is required for the benefit from some risk taking to exceed the cost.

[21] The model allows for shorting and so $\lambda \neq 0$ represents risk taking.

[22] This rule is derived algebraically in Section 6.

Figure 1: Endowment Risk: First-best ($\widehat{\lambda} = 0$), Second-best ($|\lambda^*|$), and No Risk ($\lambda = 0$)

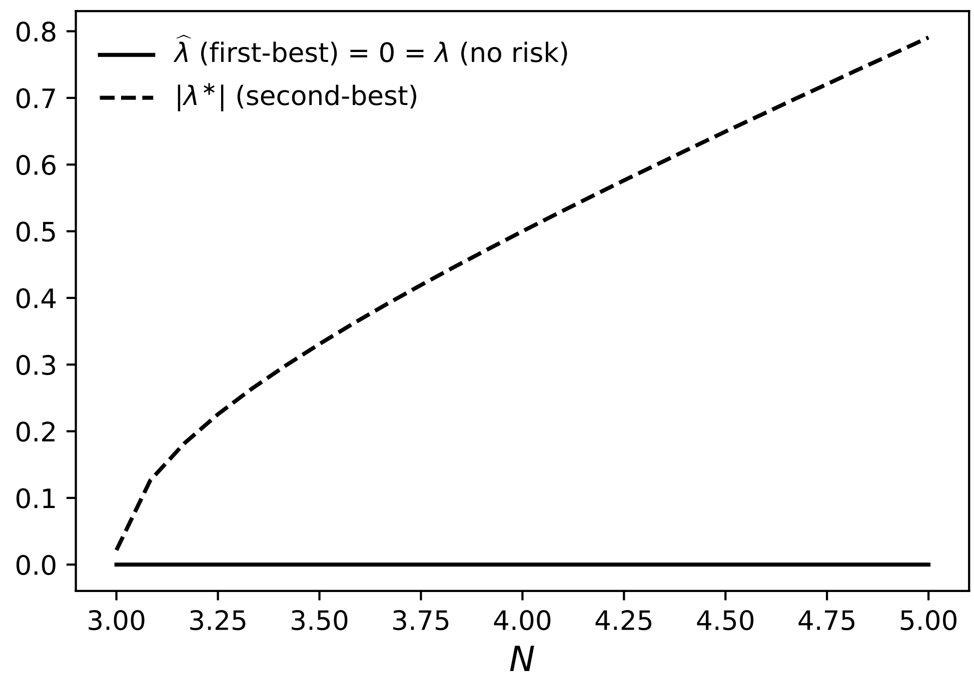
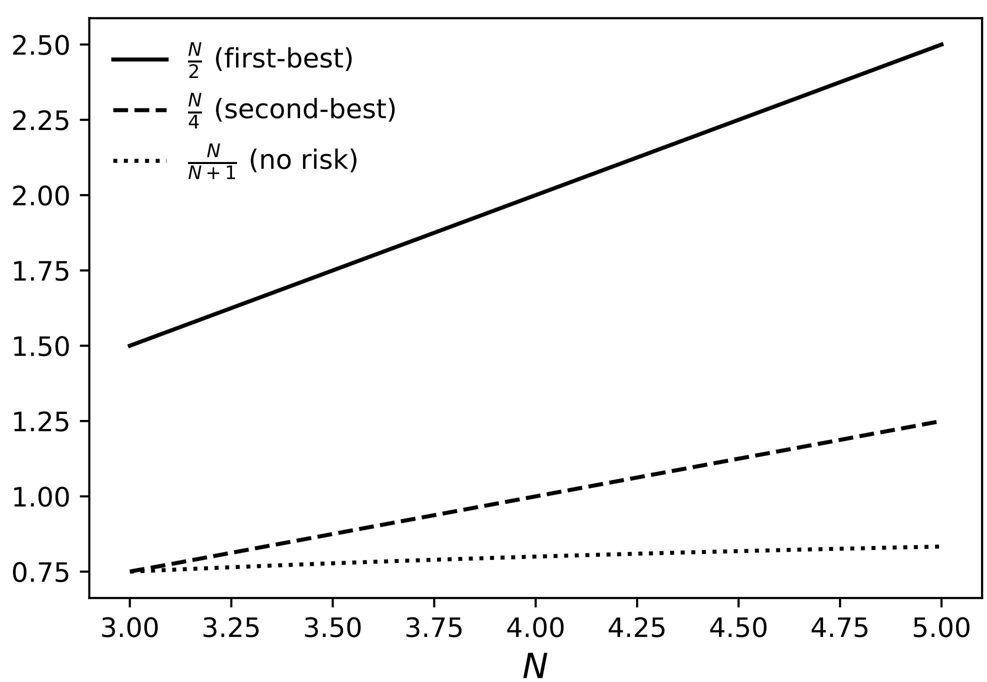


Figure 2: Total Giving: First-Best, Second-Best and No Risk



Notes for Figures 1 and 2: As in Section 6.3, $u(w) = v(w) = log(w)$ and net return to stocks, $\tilde{x}$, take values in set $\{-1., +1.0\}$ with equal probability.

give a larger, precautionary donation. More specifically, under the optimality condition derived in Section 5, the marginal benefit of reducing free-riding exceeds the marginal cost of additional "junk" risk, each measured at the point where the endowment initially takes no risk ($\lambda = 0$). Moreover, this second-best solution is required by competition (or complete information). Under the normative baseline where principal-agent conflicts do not exist by definition, rational donors, will reward (donate to) endowments that take on this risk and walk away from endowments that do not.

The second inequality, however, shows that decentralized second-best mechanism with endowment risk taking still produces a level of giving smaller than the first-best mechanism where free-riding is eliminated by force without risk taking. It is not efficient for the endowment to *fully* eliminate free-riding in second best. In fact, if the endowment got "greedy" and tried to generate more total giving by increasing $|\lambda|$ in N faster than the value of $|\lambda^*|$ shown in Figure 1, donors would walk away (under competition) or fire the endowment manager (with complete information and managerial control).

In sum, it is second-best optimal to grow endowment risk taking $|\lambda|$ in N. However, we can go one-step further: in second-best, the *share* of the endowment invested in the risky investment, $\frac{|\lambda^*|}{\left(\frac{N}{4}\right)}$, increases in the size of the endowment, $\frac{N}{4}$. Intuitively, as noted above, the first-best level of giving, $\frac{N}{2}$, grows *unbounded* in N. But, the level of giving under the "no-risk rule" ($\lambda = 0$), $\frac{N}{N+1}$, converges to *unity*. Hence, the size of the *free-riding problem*—the difference between these two quantities—grows unbounded in N. A growing endowment, therefore, must take on increasing *share* of risk in second best to mitigate the growing amount of free-riding. The second-best level of total giving, $\frac{N}{4}$, although less than first best, at least now grows unbounded in N. As we show later, it is even optimal for a large endowment to take on a large risk share even if donors are much more risk averse to changes in the endowment's value than characterized by log utility.

4 The Social (First-Best) Optimum

In the social optimum, the endowment and individual donors don't need to take a position in the dominated risky asset. Individual gifts are also symmetric:

Theorem 1. $\widehat{\lambda} = 0$, *$\widehat{\alpha}_i = 0\ \forall i$, with equal individual gifts $\widehat{g}_i = \widehat{g}$ that solves*

$$u'\left(1 - \widehat{g}\right) = Nv'\left(N\widehat{g}\right). \tag{4}$$

Intuitively, since the social planner directly controls individual gifts, the planner does not need to inefficiently distort risk taking by setting $\widehat{\lambda}$ to a non-zero value. The concavity of $u(\cdot)$ and $v(\cdot)$ implies that this solution is also unique. In the proof to Theorem 1 in Appendix B, we provide an extension that shows how first-best gifts $\widehat{g}$ vary in N.[23]

5 Nash (Second-Best) Equilibrium

Without the social planner, the presence of free-riding in the second-best setting, however, fundamentally changes the optimal risk allocation. We now solve for the more general non-cooperative solution (Nash) for the game outlined in Section 2 and then compare it against the social optimum solution (Section 4).

5.1 Stage 2: Nash Equilibrium Gifts

Starting first with the donor game in Stage 2, the definition of a Nash equilibrium is standard. Each donor i picks the tuple (g_i, α_i) that maximizes her problem (1), given the gifts made by other donors, $\overrightarrow{g_{-i}}$.[24] A Nash equilibrium is the vector of gifts $\vec{g}^*$ and the vector of personal risk taking $\vec{\alpha}^* = (\alpha_1, ..., \alpha_N)$ that maximizes the donor problem (1), $\forall i$.

Theorem 2. *The Nash equilibrium in the Stage-2 donor game is unique with $\alpha_i^* = 0$, $\forall i$, and equal gifts, $g_i^*(\lambda) = g^*(\lambda)$, conditional on λ, that solves*

$$u'(1 - g^*(\lambda)) = E\left[v'(Ng^*(\lambda) + \lambda\tilde{x})\right]. \tag{5}$$

In words, each donor wants to invest all of her personal (non-gifted) wealth, $1 - g_i$, into the risk-free asset, which, recall, pays the same expected return as the risky asset. The equilibrium is unique and symmetric, producing identical gifting policy functions, $g^*(\lambda)$.

[23] Let $R^v(w) = -\frac{w \cdot v''(w)}{v'(w)}$ denote the Arrow-Pratt coefficient of relative risk aversion for felicity function v over the public good. Then, $\widehat{g}$ is [increasing in, independent of, decreasing in] $N \iff R^v(N\widehat{g})\ [<, =, >]\ 1$. Intuitively, as the value of N increases, the positive externality of giving by one donor extends to more donors, thereby increasing the social benefit from the public good. At the same time, as N increases, the total endowment wealth, $N\widehat{g}$, increases, lowering the marginal utility from more gifts. Rewrite the condition as $v'(N\widehat{g}) = -N\widehat{g} \cdot v''(N\widehat{g})$. The left-hand side represents the positive externality effect while the second term reflects the wealth effect. If the first [second] effect dominates then $R^v(N\widehat{g}) < [>] 1$. The positive externality effect is absent in the Nash problem considered below, leading to an under-provision of the public good.

[24] Each donor only needs to know the total size of giving, which is standard in public good games. Even the Nash equilibrium concept can be replaced with a Perfect Bayesian Equilibrium under reasonable conditions, but this modification isn't necessary since total gifts are generally observable. For example, a university typically provides updated information about gifts during a capital campaign.

5.2 Stage 2: Comparative Statics of $g^*(\lambda)$

We now investigate how the equilibrium level of giving, $g^*(\lambda)$, changes as the number of donors, N, increases and as endowment risk moves away from its socially optimal value, $\widehat{\lambda} = 0$. Let $P^v(\cdot) \equiv -\frac{v'''(\cdot)}{v''(\cdot)}$ denote the coefficient of absolute prudence (Kimball, 1990) for felicity function v over the public good. Then:

Theorem 3. *The following comparative statics hold:*

1. $g^*(\lambda)$ *is decreasing in* N.
2. $P^v(Ng^*(0)) > 0 \iff \lambda = 0$ *is a* *local minimum* *of* $g^*(\lambda)$.
3. $P^v(\cdot) > 0 \implies g^*(\lambda)$ *is increasing in* $|\lambda|$ *and* $\lambda = 0$ *is the* *global minimum* *of* $g^*(\lambda)$.

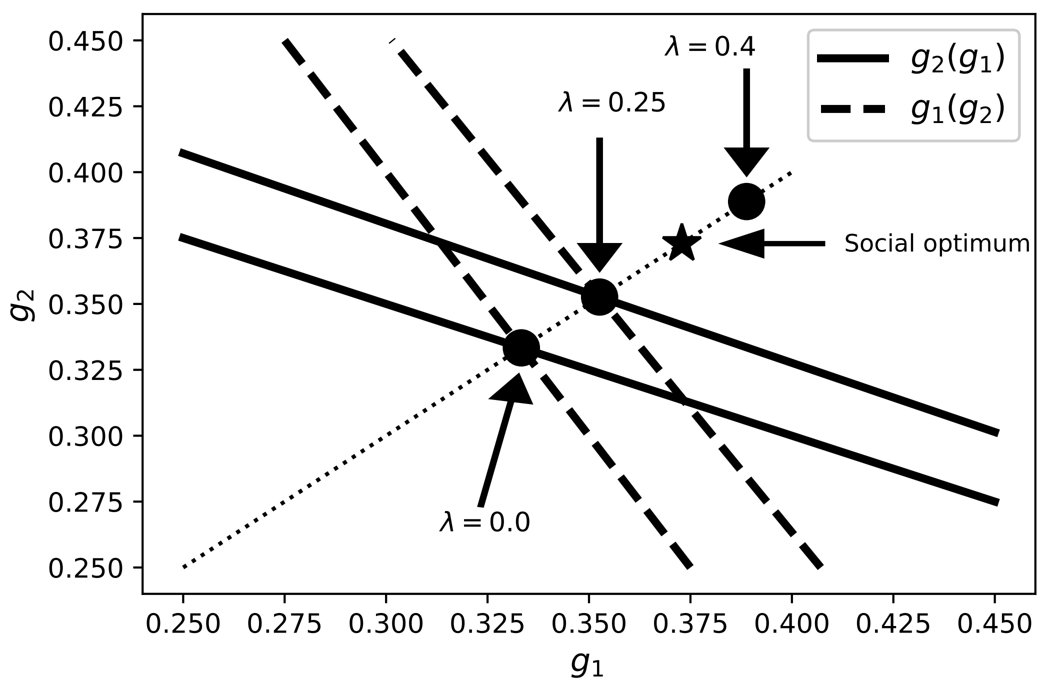
Part 1 is the standard result from the previous literature, where individual giving falls in the number of donors, which, in our model setting, is consistent with a *fixed* value of λ, including no risk taking, $\lambda = 0$. But, with positive prudence, some endowment risk taking will increase the Nash equilibrium level of giving relative to no risk taking $\lambda = 0$ (Part 2). In fact, $\lambda = 0$ produces the global minimum level of giving (Part 3).

The role of prudence can be explained with equation (5). As the endowment changes λ from zero, it introduces risk in the second term, $E[v(\cdot)]$. Donor i makes a "precautionary donation" with a larger gift, g_i, similar to "precautionary savings" with uninsurable risk (Kimball, 1990). Positive prudence is a standard assumption. DARA preferences, for example, is a sufficient (but not necessary) condition for positive prudence.

Example 1. *Consider the case of two donors ($N = 2$). Figure 3 plots the gift reaction functions of donors $i = \{1,2\}$ in gift (g_1, g_2) space with $\alpha_i = 0$. Donors have constant relative risk averse felicity (see Figure 3 notes). For the set of reaction functions labeled as "$\lambda = 0.0$" the value λ is set to zero and the intersection of the reaction functions represents the corresponding Nash equilibrium level of gifting. (By symmetry, $g_1^* = g_2^*$, all equilibria lie on the dotted 45-degree line.) The gift level lies below the indicated social optimum value. When λ is increased to 0.25, equilibrium gifts move closer to the social optimum, reducing some free-riding. When $\lambda = 0.40$, equilibrium gifts "over-shoot" the social optimum value, a point we discuss in more detail below.*

While this paper is normative in nature, Appendix A directly tests the predictions of Theorem 3 using data of U.S. university and college endowments and associated school characteristics. To summarize, gifts per potential donor (proxied by the number of full-time equivalent students) *decrease* in the number of potential donors, consistent with Nash

Figure 3: Gifts Reaction Functions for Two Donors ($N = 2$)



Notes: Shows reaction functions and corresponding equilibria for two donors $i \in \{1,2\}$, $\lambda \in \{0.0, 0.25, 0.40\}$, $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$, $\gamma = 4$, and $\alpha_i = 0$. Net return to stocks, $\tilde{x}$, take values in set $\{-1., -0.05, 0.05, 1.0\}$ with probabilities $\{0.1, 0.4, 0.4, 0.1\}$. At $\lambda = 0$, the reaction functions are linear since $(w_0 - g_i) = (g_1 + g_2)$ at optimum. For $\lambda > 0$, reaction functions are slightly nonlinear.

equilibrium giving (Theorem 3, Part 1). Moreover, gifts per donor *increase* in the share of an endowment's assets invested into risky assets, consistent with prudence (Theorem 3, Part 3). These results are strongly statistically significant and robust to various controls and data slices. This type of quantitative test is superior to trying to interpret qualitative language provided in endowment reports related to their long-term investment pools, where multiple interpretations could be made. Moreover, Theorem 3's predictions do not literally require that endowment managers understand the mechanism; rather, endowments taking on more risk receive larger individual gifts.

5.3 Stage 1: Optimal λ^*

But, is it *efficient* for the endowment to take on risk? Without free-riding, Theorem 1 shows that the optimal endowment level of risk taking is zero, $\widehat{\lambda} = 0$. With free-riding, Theorem 3 shows that endowment risk taking reduces free-riding. So, under what condition does the benefit from less free-riding exceed the cost from taking on dominated risk?

In Stage 1, the endowment solves for its optimal investment policy λ^* by maximizing problem (3), given the Nash gift policy functions $g^*(\lambda)$ determined in Stage 2. Notice that *if* the endowment fails to take on any risk ($\lambda = 0$) then Nash gifts are implicitly determined simply by the relationship,

$$u'(1-g^*) = v'(Ng^*). \tag{6}$$

Compare this relationship to equation (4) in Theorem 1 that derives the social optimal level of giving. The only difference is the presence of N that multiples the marginal utility of the public good on the right-hand side in equation (4). Intuitively, since the public good (endowment) is non-rivalrous in consumption, the socially optimal solution linearly increases the marginal utility of the public good. It is easy to see that $g^*(\lambda = 0) < \widehat{g_i}^*$, the now-familiar public goods free-riding problem first identified by Samuelson (1954).[25]

For exposition, suppose $u(\cdot) = v(\cdot)$. (Footnotes and Appendix B generalize the results to $u(\cdot) \neq v(\cdot)$). Denote $A(\cdot) \equiv -\frac{u''(\cdot)}{u'(\cdot)}$ and $P(\cdot) \equiv -\frac{u'''(\cdot)}{u''(\cdot)}$ as the coefficients of absolute risk aversion and absolute prudence, respectively. Recall each donor has wealth $w = 1$ before making a gift, and so the donor count N equals total wealth.

Theorem 4. *$\lambda = 0$ is a local <u>minimum</u> solution to the endowment problem that maximizes equation (3) $\iff P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right)$ when $u(\cdot) = v(\cdot)$.[26] Hence, $|\lambda^*| > \widehat{\lambda} = 0$.*

[25] Indeed, the expression shown in Theorem 1 is commonly known as the "Samuelson condition."

[26] As shown in the proof to Theorem 4 in Appendix B, for the general case where $u(\cdot)$ and $v(\cdot)$ might be

Theorem 4 shows that the endowment's optimal investment policy λ^* is to take on positive risk (by longing or shorting the risky asset[27]) under a new condition where the felicity's level of absolute prudence is sufficiently larger than its absolute risk aversion. For a finite value of N, this condition is slightly stronger than the relationship between prudence and risk aversion that is equivalent to standard Decreasing Absolute Risk Aversion (DARA) preferences, i.e., $P\left(\frac{N}{N+1}\right) > A\left(\frac{N}{N+1}\right)$.[28] However, this new condition quickly converges to DARA in N since the multiplier, $\frac{N+1}{N-1}$, converges to 1.

To understand this relationship, first consider the term $\frac{N}{N+1}$ inside of the $P(\cdot)$ and $A(\cdot)$ operators in Theorem 4. At $\lambda = 0$, equation (6) with $u(\cdot) = v(\cdot)$ implies that the individual Nash equilibrium gift $g^*(0) = \frac{1}{N+1}$. Hence, total gifts equal $\frac{N}{N+1}$ at $\lambda = 0$.

At large N, therefore, the Theorem-4 condition simply requires DARA preferences, $P\left(\frac{N}{N+1}\right) > A\left(\frac{N}{N+1}\right)$, at the total levels of gifts with no endowment risk taking, $\lambda = 0$. Intuitively, as shown in the Theorem 3, positive prudence, $P(\cdot) > 0$, implies that endowment risk taking, $|\lambda^*| > 0$, reduces free-riding, thereby capturing the marginal benefit of additional risk taking. However, as shown in Theorem 1 risk taking is not optimal without free-riding. The coefficient of relative risk aversion, $A(\cdot)$, captures the marginal cost of additional risk. It is optimal, therefore, for the endowment to take on risk if the marginal benefit, $P\left(\frac{N}{N+1}\right)$, exceeds the marginal cost, $A\left(\frac{N}{N+1}\right)$, calculated at the level of gifts with no risk taking. These results generalize to the case where N is not large and $u(\cdot) \neq v(\cdot)$, although with some additional notation.[29] Simulation code, made available

different, the necessary and sufficient condition is:

$$(N-1)\,P^v\left(Ng^*(0)\right) > A^u\left(1-g^*(0)\right) + NA^v\left(Ng^*(0)\right),$$

where P^v is the absolute prudence of v. A^u and A^v are the absolute risk aversions for u and v, respectively. See Footnote 29 for additional discussion. We focus on the equality case in the text for exposition, thereby dropping the superscripts on P and A.

[27] Our characterization of risky returns does not distinguish between long or short positions. Any deviation from zero represents risk taking.

[28] DARA is "a very intuitive condition" (Eeckhoudt et al., 2005) and includes commonly-used preferences such as Constant Relative Risk Aversion. DARA is necessary and sufficient for the absolute amount of risk taking to increase in wealth, along with many other standard properties (Gollier, 2004).

[29] At smaller N, the multiplier, $\frac{N+1}{N-1}$, is more relevant. Consider the more general Theorem-4 condition from footnote 26, where we multiply each side by $\frac{1}{2}E\left[\tilde{x}^2\right]$:

$$(N-1)\,\frac{1}{2}E[\tilde{x}^2]P^v\left(Ng^*(0)\right) > \frac{1}{2}E[\tilde{x}^2]A^u\left(1-g^*(0)\right) + N\frac{1}{2}E[\tilde{x}^2]A^v\left(Ng^*(0)\right).$$

The left-hand side, which is the approximation for the precautionary equivalent premium (Kimball, 1990) for small, zero-mean risks, is exactly equal to the willingness of $N-1$ donors to increase their precautionary donations in response to additional risk taking by the endowment in Nash equilibrium, i.e., the benefit of reducing free-riding to an individual donor. The right-hand side, which is the Arrow-Pratt approximation for the risk premium, is exactly equal to the premium required by an individual donor to take on the additional risk, i.e., the cost of additional risk taking. More specifically, the right-hand side is equal to the online, considers the case of $E\left[\tilde{x}\right] > r = 0$, consistent with a positive equity premium.[30]

5.4 Comparison Against the Social Equilibrium

The Nash (second-best) expected utility, however, calculated at the optimal endowment investment policy $\Omega\left(\lambda^*\right)$ with $|\lambda^*| > 0$, can never produce the socially-optimal (first-best) level of expected utility. At only a slight abuse of notation, denote $\Omega_{SO}\left(\lambda\right)$ as as the value of expected utility $\Omega\left(\lambda\right)$ in equation (3), but where the social planner picks λ *and* individual gifts in Stage 1. Then:

Theorem 5. *Under the Theorem-4 condition,* $\Omega_{SO}\left(\widehat{\lambda} = 0\right) > \Omega\left(\lambda^*\right) > \Omega\left(\lambda = 0\right)$*, where* $|\lambda^*| > 0$*.*

The first inequality shows that the socially optimum solution, where social planner directly picks gifts and the endowment takes no risk, produces larger donor expected utility than the second-best Nash optimum where the only available instrument is for the endowment to take on investment risk. (This inequality does not require the Theorem-4 condition.) Once in the second-best setting, the second inequality shows that positive risk taking is optimal under the Theorem-4 condition, as previously proven in Theorem 4.

Let's return to Example 1. The case of $\lambda = 0.40$ in Figure 3 shows that the Nash equilibrium level of gifts can over-shoot the social optimum level. Indeed, there exists a value of λ that fully eliminates free-riding. Importantly, however, this value of λ is generally *not* second-best optimal, as it does not maximize the Stage-1 endowment problem (3). Intuitively, fully solving the gifts free-riding problem in the second-best Nash game would require distorting the endowment's dominated risk investment λ too much. In fact, in this example, recall that $\gamma = 4$ and $N = 2$ ($N = 2$ allowed us to plot the reaction functions in two dimensions in Figure 3). As proven in Section 6, with $\gamma = 4$, there must be at least 10 donors ($N \geq 10$) for the Theorem-4 condition to be satisfied with HARA utility (of which Constant Relative Risk Aversion is a special case). In fact, the second-best level of endowment risk taking in Example 1 is $\lambda^* = 0$. In words, the endowment should not take on *any* risk since, with just two donors, the value of reducing any free-riding is outweighed by the dominated risk costs associated with an even small investment.

direct cost associated with more risk taking by the endowment as a whole, $N\frac{1}{2}E[\tilde{x}^2]A^v\left(Ng^*\left(0\right)\right)$, and in the donor's "private wealth", $\frac{1}{2}E[\tilde{x}^2]A^u\left(1 - g^*\left(0\right)\right)$. Even though $\alpha^* = 0$, the envelope theorem implies that, in effect, a donor's private wealth, up to the individual gift level, $g^*\left(0\right)$, is exposed to additional risk.

[30] The key results presented herein remain qualitatively unchanged but generally become quantitatively stronger. The reason is that *first* term in equation (5) now contains some risk since $\alpha^* > 0$. This risk creates a competing prudence effect for the endowment, which must increase λ even more to offset.

Now assume $N = 10$. Figure 4 plots the value of expected utility $\Omega(\lambda)$ from equation (3), where the equilibrium Stage-1 policy functions $\alpha^*(\lambda)$ and $g^*(\lambda)$ are Nash. Notice that the expected utility peaks at a value λ greater than zero but falls at larger values of λ. Increasing λ above zero reduces free-riding, thereby increasing donor expected utility. But, raising λ too much, reduces donor utility by distorting risk taking too much. In sharp contrast, Figure 5 shows the donor expected utility for the same donor problem where the social planner can directly pick gifts. Notice that expected utility now peaks at $\lambda = 0$, consistent with Theorem 1. In the first-best setting, there is no need to take on otherwise inefficient risk taking since the social planner can directly solve free-riding.

6 Examples

We now present three examples, where $u(\cdot) = v(\cdot)$, starting with very general HARA preferences before narrowing, while proving the key derivations presented in Section 3.

6.1 HARA Utility

Consider the HARA class of felicity functions,

$$u(w) = \zeta\left(\eta + \frac{w}{\gamma}\right)^{1-\gamma},$$

on the domain $\eta + \frac{w}{\gamma} > 0$. The first three derivatives are:

$$\begin{aligned} u'(w) &= \zeta\frac{1-\gamma}{\gamma}\left(\eta + \frac{w}{\gamma}\right)^{-\gamma}, \\ u''(w) &= -\zeta\frac{1-\gamma}{\gamma}\left(\eta + \frac{w}{\gamma}\right)^{-\gamma-1}, \text{ and,} \\ u'''(w) &= \zeta\frac{(1-\gamma)(1+\gamma)}{\gamma^2}\left(\eta + \frac{w}{\gamma}\right)^{-\gamma-2}. \end{aligned}$$

We naturally assume $\zeta\frac{1-\gamma}{\gamma} > 0$ such that $u'(w) > 0$ and $u''(w) < 0$.

The first-best (socially optimal) investment policy is, of course, $\widehat{\lambda} = 0$ (Theorem 1). The first-best individual gift level is

$$\widehat{g} = \frac{(N)^{1/\gamma} + \left((N)^{1/\gamma} - 1\right)\gamma\eta}{(N)^{1/\gamma} + N}. \tag{7}$$

Figure 4: Expected Utility: Nash

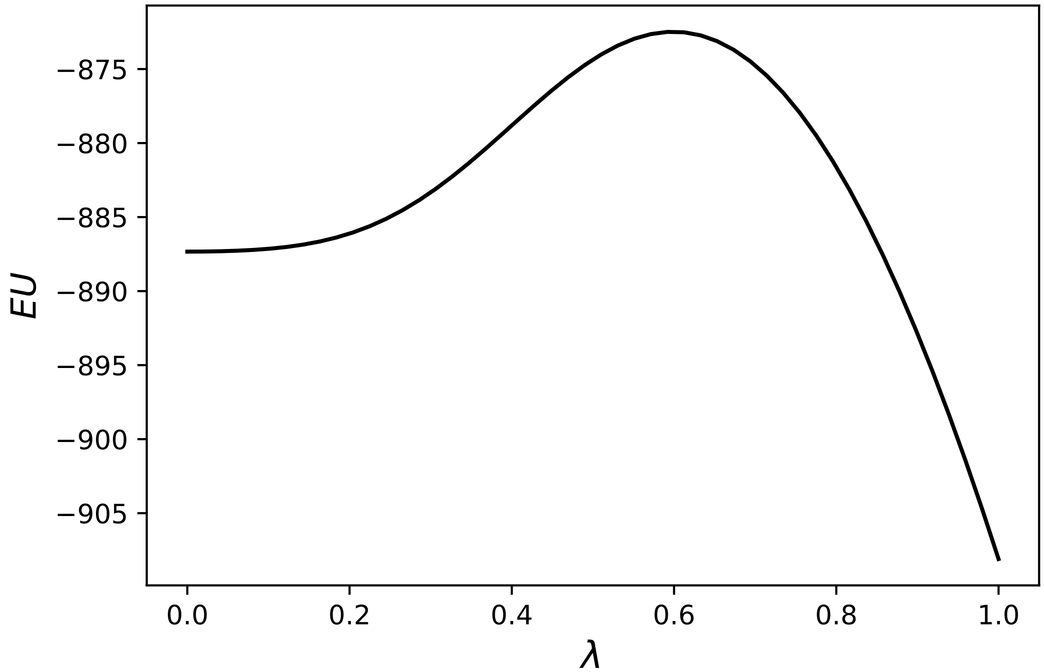
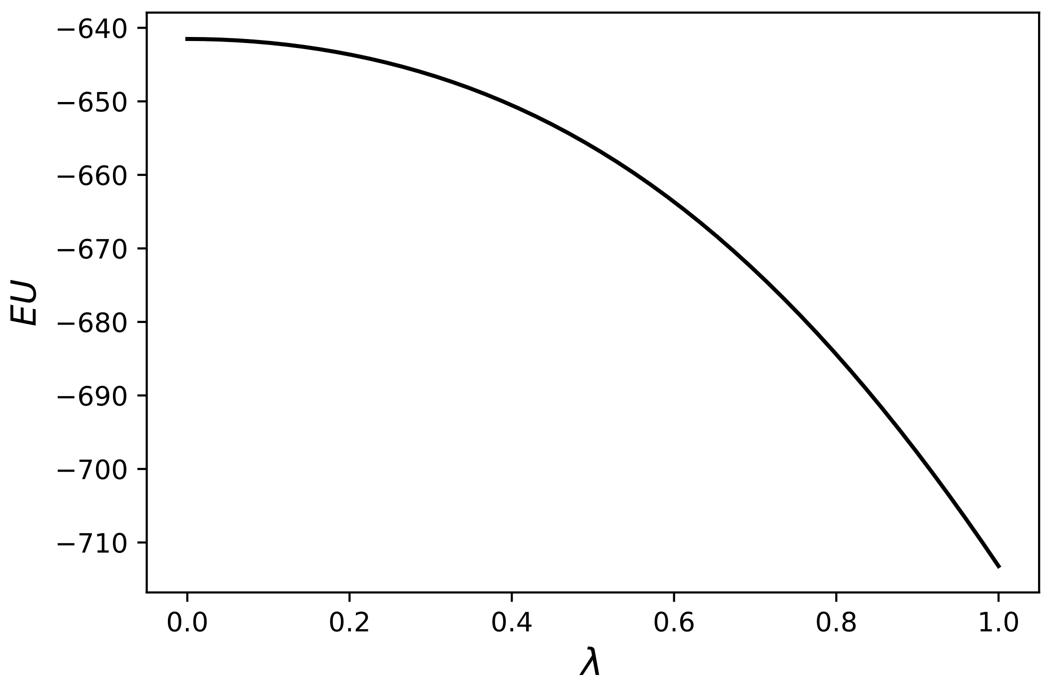


Figure 5: Expected Utility: Social Optimum



Notes for Figures 4 and 5: $N = 10$, $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$, and $\gamma = 4$. Net return to stocks, $\tilde{x}$, take values in set $\{-1., -0.05, 0.05, 1.0\}$ with probabilities $\{0.1, 0.4, 0.4, 0.1\}$.

The coefficients of absolute risk aversion and prudence are given by

$$A(w) = \left(\eta + \frac{w}{\gamma}\right)^{-1}, \text{ and,}$$
$$P(w) = \frac{1+\gamma}{\gamma}\left(\eta + \frac{w}{\gamma}\right)^{-1},$$

respectively. Then the Theorem-4 condition,

$$P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right),$$

is equivalent to

$$0 < \gamma < \frac{N-1}{2}. \tag{8}$$

With the value $\gamma = 4$, as used in Example 1, it takes just 10 donors for it to be optimal for the endowment to take on investment risk, $|\lambda^*| > 0$. Even smaller values of N are required at smaller values of γ (less concavity). At large N, this condition simply becomes $0 < \gamma < \infty$, which is equivalent to DARA preferences.

6.2 CRRA Utility

For the HARA subset of constant relative risk aversion (CRRA) felicity functions ($\eta = 0$), the first-best (social optimally) gift level shown in equation (7) simplifies to:

$$\widehat{g}\left(\widehat{\lambda} = 0\right) = \frac{N^{1/\gamma}}{N^{1/\gamma} + N} \tag{9}$$

To determine the Theorem-4 condition, let $u(\cdot)$ and $v(\cdot)$ take on potentially different relative risk aversion parameters, γ_u and γ_v. Equation (8) is then replaced by:

$$0 < \gamma_u + \gamma_v < N - 1 \tag{10}$$
$$0 < \gamma_v. \tag{11}$$

Equations (10) and (11) indicate that the more linear the preferences, the lower the value N required for $|\lambda^*| > 0$. These conditions support flexible preferences.

Remark (Quasi-linear utility)**.** *Preferences can be quasi-linear. In the extreme, preferences over non-charitable consumption can be linear, $\gamma_u = 0$, and preferences over charitable consumption* *can be arbitrarily close to linear, $\gamma_v \to 0^+$. If $\gamma_v = 0$ then it would be socially optimal for each donor to donate all of her wealth above a small non-charitable value.*

The model also accommodates nearly "deterministic" planning goals.

Remark ("Deterministic" future expenses)**.** *If the endowment's sponsor's future spending goals are fully fixed ("deterministic"), then $\gamma_v = \infty$. In this case, condition (10) cannot hold for any finite value of N, and so the optimal endowment risk is zero, $\lambda^* = 0$. However, $\gamma_v = \infty$ also implies that non-charitable consumption, $1 - g_i$, converges to zero even in Nash equilibrium. More realistically, suppose that future spending goals by the endowment's sponsor are at least somewhat flexible. Then, the value of γ_v is finite and there exists a finite value of N where endowment risk taking is optimal, $|\lambda^*| > 0$. For example, suppose $\gamma_u = 4$ and $\gamma_v = 20$, suggesting very high risk aversion to the endowment's asset value falling short of expectation. Then, an endowment with $N \leq 25$ donors optimally takes no risk while an endowment with $N \geq 26$ does.*

6.3 Log Utility with a Two-Point Risk Distribution

Now consider the CRRA subset of log felicity ($\gamma_u = \gamma_v = 1$). By equation (9), the first-best level of gifts is simply

$$\widehat{g}\left(\widehat{\lambda} = 0\right) = \frac{1}{2}.$$

To compute the corresponding second-best Nash equilibrium, the Theorem-4 condition is satisfied for all $N > 3$ by equation (8). We can also obtain analytic solutions for individual gifts of the Nash game by specifying a distribution for $\tilde{x}$. Suppose $\tilde{x}$ follows a two-point distribution that takes the values $+1$ and -1 with equal probability, $\frac{1}{2}$. Then, the Nash equilibrium donation is

$$g^*(\lambda) = \frac{N + \sqrt{N^2 + 4N(N+1)\lambda^2}}{2N(N+1)}. \tag{12}$$

Equation (12) implies that *if* λ were mistakenly set to zero (no endowment risk taking), then $g^*(\lambda = 0) = \frac{1}{N+1}$, which is less than the social optimal value of $\frac{1}{2}$, with $N > 1$. Individual Nash gifts converge to zero in N while total gifts $\frac{N}{N+1}$ converges to just unity. But, the endowment can do better by choosing the second-best λ to grow with N:

Example 2. *Suppose felicity is log and let $\tilde{x}$ follow a two-point distribution that takes the values $+1$ and -1 with equal probability. Then, $|\lambda^*| = \frac{\sqrt{N(N-3)}}{4}$ and $g^*(\lambda^*) = \frac{1}{4}$.*

Notice that the individual second-best donation is now a constant $\frac{1}{4}$. So, total donations $\frac{N}{4}$ rise in N but are still below the first-best level $\frac{N}{2}$. Setting $|\lambda^*|$ at a larger value than shown in Example 2 would generate generate larger gifts (Theorem 2) but cause too much risk distortion, reducing donor expected utility (Theorem 4).

6.4 CARA Utility

Our final example tests the boundary of the Theorem-4 condition, which, recall, is sufficient *and* necessary for the endowment to take on risk in the Nash game, $|\lambda^*| > 0$.

Consider the case of Constant Absolute Risk Aversion (CARA) felicity:

$$u(w) = -e^{-\gamma w}$$

The socially optimal level of individual gifts are:

$$\widehat{g}(\widehat{\lambda} = 0) = \frac{1}{N+1}\left(1 + \frac{\ln N}{\gamma}\right).$$

For the Nash game, to obtain an analytic solution, assume that $\tilde{x}$ is normally distributed with expectation 0 and and variance σ^2. The Nash gift policy function is

$$g^*(\lambda) = \frac{1}{N+1}\left(1 + \frac{1}{2}\gamma\sigma^2\lambda^2\right). \tag{13}$$

Like the CRRA example considered above, *if* λ were set to zero, then individual gifts $g^*(\lambda = 0)$ fall to zero in N and the value of the total gift, Ng^*, approaches the value of just unity. Moreover, individual gifts are below the socially optimal level of gifts.

However, CARA presents a problem not previously found with CRRA. In particular, CARA felicity implies that $P\left(\frac{N}{N+1}\right) = \gamma = A\left(\frac{N}{N+1}\right) < \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right)$. Hence, CARA violates the Theorem-4 condition, even if "just barely" at large N. As a result, in Stage 1, it is no longer optimal for the endowment to take on more risk to increase individual gifts.

Example 3. *Suppose felicity takes the CARA form and let $\tilde{x}$ follow a normal distribution with expectation* 0 *and variance σ^2. Then, $\lambda^* = 0$ and $g^*(\lambda^*) = \frac{1}{N+1}$.*

Of course, it is important to remember that while CARA felicity is popular for producing analytic solutions, it also implies an implausible attitude toward risk aversion discarding the usual Inada condition. So, CARA predicts that each coauthor of this paper would hold the same *dollar* amount in risky assets as Bill Gates, by shorting the risk-free asset by, in our individual cases, quite sadly, a nearly identical amount!

7 The Size Effect

The main focus of this paper is normative in nature, that is, on deriving *optimal* endowment risk taking rather than attempting to explain *actual* practice. Still, we now show that our model produces a key empirical relationship between endowment size and optimal risk taking—the cross-sectional "size effect"—where risk taking increases in endowment size (NCSE (2017), Figure 3.2), including liquidity risk by the largest endowments (ibid.). However, contrary to conventional wisdom, a large endowment optimally takes on substantial risk in our model even *without* the presence of fixed costs (e.g., dedicated asset managers) and even if it does not have access to unique asset classes that produce superior expected returns. Conversely, it is also optimal for *smaller* endowments to take on less risk even in the presence of modern outsourced "endowment style" turnkey investment solutions that are supposed to absorb these fixed costs and provide access to superior returns. We first consider our baseline model with pure altruism and then show how the results extend to the case of "impure altruism."

7.1 Pure Altruism: Log Utility

Denote $G(\lambda^*) = \vec{g}(\lambda^*) \cdot \vec{1}$ as sum of all gifts in Nash equilibrium, equal to the size of the total endowment. The ratio $\frac{\lambda^*}{G(\lambda^*)}$ gives the second-best *optimal risk share*, that is, the optimal share of the total endowment invested into risky assets.

For the two-point log utility example in Section 6.3, the optimal risk share is

$$\frac{|\lambda^*|}{G^*(\lambda^*)} = \frac{|\lambda^*|}{Ng^*(\lambda^*)} = \frac{\sqrt{N(N-3)}/4}{N/4} = \sqrt{1-3/N},$$

where, recall, $N > 3$. Notice that the optimal risk share increases in N, converging to 1 in N. Intuitively, Figure 2 shows that the *gap* between the first-best total level of giving (solid line) and the constrained no-risk ($\lambda = 0$) total level of giving (dotted line) increases in N. The second-best strategy, therefore, is to not only increase the value $|\lambda|$ in N but to increase its value *faster* than the size of the endowment G^* itself.

Figure 6 plots the endowment's second-best optimal risk share for the two-point log utility case (γ = 1.0). For comparison, additional CRRA values ($\gamma_u = \gamma_v = \gamma$ = 2.0, 4.0) are also shown, consistent with greater risk aversion. Notice the graphs in Figure 6 turn positive at different values of N. As noted in Section 6.2, the minimum value of N required for the Theorem-4 optimality condition to hold increases in γ. In each case, the optimal risk share converges to unity (complete risk taking) in N. In sum, a higher level of risk aversion only increases the minimum value of N required for a positive optimal risk share and then the speed at which the optimal risk share converges to 1.0.

7.2 Extension: Impure Altruism

As previously noted, our key insights only requires that *some* giving is to a public good; a large component can still be for private consumption. Consider a modification of the original public goods donor problem (1) that now accommodates impure altruism:

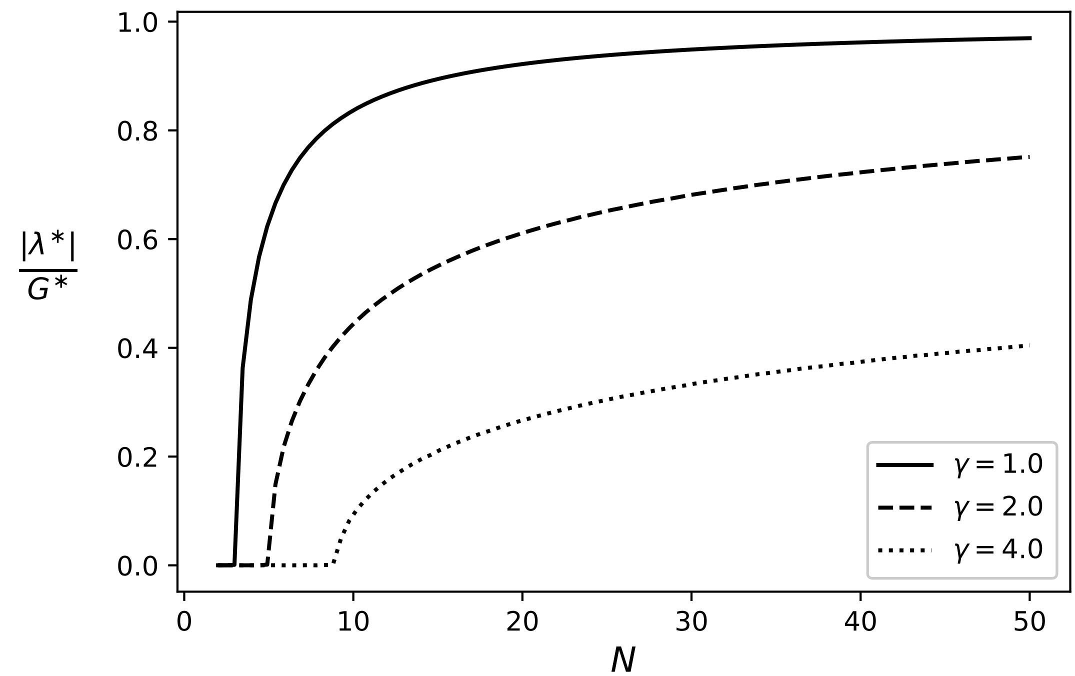
$$EU_i\left(g_i, \alpha_i | \lambda, \overrightarrow{g_{-i}}\right) = E\left[u\left(1 + \alpha_i \tilde{x} - g_i\right)\right] + E\left[v\left(g_i + \frac{\lambda}{N}\tilde{x} + \rho\left[\sum_{j=1, j\neq i}^{N} g_j + \lambda\frac{(N-1)}{N}\tilde{x}\right]\right)\right], \tag{14}$$

where ρ represents the degree of altruism. With $\rho = 1$, we get the original donor problem (1), where a donor i's gift, g_i, and its subsequent investment return, produces the same felicity within $v(\cdot)$ as gifts and subsequent investment returns from other donors. It is this substitutability of giving that produces the free-rider problem. With $\rho = 0$, donors only value their own gift and subsequent investment return, consistent with pure private consumption.[31] Intermediate values of ρ correspond to "impure altruism."

Similar to Figure 6 for the pure altruism problem ($\rho = 1$), Figure 7 plots the endowment's second-best optimal risk share from the three-stage Nash problem outlined in Section 5 but with the *impure* altruism donor problem (14) now used in Stage 2. We set $\rho = 0.25$, consistent with a significant level of impurity in altruism. Let's now compare the results in Figure 7 against Figure 6 where $\rho = 1.0$ (full altruism). For a given risk aversion γ, notice that impure altruism increases the minimum value of N required for a positive optimal risk share and slows the speed at which the optimal risk share converges to 1.0. Intuitively, since the risky asset with return $\tilde{x}$ is second-order stochastically dominated by the risk-free asset, a positive optimal risk share is inefficient without the free-riding associated with pure altruism. Less altruism produces less free-riding at a given level of N, increasing the minimum N required for positive risk taking and reducing its slope in N. Indeed, the risk share "flat lines" at zero when $\rho = 0$ for any λ.

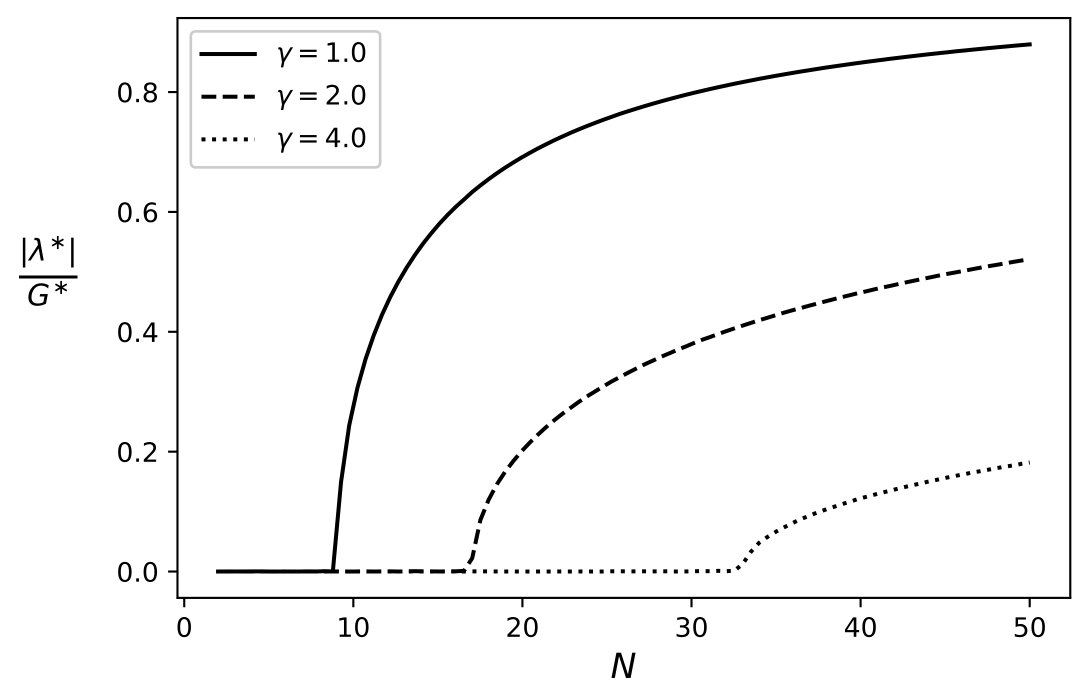
[31] Notice that λ is partitioned equally across donors, consistent with equal donations in the symmetric Nash equilibrium that is known in Stage 1.

Figure 6: Endowment Second-best Optimal Risk Share: Pure Altruism



Notes: Considers the pure altruism donor problem (1) with $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$ for different values of γ shown. Net return to stocks, $\tilde{x}$, take values in set $\{-1., 1.0\}$ with equal probability, as in Section 6.3.

Figure 7: Endowment Second-best Optimal Risk Share: Impure Altruism



Notes: Considers the impure altruism donor problem (14) with $u(w) = v(w) = \frac{w^{1-\gamma}}{1-\gamma}$ for different values of γ shown and $\rho = 0.25$. $\tilde{x}$ takes values in set $\{-1., 1.0\}$ with equal probability, as in Section 6.3.

8 Discussion and Extensions

This section compares our results to related literature, including asset-liability matching and charitable giving under pure altruism. We also address how our model provides a framework for addressing other real-world endowment frictions, in addition to informational asymmetries (moral hazard) addressed earlier.

Asset-Liability Matching. Our results differ sharply from the asset-liability matching (or "liability-driven investing," LDI) convention that is appropriate with pension plans (Brown and Wilcox, 2009; Novy-Marx and Rauh, 2011), which predicts that fairly deterministic liabilities should be determined with safe asset returns. Why the big difference? In the extreme case with pure "warm glow" (no altruism), our model and LDI make similar predictions. With donor altruism, however, our model indicates a role for risk taking in the context of endowment investing to reduce free-riding. In contrast, pension claims are paid by firms (out of firm revenues) or governments (out of taxes) as part of compensation; neither party seeks voluntary donations to pay pension claims.

Too Little Giving with Pure Altruism. The standard argument in the charitable giving literature for "impure altruism" is that the observable level of charitable giving is too large to be consistent with pure altruism, especially with a large number of donors N (Andreoni, 1988a). This result can be seen in our previous log-log ($\gamma_u = \gamma_v = 1.0$) example (Sections 3 and 6). *Without* risk taking ($\lambda = 0$), individual donations $\frac{1}{N+1}$ converge to zero in N and total donations $\frac{N}{N+1}$ converge to 1. However, in our model, *if* endowments take the second-best level of risk, individual donations remain fixed at $\frac{1}{4}$ and so total giving $\frac{N}{4}$ grows *linearly* in N, i.e., divergence.[32] The sizeable amount of giving to endowments could, therefore, be compatible with pure altruism, even though less than first-best.

Tax Deduction of Donations. Since the War Revenue Act of 1917, the United States has allowed a tax deduction for charitable donations. A charitable deduction can easily be incorporated into our model by reducing donor $i's$ private consumption in equation (1) by $(1 - \tau)g_i$ instead of g_i, where τ is the donor's marginal tax rate. Our results would not materially change unless the tax deduction could fully eliminate the free-riding problem. However, *fully* resolving the free-riding problem with a deduction is generally inefficient (indeed, likely even impossible). First, the value of τ would have to be a function of the

[32] For brevity, we don't report figures for the non-log case herein. However, at levels of risk aversion greater than unity (e.g., $\gamma_u = \gamma_v > 1.0$), it can be shown that individual donations fall in N but at a rate slower than the value of N itself, thereby allowing total donations to still grow unbounded in N, although less than linearly.

number of donors N to a specific charity. As N increases for a given charity, the optimal tax deduction of g_i would converge to a *credit*, nearly eliminating a donor's after-tax cost of giving. Second, the lost tax revenue must be replaced using distorting taxes levied on a smaller tax base, increasing the nonlinear dead-weight loss from raising revenue.

Capital Campaigns. New donations are often raised as part of capital campaigns. These campaigns are usually preceded by "a study" or "quiet period" with some "testing of the waters" from some limited donors before stating an achievable target for a larger campaign. Hence, the target is endogenous to the Nash game and so achieving it is not a sign that free-riding does not exist. Rather, the very presence of capital campaigns reflect the presence of free-riding. A capital campaign might attempt to reduce free-riding through a coordination of giving, a miniature version of Coasian bargaining.[33] By creating a list of donor-named gift opportunities, a capital campaign might also attempt to increase the *impurity* of altruistic giving, thereby reducing free-riding.

Multiple Endowment Funds. At large universities, for example, most assets are managed within long-term investment pools that take on substantial risk. But there sometimes exists smaller side pocket funds that are managed separately and more conservatively. These funds can be interpreted within our framework. In some cases, these funds are associated with a charitable gift annuity (where the donor receives back a stream of income until he or she dies) that must legally be managed conservatively as an insurance product. In other cases, these funds hold gifts for a specific cause for which a donor substantially cares about but for which there might be very little free-riding (e.g., a donor's personal experience with a specific disease motivates a gift to find a cure, or the fixed and variable expenses of a named building or program). Echazu and Nocetti (2015), for example, argues that the pure altruism might be more scalable in the traditional giving model if targets of giving are heterogeneous and each target can be assigned to a small pool of donors. Recall that a small N in our model implies more conservative risk taking.

Committed or Mixed Spending Needs. Closely related, the entire endowment might be closely committed to specific funding needs even without side pocket funds. This setup

[33] To the extent that a capital campaign does, however, reduce free-riding, it is then optimal to take on less risk. Actual measurement is confounded by data limitations, as major data trackers of university endowments—including Commonfund, NACUBO and VSE—do not distinguish assets raised from capital campaigns from other assets. An endogeneity problem also arises where charities facing greater free-riding are more likely to engage in multiple mechanisms to reduce it. Moreover, with a large N, the marginal benefit of a formal capital campaign might exceed its fixed costs, but the overall impact on free-riding could still be small relative to the endowment size.

sometimes occurs with endowments supported by a small number of donors such as a small school. Recall, that our model predicts that endowments supported by a small N should, in fact, take less risk. Moreover, Section 6.2 considered the case in which donors are potentially very averse to changes in endowment supported spending. If donors are infinitely risk averse to such changes ($\gamma_v = \infty$) then no investment risk is optimal. If risk aversion is large and finite, then N must be sufficiently large for risk taking to be optimal. At the same time, managers of endowments who are supported by a sufficiently large value of N might be taking too little risk (moral hazard). Alternatively, a single fund might support mixed spending needs, with existing assets committed to existing spending and new donations earmarked to expanding the mission. Still, nothing materially changes in our framework. Unless the number of donors N is small (more generally, γ_v is very large relative to N), it remains optimal for the endowment to take on risk.[34]

Private Foundations. Our model does not apply to private foundations that are not supported by several donors. For example, the Carnegie Foundation was established with the wealth of Andrew Carnegie and currently does not take private donations. Still, its $3.5 billion endowment takes on risk. More generally, suppose, that we think of a private foundation as deriving legacy value for the single donor and her family, where, in effect, $N = 1$. In Section 2, the risk-free asset second-order stochastically dominates the risky asset, and so this foundation would not take on any risk. However, recall that this assumption was made for exposition: our intent was to cleanly demonstrate the optimality of risk taking in the presence of free-riding under conditions that would not produce risk taking without free-riding. In contrast, a private foundation with *flexible* future goals might optimally take on risk precisely because equities generally out-perform risk-free assets, similar to the predictions of an asset-liability matching model.

9 Conclusions

This paper revisits the large literature on endowment investing with a more complete micro-foundation that includes a donor's objective function. We derive a condition—which quickly converges to standard DARA preferences in the number of donors—where the value of reducing free-riding exceeds the cost of additional risk. Risk taking is Pareto

[34] A stock of previous donations G_{past} could be added to the last term in equation (1) without materially changing our key qualitative results. In Nash equilibrium, each donor takes the donations of others as given regardless of source. Adding G_{past} would produce a wealth effect but free-riding would still emerge. For positive risk taking to be optimal, only the required value of γ_v relative to the number of contemporaneous donors N would be impacted.

improving and required by competition among endowments for donations, or, even more generally, by imperfect competition with complete information.

A strong cross-sectional "size effect" also emerges, where endowment risk taking increases in the size of the endowment. It is efficient for a large endowment to take on substantial risk even without the presence of fixed costs and even if the endowment does not have access to unique risk asset classes relative to its donors. In fact, risk taking is optimal even if the endowment's sponsors are very risk averse to changes in spending and use expensive investments that don't over-perform cash on average. Similarly, it is optimal for smaller endowments to take on less risk, even in the presence of more modern outsourced "endowment style" turnkey investment solutions. As a result, the strong size effect found in the data could be close to optimal.

A key difference between our contribution and the Modigliani-Miller theorem is that we also incorporate the free-riding problem that makes the endowment problem fundamentally different from corporate finance. Of course, as with the original M-M theorem, the real world is more complicated. But, like the original M-M theorem in corporate finance, our framework allows for a more comprehensive interpretation of real-world complexities related to endowment investing. In so doing, our results challenge the conventional thinking about the relationship between endowment risk taking and agency conflicts. A *low* level of risk taking by a large endowment likely indicates a moral hazard problem. We show that numerous other real-world complications can also be interpreted within our framework.

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Appendices

A Empirical Evidence on Giving Mechanism (Section 5.2)

Section 7 proved how the model's equilibrium endowment risk share increases in endowment size, consistent with the most salient empirical fact of endowment asset allocation. This section provides more direct evidence of the comparative statics mechanism derived in Section 5.2 by empirically estimating how donor giving to U.S. university and college endowments responds to the number of donors as well as the endowment's risk taking.

We obtained confidential data of university and college endowment asset information between the years 2009 and 2014 for a total of 4,256 endowment asset allocations across these years.[35] The data includes most U.S. colleges and university endowments, including all Ivy League schools as well even smaller schools such as seminaries and art schools. Asset holdings are decomposed into five major asset classes: domestic equities, international equities, private equity, bonds and cash. Annual gifts can be assessed from the survey data as can student enrollment and other demographic information.[36]

We proxy potential donor size by the number of full-time equivalent (FTE) students. To accommodate preference heterogeneity, we also incrementally add indicator (dummy) variables for the type of school, including private / public, elite / non-elite,[37] and college / university. We construct the empirical risk share (a value between 0 and 100 percent) by summing the value of an endowment's assets invested in domestic equities, international equities and private equity and then dividing this sum by the value of all assets.

Table 1 provides a summary of the data, pooled across years 2009 and 2014. Notice the presence of the "size effect" discussed in Section 7, where larger endowments take on more risk. For example, at the median (50th quantile), our constructed risk share increases from 65.9 percent in small endowments to 86.3 percent in large endowments. For the sake of comparison, the last row in Table 1 recomputes the risk share by including only international and alternative investments ("High Risk Share"), thereby excluding domestic equities. Notice that a considerable amount of risk taking in larger endowments takes the form of tilting toward international (including emerging markets) and alternative invest-

[35] Data provided by NACUBO and Commonfund. For years prior to 2009, their survey methodology changed between 2008 and 2009, making data before 2009 incompatible for our uses.

[36] Across schools, the median number of annual full-time equivalent students is 15,370, with 1,830 at the 10th quantile and 63,386 at the 90th quantile. The median gift per FTE is $3,118 while the mean value is $10,975.

[37] Defined as "Ivy League Plus" schools often used in the literature, including Brown, Columbia, Cornell, Dartmouth, Harvard, Penn, Princeton, Yale, Stanford, MIT, CalTech, University of Chicago, Duke, and Northwestern.

Table 1: Summary Statistics (Pooled Across Years)

Endowment Size	$0-25M	$25-50M	$50-100M	$100-500M	$500M-$1B	$1B+
Number of Schools	720	667	899	1280	353	337
Private	409	463	677	922	227	214
Elite	0	0	4	0	0	45
College	360	197	364	416	98	50
Risk Share (quantile)						
10^{th}	45.7	55.9	61	65.2	71.8	78
25^{th}	57.5	66	68.7	72.7	80	82.1
50^{th}	65.9	73.8	75.7	79.5	86	86.3
75^{th}	72.6	80.3	81.2	85.5	89.5	89.5
90^{th}	79.8	86.3	86	89.5	92	92.7
High Risk Share						
Median	14.7	33	40.9	52	66.1	69.4

ments, which have, historically, produced higher volatility than than domestic equities. To avoid any appearance of data mining, the subsequent analysis focuses on the broader measure of risk ("Risk Share") that includes domestic equities. The key results become stronger if we, instead, focused on the more narrow "High Risk Share" variable.[38]

Table 2 shows the outcomes of regressing the log of gifts per FTE student on the log of number of students ("log(# Students)") and the constructed risk share ("Risk Share"). The coefficients on both variables are consistent with our model. The estimated coefficient on "log(students)" indicates that a one percent increase in FTE students reduces gifts per FTE by around -0.4 percent. Moreover, the coefficient on "Risk Share" indicates that a ten point increase in the proportion of the endowment invested in risky assets increases gifts per FTE by about 0.4 percent.

Figure 8 provided additional robustness checks on the significance of the "Risk Share" coefficient. It shows the results of Model 3 by year ("All") along with 99-percent confidence intervals. Despite reducing the degrees of freedom relative to the regression pooled over time presented in Table 2, the annual results are highly significant. This significance is robust to additional data slices, including focusing on smaller endowments, schools with smaller FTE counts, both of these slices combined, and then both of these slices combined with a control for outliers. Figures 9 and 10 show that residual errors are nearly

[38] Similarly, our results below become stronger with more sophisticated risk adjustments including using risk-weights similar to those commonly used in banking for regulatory purposes, e.g., Basel III.

Table 2: Dependent Variable: log(Gifts Per Student)

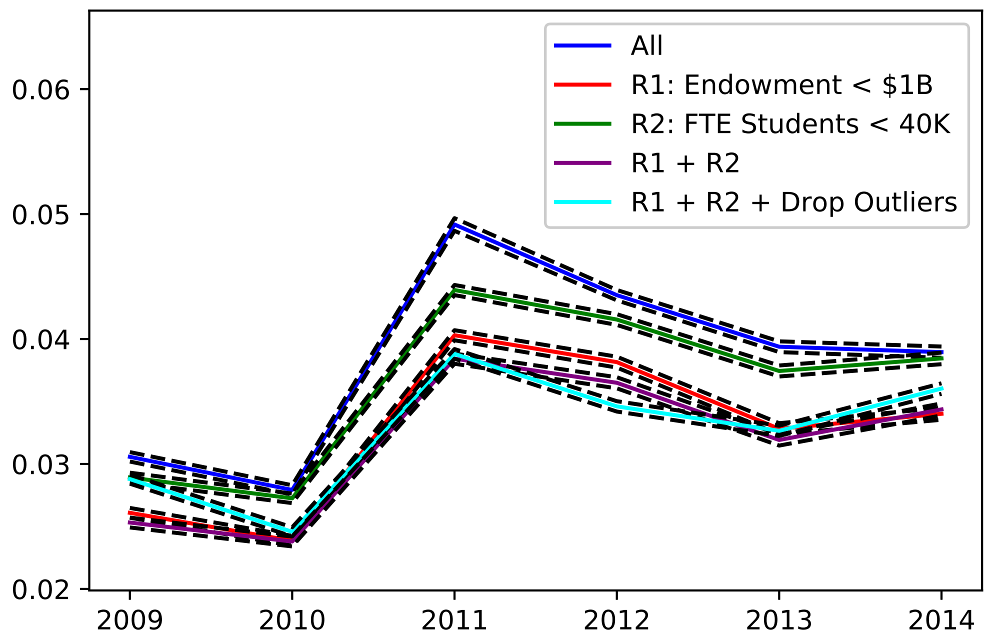
Model	1	2	3	4
Intercept	7.27***	5.91***	6.35***	6.53***
	(35.7)	(22.59)	(24.27)	(23.37)
log(# Students)	-0.48***	-0.34***	-0.38***	-0.4***
	(-24.06)	(-13.4)	(-14.96)	(-14.66)
Risk Share	0.04***	0.04***	0.04***	0.04***
	(22.41)	(21.25)	(20.94)	(20.96)
Private		0.57***	0.46***	0.46***
		(8.23)	(6.69)	(6.71)
Elite			2.4***	2.4***
			(10.37)	(10.37)
College				-0.1
				(-1.8)

Observations = 4256.
t-values in parentheses.
$^{*}p < 0.05$, $^{**}p < 0.01$, $^{***}p < 0.001$.

normally distributed, with a slight negative skew, allowing the confidence intervals to be computed directly from the standard errors.

Given the other controls for preference heterogeneity, the identification of the "Risk Share" coefficient in Table 2 and Figure 8 comes from assuming that some investment managers are engaging in some moral hazard by under-investing in risk (see the related discussion in Section 1). If risk shares were second-best optimal, then the "Risk Share" control could not be separately identified relative to the "log(# Students)." The "Risk Share" coefficient, therefore, might be biased downward relative to the coefficient that would be estimated from a natural experiment that was more observable to donors. We leave those considerations to future work.

Figure 8: Marginal Contribution of Risk Share on
Giving Per Student for Model 3 in Table 2, by Year



Explanation: Shows annual Risk Share coefficient (solid line) with 99% confidence interval (dotted black lines) for Model 3 shown in Table 2.[39]
In the figure legend, "All" corresponds to the data shown in Table 2, although unpooled to show annual results. Results for additional single data Restrictions are also shown, including "R1" (endowments with less than $1 billion in assets), "R2" (schools with less than 40,000 full-time equivalent students), "R3" (schools with endowments with at least 40% of Risk Share), "R1, R2 and R3" (all restrictions combined) and "Drop Outliers" (drop data points outside of three-standard deviations of fitted value). Results are very robust to other models shown in Table 1.
Data Source: Authors calculations based on Commonfund data.

Figure 9: Residuals between Actual and Fitted

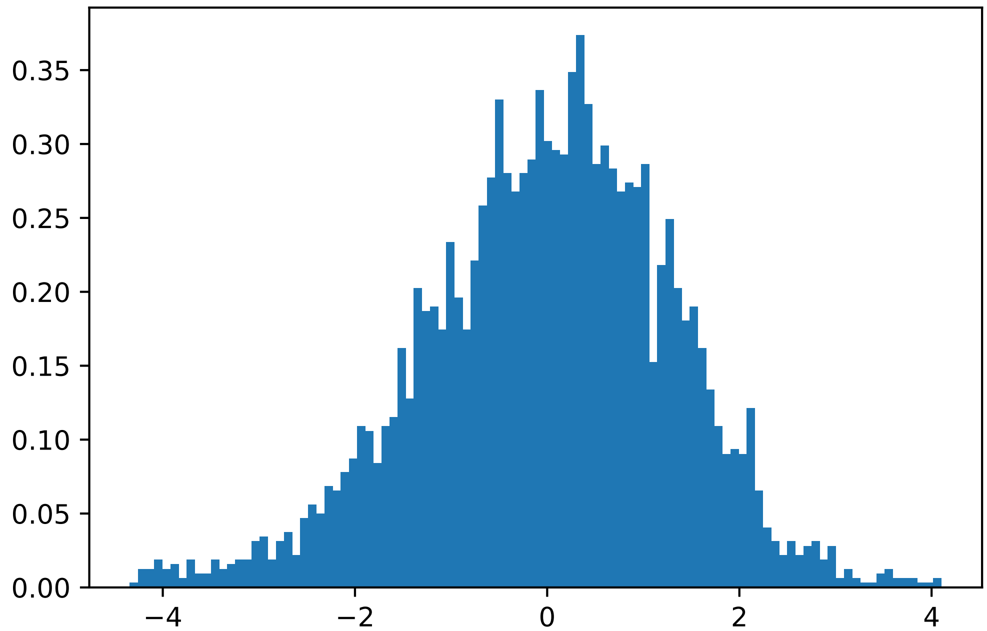
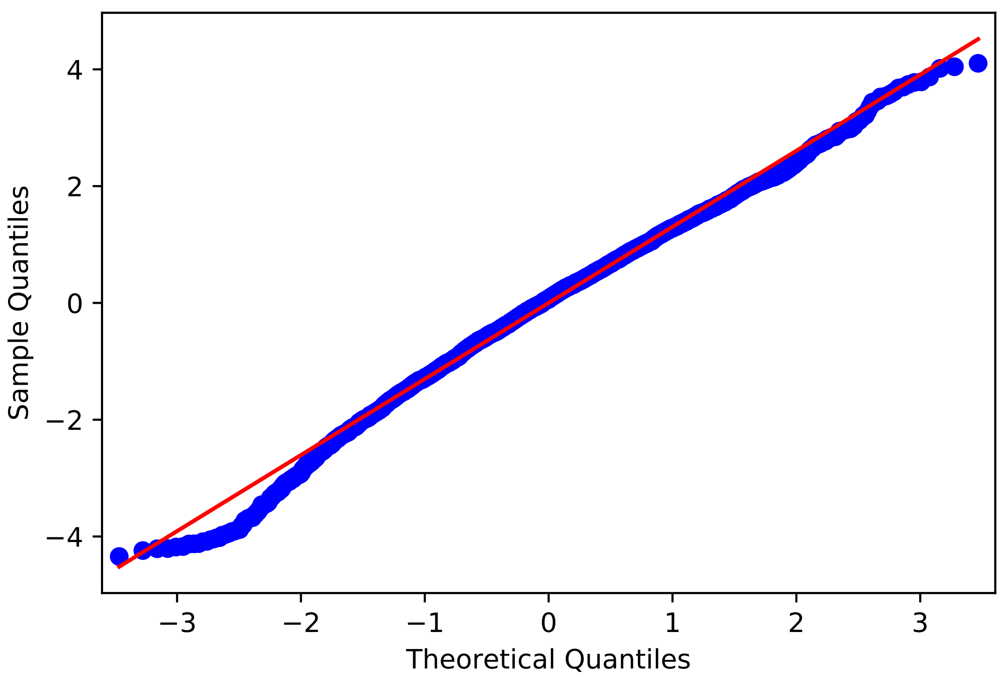


Figure 10: QQ Plot



Explanation: Figure 9 shows that the residuals between actual and the fitted Model 3 (using the last case shown in Figure 8) are nearly normally distributed. Figure 10 shows a QQ plot for these residuals. If the blue line overlapped perfectly with the red line then the residuals would be perfectly normally distributed. Data Source: Author calculations based on Commonfund data.

B Proofs

Theorem 1

Stage 2 For given donation vector $\vec{g}$ and investment level λ of the endowment fund, the optimal private investment decision vector $(\widehat{\alpha}_1, ..., \widehat{\alpha}_N)$ is the solution to the following maximization problem

$$\max_{\alpha_1,...,\alpha_N} \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right) = \sum_{i=1}^{N} E\left[u\left(1+\alpha_i\tilde{x}-g_i\right)\right] + NE\left[v\left(\sum_{i=1}^{N} g_i + \lambda\tilde{x}\right)\right].$$

The FOCs are

$$\frac{\partial \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right)}{\partial \alpha_i} = E\left[\tilde{x}u'\left(1+\alpha_i\tilde{x}-g_i\right)\right] = 0.$$

The SOCs are

$$\frac{\partial^2 \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right)}{\partial \alpha_i^2} = E\left[\tilde{x}^2 u''\left(1+\alpha_i\tilde{x}-g_i\right)\right] < 0,$$

and thus satisfied. For $\alpha_i = 0$, we derive

$$\frac{\partial \sum_{i=1}^{N} EU\left(\lambda, \vec{\alpha}, \vec{g}\right)}{\partial \alpha_i} \mid_{\alpha_i=0} = 0.$$

$\widehat{\alpha}_i = 0$ for all $i = 1, .., N$ is thus the unique global maximum.

For a given investment level λ of the endowment fund, the social planner picks the gift vector $(\widehat{g}_1, ..., \widehat{g}_N)$ to maximize

$$\sum_{i=1}^{N} EU_i\left(\lambda, \vec{\widehat{\alpha}} = \vec{0}, \vec{g}\right) = \sum_{i=1}^{N} u\left(1-g_i\right) + NE\left[v\left(\sum_{i=1}^{N} g_i + \lambda\tilde{x}\right)\right].$$

The FOCs are

$$\frac{\partial \sum_{i=1}^{N} EU_i\left(\lambda, \vec{0}, \vec{g}\right)}{\partial g_i} = -u'\left(1-g_i\right) + NE\left[v'\left(\sum_{i=1}^{N} g_i + \lambda\tilde{x}\right)\right] = 0.$$

The SOCs are satisfied since

$$\frac{\partial^2 \sum_{i=1}^{N} EU_i\left(\lambda,\vec{0},\vec{g}\right)}{\partial g_i^2} = u''\left(1-g_i\right) + NE\left[v''\left(\sum_{i=1}^{N} g_i + \lambda\tilde{x}\right)\right] < 0.$$

The unique optimal donation $\widehat{g}_i\left(\lambda\right)$ is thus the solution to the FOC

$$u'\left(1-\widehat{g}_i\left(\lambda\right)\right) = NE\left[v'\left(\sum_{i=1}^{N} \widehat{g}_i\left(\lambda\right) + \lambda\tilde{x}\right)\right].$$

This condition implies $\widehat{g}_1\left(\lambda\right) = ... = \widehat{g}_N\left(\lambda\right) = \widehat{g}\left(\lambda\right)$ and thus

$$u'\left(1-\widehat{g}\left(\lambda\right)\right) = NE\left[v'\left(N\widehat{g}\left(\lambda\right) + \lambda\tilde{x}\right)\right].$$

Stage 1 The optimal investment decision of the endowment fund $\widehat{\lambda}$ is the solution to the following maximization problem

$$\max_{\lambda} \Omega_{SO}\left(\lambda\right) = u\left(1-\widehat{g}\left(\lambda\right)\right) + E\left[v\left(N\widehat{g}\left(\lambda\right) + \lambda\tilde{x}\right)\right].$$

The FOC is

$$\Omega_{SO}'\left(\lambda\right) = -\widehat{g}'\left(\lambda\right)u'\left(1-\widehat{g}\left(\lambda\right)\right) + E\left[\left(N\widehat{g}'\left(\lambda\right) + \tilde{x}\right)v'\left(N\widehat{g}\left(\lambda\right) + \lambda\tilde{x}\right)\right] = 0.$$

Substitution of the FOC at Stage 2 yields

$$\Omega_{SO}'\left(\lambda\right) = E\left[\tilde{x}v'\left(N\widehat{g}\left(\lambda\right) + \lambda\tilde{x}\right)\right] = 0.$$

The concavity of $v\left(\cdot\right)$ implies

$$\begin{aligned}
\Omega_{SO}'\left(\lambda\right) &> 0 \text{ for all } \lambda < 0, \\
\Omega_{SO}'\left(0\right) &= 0, \text{ and,} \\
\Omega_{SO}'\left(\lambda\right) &< 0 \text{ for all } \lambda > 0.
\end{aligned}$$

Expected utility is thus globally concave in λ and $\widehat{\lambda} = 0$ is the unique global maximum. Last, the FOC for $\widehat{g}\left(\widehat{\lambda} = 0\right)$ then yields

$$u'\left(1-\widehat{g}\left(0\right)\right) = Nv'\left(N\widehat{g}\left(0\right)\right).$$

Note: If $u \equiv v$, then $\widehat{g}(0) > \frac{1}{N+1}$ as

$$u'\left(1-\frac{1}{N+1}\right) < Nu'\left(N \cdot \frac{1}{N+1}\right)$$

and $u'' < 0$.

Extension: $\widehat{g}$ is [increasing in, independent of, decreasing in] $N \iff R^v(N\widehat{g})\ [<,=,>]\ 1$.

Proof. Implicitly differentiating the FOC for the socially optimal gift, $\widehat{g}$, with respect to N yields

$$-\widehat{g}_N \cdot u''(1-\widehat{g}) = v'(N\widehat{g}) + N(g + N\widehat{g}_N) v''(N\widehat{g}),$$

with $\widehat{g}_N = \frac{\partial \widehat{g}}{\partial N}$. This implies

$$\widehat{g}_N = -\frac{v'(N\widehat{g}) + N\widehat{g} \cdot v''(N\widehat{g})}{u''(1-\widehat{g}) + N^2 v''(N\widehat{g})},$$

and thus

$$\begin{aligned} sign(\widehat{g}_N) &= sign\left(v'(N\widehat{g}) + N\widehat{g} \cdot v''(N\widehat{g})\right) \\ &= sign(1 - R^v(N\widehat{g})). \end{aligned}$$

Theorem 2

Without loss of generality we consider investor 1. Given the donations of all other investors $\vec{g}_{-1} = (g_2, ..., g_N)$, their investment levels $\vec{\alpha}_{-1} = (\alpha_1, ..., \alpha_N)$, and the investment level λ of the endowment, investor 1's best response function $\alpha_1^*(\lambda, \vec{\alpha}_{-1}, \vec{g}_{-1})$ is given by the solution to the following maximization problem

$$\begin{aligned} \alpha_1^*(\lambda, \vec{\alpha}_{-1}, \vec{g}_{-1}) &\in \arg\max_{\alpha_1} EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}) \text{ with} \\ EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1}) &= E\left[u(1 + \alpha_1 \tilde{x} - g_1)\right] + E\left[v\left(g_1 + \sum_{i=2}^{N} g_i + \lambda \tilde{x}\right)\right]. \end{aligned}$$

The FOC for the best response function is

$$\frac{\partial EU_1(\lambda, \alpha_1, \vec{\alpha}_{-1}, g_1, \vec{g}_{-1})}{\partial \alpha_1} = E\left[\tilde{x} u'(1 + \alpha_1 \tilde{x} - g_1)\right] = 0.$$

The SOC for the best response function holds as

$$\frac{\partial^2 EU_1\left(\lambda,\alpha_1,\vec{\alpha}_{-1},g_1,\vec{g}_{-1}\right)}{\partial\alpha_1^2}=E\left[\tilde{x}^2u''\left(1+\alpha_1\tilde{x}-g_1\right)\right]<0 \text{ for all } \left(\lambda,\alpha_1,\vec{\alpha}_{-1},g_1,\vec{g}_{-1}\right).$$

Evaluating the FOC at $\alpha_1=0$ yields

$$\frac{\partial EU_1\left(\lambda,\alpha_1,\vec{\alpha}_{-1},g_1,\vec{g}_{-1}\right)}{\partial\alpha_1}\mid_{\alpha_1=0}=0 \text{ for all } \left(\lambda,\vec{\alpha}_{-1},g_1,\vec{g}_{-1}\right).$$

$\alpha_1^*\left(\lambda,\vec{\alpha}_{-1},g_1,\vec{g}_{-1}\right)=0$ is thus the unique global maximum. Analogously, $\alpha_i^*\left(\lambda,\vec{\alpha}_{-i},g_i,\vec{g}_{-i}\right)=0$ for all i and it is thus the unique global maximum of the best response function of investor i. Thus, $\alpha_i^*\left(\lambda,g_i,\vec{g}_{-i}\right)=0$ for all i is the unique Nash equilibrium for all $(\lambda,g_i,\vec{g}_{-i})$.

Given the donations $\vec{g}_{-1}$ of all other investors, investor 1's best response function $g_1^*\left(\lambda,\vec{g}_{-1}\right)$ is given by the solution to the following maximization problem

$$\begin{aligned}
g_1^*\left(\lambda,\vec{g}_{-1}\right) &\in \arg\max_{g_1} EU_1\left(\lambda,g_1,\vec{g}_{-1}\right) \text{ with}\\
EU_1\left(\lambda,g_1,\vec{g}_{-1}\right) &= u\left(1-g_1\right)+E\left[v\left(g_1+\sum_{i=2}^N g_i+\lambda\tilde{x}\right)\right].
\end{aligned}$$

The FOC for the best response function is

$$\frac{\partial EU_1\left(\lambda,g_1,\vec{g}_{-1}\right)}{\partial g_1}=-u'\left(1-g_1\right)+E\left[v'\left(g_1+\sum_{i=2}^N g_i+\lambda\tilde{x}\right)\right]=0.$$

The SOC for the best response function holds as

$$\frac{\partial^2 EU_1\left(\lambda,g_1,\vec{g}_{-1}\right)}{\partial g_1^2}=u''\left(1-g_1\right)+E\left[v''\left(g_1+\sum_{i=2}^N g_i+\lambda\tilde{x}\right)\right]<0.$$

Therefore, there exists a unique solution $g_1^*\left(\lambda,\vec{g}_{-1}\right)$ to the above optimization problem which is determined by the FOC. We denote the Nash equilibrium by $\left(g_1^*\left(\lambda\right),...,g_N^*\left(\lambda\right)\right)$ which satisfies

$$\begin{aligned}
u'\left(1-g_1^*\left(\lambda\right)\right) &= u'\left(1-g_i^*\left(\lambda\right)\right)\\
&= E\left[v'\left(\sum_{i=1}^N g_i^*\left(\lambda\right)+\lambda\tilde{x}\right)\right]
\end{aligned}$$

for all i. Thus $g_1^*(\lambda) = ... = g_N^*(\lambda) = g^*(\lambda)$ which is given by the FOC

$$u'(1 - g^*(\lambda)) = E\left[v'(Ng^*(\lambda) + \lambda\tilde{x})\right].$$

Theorem 3

1. Implicitly differentiating the FOC for the Nash equilibrium gift policy functions, $g^*(\lambda)$, with respect to N yields

$$-g_N^*(\lambda) u''(1 - g^*(\lambda)) = (g^*(\lambda) + Ng_N^*(\lambda)) E\left[v''(Ng^*(\lambda) + \lambda\tilde{x})\right],$$

   with $g_N^*(\lambda) = \frac{\partial g^*(\lambda)}{\partial N}$. This implies

$$g_N^*(\lambda) = -\frac{g^*(\lambda) E[v''(Ng^*(\lambda) + \lambda\tilde{x})]}{u''(1 - g^*(\lambda)) + NE[v''(Ng^*(\lambda) + \lambda\tilde{x})]} < 0.$$

2. We derive the first- and second-order effects of changes in the investment policy λ on the Nash gift policy functions $g^*(\lambda)$. Implicitly differentiating the FOC for the Nash equilibrium with respect to λ yields

$$-g^{*\prime}(\lambda) u''(1 - g^*(\lambda)) = E\left[\left(Ng^{*\prime}(\lambda) + \tilde{x}\right) v''(Ng^*(\lambda) + \lambda\tilde{x})\right],$$

   i.e.,

$$g^{*\prime}(\lambda) = -\frac{E[\tilde{x}v''(Ng^*(\lambda) + \lambda\tilde{x})]}{u''(1 - g^*(\lambda)) + NE[v''(Ng^*(\lambda) + \lambda\tilde{x})]}.$$

   Evaluating this equation at $\lambda = 0$ yields $g^{*\prime}(0) = 0$. Taking the second deriative of the FOC of the Nash equilibrium with respect to λ yields

$$\begin{aligned} & -g^{*\prime\prime}(\lambda) u''(1 - g^*(\lambda)) + \left(g^{*\prime}(\lambda)\right)^2 u'''(1 - g^*(\lambda)) \\ = \; & Ng^{*\prime\prime}(\lambda) E\left[v''(Ng^*(\lambda) + \lambda\tilde{x})\right] + E\left[\left(Ng^{*\prime}(\lambda) + \tilde{x}\right)^2 v'''(Ng^*(\lambda) + \lambda\tilde{x})\right]. \end{aligned}$$

   Evaluating this equation at $\lambda = 0$ yields

$$-g^{*\prime\prime}(0) u''(1 - g^*(0)) = Ng^{*\prime\prime}(0) v''(Ng^*(0)) + E\left[\tilde{x}^2\right] v'''(Ng^*(0)),$$

   which implies

$$g^{*\prime\prime}(0) = -\frac{E\left[\tilde{x}^2\right] v'''(Ng^*(0))}{u''(1 - g^*(0)) + Nv''(Ng^*(0))}.$$

$\lambda = 0$ is a local minimum if and only if $g^{*\prime\prime}(0) > 0$. This holds if and only if $v'''(Ng^*(0)) > 0$ which is identical to the condition $P^v(Ng^*(0)) > 0$.

3. Now suppose $P^v(\cdot) > 0$. This implies

$$\begin{aligned} E\left[\tilde{x}v''(Ng^*(\lambda) + \lambda\tilde{x})\right] &< 0 \text{ for all } \lambda < 0, \text{ and,} \\ E\left[\tilde{x}v''(Ng^*(\lambda) + \lambda\tilde{x})\right] &> 0 \text{ for all } \lambda > 0, \end{aligned}$$

and thus

$$\begin{aligned} g^{*\prime}(\lambda) &< 0 \text{ for all } \lambda < 0, \\ g^{*\prime}(0) &= 0, \text{ and,} \\ g^{*\prime}(\lambda) &> 0 \text{ for all } \lambda > 0. \end{aligned}$$

$\lambda = 0$ is thus the global minimum of $g^*(\lambda)$.

Theorem 4

The endowment fund selects the optimal investment strategy λ^* by maximizing the expected utility of a single donor, $\Omega(\lambda)$. It is thus given by the solution to the following maximization problem

$$\lambda^* \in \arg\max_{\lambda} \Omega(\lambda) = u(1 - g^*(\lambda)) + E[v(Ng^*(\lambda) + \lambda\tilde{x})].$$

The first derivative is

$$\Omega'(\lambda) = -g^{*\prime}(\lambda)\, u'(1 - g^*(\lambda)) + E\left[\left(Ng^{*\prime}(\lambda) + \tilde{x}\right) v'(Ng^*(\lambda) + \lambda\tilde{x})\right].$$

Substitution of the condition for the Nash equilibrium at Stage 2 yields

$$\Omega'(\lambda) = E\left[\left((N-1)\, g^{*\prime}(\lambda) + \tilde{x}\right) v'(Ng^*(\lambda) + \lambda\tilde{x})\right].$$

Evaluating this derivative at $\lambda = 0$ yields $\Omega'(0) = 0$.

The second derivative is given by

$$\begin{aligned} \Omega''(\lambda) &= (N-1)\, g^{*\prime\prime}(\lambda)\, E\left[v'(Ng^*(\lambda) + \lambda\tilde{x})\right] \\ &\quad + E\left[\left((N-1)\, g^{*\prime}(\lambda) + \tilde{x}\right)\left(Ng^{*\prime}(\lambda) + \tilde{x}\right) v''(Ng^*(\lambda) + \lambda\tilde{x})\right]. \end{aligned}$$

Evaluating this second derivative at $\lambda = 0$ yields

$$\begin{aligned}
\Omega''(0) &= (N-1)\, g^{*\prime\prime}(0)\, v'(Ng^*(0)) + E\left[\tilde{x}^2\right] v''(Ng^*(0)) \\
&= -(N-1)\frac{E\left[\tilde{x}^2\right] v'''(Ng^*(0))}{u''(1-g^*(0)) + Nv''(Ng^*(0))} v'(Ng^*(0)) + E\left[\tilde{x}^2\right] v''(Ng^*(0)) \\
&= E\left[\tilde{x}^2\right]\left(-\frac{(N-1)\, v'''(Ng^*(0))\, v'(Ng^*(0))}{u''(1-g^*(0)) + Nv''(Ng^*(0))} + v''(Ng^*(0))\right).
\end{aligned}$$

$\lambda = 0$ is a local minimum if and only if $\Omega''(0) > 0$. With the FOC of the Nash equilibrium, $u'(1-g^*(0)) = v'(Ng^*(0))$, we derive that $\Omega''(0) > 0$ if and only if

$$(N-1)\, P^v(Ng^*(0)) > A^u(1-g^*(0)) + NA^v(Ng^*(0)).$$

Corollary 1. *If $u(\cdot) = v(\cdot)$, then $g^*(0) = \frac{1}{N+1}$ and $\lambda = 0$ is a local minimum if and only if*

$$P\left(\frac{N}{N+1}\right) > \frac{N+1}{N-1} \cdot A\left(\frac{N}{N+1}\right).$$

Theorem 5

The second inequality, $\Omega(\lambda^*) > \Omega(\lambda = 0)$ with $|\lambda^*| > 0$, is the result of Theorem 4. For the first inequality, note that the socially optimal $\widehat{\lambda} = 0$, while the Nash equilibrium implies $|\lambda^*| > 0$. (The value $\alpha_i = 0$ is optimal both socially and in the Nash equilibrium.)

We then have

$$\begin{aligned}
\Omega_{SO}\left(\widehat{\lambda} = 0\right) &= EU\left(\widehat{\lambda} = 0, \widehat{g}\left(\widehat{\lambda} = 0\right)\right) \\
&> EU\left((\lambda^*, \widehat{g}(\lambda^*)\right) \\
&> EU\left((\lambda^*, g^*(\lambda^*)\right) = \Omega(\lambda^*),
\end{aligned}$$

where $|\lambda^*| > 0$. The first inequality comes from knowing that the socially optimal value of $\widehat{\lambda} = 0$ maximizes donor expected utility, equation (3). Hence, any other choice of $\lambda \neq 0$, including the value in the Nash equilibrium, λ^*, must produce a smaller expected utility, if inserted into the social optimal gift policy function. The second inequality follows from the fact that the social optimum problem maximizes donor expected utility, thereby producing a larger expected utility than in the Nash equilibrium, conditional on the same value of λ.

Example 2

Stage 2 The Nash equilibrium gifts $g^*(\lambda)$ are given by the condition

$$u'(1-g^*(\lambda)) = E\left[v'(Ng^*(\lambda)+\lambda\tilde{x})\right].$$

Solving this condition for $u(w) = \ln(w)$ and $\tilde{x}$ following a two-point distribution that takes the values $+1$ and -1 with equal probability yields equation (12)

$$g^*(\lambda) = \frac{N+\sqrt{N^2+4N(N+1)\lambda^2}}{2N(N+1)}.$$

The first derivative of the gift policy function is

$$g^{*\prime}(\lambda) = \frac{2\lambda}{\sqrt{N^2+4N(N+1)\lambda^2}}.$$

Stage 1 The optimal investment strategy λ^* is given by the solution to the maximization problem

$$\lambda^* \in \arg\max_{\lambda} \Omega(\lambda) = u(1-g^*(\lambda)) + E\left[v(Ng^*(\lambda)+\lambda\tilde{x})\right].$$

For $u(w) = \ln(w)$ and $\tilde{x}$ following a two-point distribution we derive

$$\Omega(\lambda) = \ln(1-g^*(\lambda)) + \frac{1}{2}\left(\ln(Ng^*(\lambda)+\lambda) + (Ng^*(\lambda)-\lambda)\right).$$

Note that both $g^*(\lambda)$ and $\Omega(\lambda)$ are symmetric in λ. We thus focus on $\lambda \geq 0$. Furthermore, the domain restriction $\lambda < N$ ensures that $\Omega(\lambda)$ is well-defined, i.e. $1-g^*(\lambda) > 0$ and $Ng^*(\lambda)-\lambda > 0$.

Substituting the condition for the Nash equilibrium gifts at Stage 2 into the first derivative yields

$$\begin{aligned} \Omega'(\lambda) &= E\left[\left((N-1)g^{*\prime}(\lambda)+\tilde{x}\right)v'(Ng^*(\lambda)+\lambda\tilde{x})\right] \\ &= \frac{N(N-1)g^{*\prime}(\lambda)g^*(\lambda)-\lambda}{(Ng^*(\lambda)+\lambda)(Ng^*(\lambda)-\lambda)}. \end{aligned}$$

The denominator is strictly positive. Solving the FOC $\Omega'(\lambda) = 0$ yields the solutions $\lambda = 0$ and $\lambda = \frac{\sqrt{N(N-3)}}{4}$.

Moreover, it can be shown that

$$\begin{aligned}
\Omega'(0) &= 0, \\
\Omega'(\lambda) &> 0 \text{ for all } 0 < \lambda < \frac{\sqrt{N(N-3)}}{4}, \\
\Omega'\left(\frac{\sqrt{N(N-3)}}{4}\right) &= 0, \text{ and,} \\
\Omega'(\lambda) &< 0 \text{ for all } \frac{\sqrt{N(N-3)}}{4} < \lambda < N.
\end{aligned}$$

Taking into account the symmetry of $\Omega(\lambda)$, we conclude that $\lambda = 0$ is a local minimum and the global maximum is attained at $|\lambda^*| = \frac{\sqrt{N(N-3)}}{4}$.

Evaluating the Nash equilibrium gifts at λ^* implies

$$g^*(\lambda^*) = \frac{1}{4}.$$

Example 3

Stage 2 The Nash equilibrium gifts $g^*(\lambda)$ are given by the condition

$$u'(1 - g^*(\lambda)) = E\left[v'(Ng^*(\lambda) + \lambda\tilde{x})\right].$$

Solving this condition for $u(w) = -e^{-\gamma w}$ and $\tilde{x}$ following a normal distribution with expectation 0 and variance σ^2 yields equation (13)

$$g^*(\lambda) = \frac{1}{N+1}\left(1 + \frac{1}{2}\gamma\sigma^2\lambda^2\right).$$

Note that $E\left[e^{-\gamma\lambda\tilde{x}}\right] = e^{\frac{1}{2}\gamma^2\sigma^2\lambda^2}$. The first derivative of the gift policy function is

$$g^{*\prime}(\lambda) = \frac{\gamma\sigma^2\lambda}{N+1}.$$

Stage 1 The optimal investment strategy λ^* is given by the solution to the maximization problem

$$\lambda^* \in \arg\max_{\lambda} \Omega(\lambda) = u(1 - g^*(\lambda)) + E[v(Ng^*(\lambda) + \lambda\tilde{x})].$$

For $u(w) = -e^{-\gamma w}$ and $\tilde{x}$ following a normal distribution with expectation 0 and variance σ^2 we derive

$$\Omega(\lambda) = -2e^{-\frac{\gamma}{N+1}\left(N-\frac{1}{2}\gamma\sigma^2\lambda^2\right)}.$$

The first derivative yields

$$\Omega'(\lambda) = -\frac{2\gamma^2\sigma^2}{N+1}\lambda e^{-\frac{\gamma}{N+1}\left(N-\frac{1}{2}\gamma\sigma^2\lambda^2\right)}.$$

This implies

$$\begin{aligned} \Omega'(\lambda) &> 0 \text{ for all } \lambda < 0, \\ \Omega'(0) &= 0, \text{ and,} \\ \Omega'(\lambda) &< 0 \text{ for all } \lambda > 0. \end{aligned}$$

$\lambda^* = 0$ is thus the unique maximum. Evaluating the Nash equilibrium gifts at $\lambda^* = 0$ implies

$$g^*(\lambda^*) = \frac{1}{N+1}.$$