On the Shape of High Order Additive and Multiplicative Utility Premiums

Henri Loubergé*and Béatrice Rey†

February 28, 2022

Abstract

The utility premium of Friedman and Savage (1948) attracted renewed attention over the past years as a non monetary measure of risk aversion, in particular when high order changes in risk are considered. In this paper, our motivation is to complete the literature on the shape of utility premiums as a function of wealth by considering high order changes in risk for additive and multiplicative risks. When additive risks are addressed, the shape is well-defined and in accordance with intuition. Our main contribution is however to address the case of multiplicative risks, a class of risks frequently met in financial applications. In this case two wealth components need to be considered: sure wealth and the wealth exposed to multiplicative risks. We analyse the shape of the utility premium with respect to these two components and find that the well-defined and intuitive results hold only in the sure wealth case. When the wealth exposed to multiplicative risks varies, the shape of the multiplicative utility premium depends on benchmark values for the relative or partial risk aversion coefficients. The special case of CRRA utility is also considered.

JEL classification: D81

Keywords: Additive vs. multiplicative risks, Constant relative risk aversion, Higher order risk aversion, Expected utility, Stochastic dominance, Utility premium.

*Geneva School of Economics and Management, University of Geneva, Uni Mail, Pont d'Arve 40, 1211 Geneva 4, Switzerland. E-mail: Henri.Louberge@unige.ch

†Corresponding author, Université de Lyon, Université Lumière Lyon 2, GATE UMR 5824, F-69130 Ecully, France. E-mail: rey-fournier@gate.cnrs.fr, Adress: Laboratoire Gate, 93 Chemin des Mouilles, 6930 Ecully, France.

1 Introduction

In the theory of decision under risk and uncertainty, the term "premium" is generally associated to the "risk premium". The concept was originally introduced by Friedman and Savage (1948) to measure risk aversion, but mainly developed by Arrow (1965) and Pratt (1964), who addressed situations where a decision-maker (DM) faces a single exogenous risk. It reflects the price that a risk averse DM is prepared to pay to avoid such a risk. The risk premium as an analytical tool has become highly popular in the economic and financial literature since these seminal contributions. It was also extended to endogenous and to multiple risks, in order to increase the domain of its use. However, in the same seminal article, Friedman and Savage (1948) proposed a different way to measure the same degree of risk aversion. They introduced the "utility premium", measuring the loss of welfare faced by the same DM exposed to a single exogenous risk. The utility premium concept met much less success than the risk premium in the economic and financial literature. The main reason is that it is measured in utility units instead of being expressed in monetary units, like the risk premium. The von Neumann-Morgenstern (1947) utility function routinely used in this literature being "unique up to a linear positive transformation", this prevents direct interpersonal comparison of risk aversion degrees, in contrast to the risk premium that spurred a large strand of recent literature focusing on "comparative nth-degree risk aversion": see Jindapon and Neilson (2007), Liu and Meyer (2013), Liu and Neilson (2019), and Jindapon et al. (2021)[1].

However, interest in the Friedman-Savage utility premium grew more recently, due to the seminal work of Eeckhoudt and Schlesinger (2006) on risk attitudes defined as behavior with respect to "risk apportionment" at different orders[2]. In particular, the two authors address the link between the shape of the utility premium and risk apportionment[3]. They point out that the utility premium is decreasing in wealth under risk apportionment of order three (prudence) and convex in wealth under risk apportionment of order four (temperance). They also introduce a second-order measure of the utility premium − later defined as the temperance utility premium by Courbage and Rey (2010) − and they show that this measure is decreasing and convex under, respectively, risk apportionment of orders five and six.

Starting from there, the utility premium concept has then proved highly useful to analyze and predict behavior towards risk. First, Crainich and Eeckhoudt (2008) use the utility premium concept to measure the loss of welfare due to misapportionment of a small

[1] See, however, Huang and Stapleton (2015) and Wong (2018) for comparative risk aversion and comparative prudence based on the utility premium. See also Li and Liu (2014) and Heinzel (2019).

[2] Risk apportionment holds when the DM prefers to spread the risks and sure losses over different states of nature, instead of facing them grouped in one single state of nature. For instance, when exposed to a zero-mean risk, the DM prefers to avoid facing this risk in a situation where she also faces a sure loss of wealth.

[3] Actually, they use the negative of the utility premium as defined usually. See their equation (1). For clarity, we translate their result into the usual definition

risk $\tilde{\epsilon}$ – a measure later defined as the prudence utility premium by Courbage and Rey (2010). They show that this measure is tightly linked to their proposed monetary measure of downside risk aversion (i.e., prudence). Then, Eeckhoudt and Schlesinger (2009) show that the shape of the original Friedman-Savage (1948) utility premium is useful to understand precautionary saving, in the additive risk case, as well as in the multiplicative risk case. A demand for precautionary saving appears in both cases if the utility premium is decreasing in wealth. Courbage and Rey (2010) clarify Eeckhoudt and Schlesinger (2006) and Crainich and Eeckhoudt (2008) by extending the Friedman-Savage (2^{nd}-order) utility premium to the 3^{rd} and 4^{th} orders, explicitly defining the prudence utility premium and the temperance utility premium. They show that the three premiums are vulnerable to a sure loss and to a background risk. This implies that all these premiums are decreasing in initial wealth and convex. In a later paper (Courbage and Rey, 2019), they consider several independent risks, instead of one or two risks as in previous work. They show that, contrary to intuition, temperance is sufficient to guarantee that spreading N risks over N states of nature is preferred to facing the N risks in one state, or two states, or three states, or $n \leq N - 1$ states. This implies the N-superadditivity of the utility premium, given temperance: the welfare loss measured by the utility premium is larger when N risks are faced together than the sum of the welfare losses due to facing each of the N risks individually. Ebert et al. (2017) and Courbage et al. (2018) consider high-order changes in risks and their impact on the utility premium. In particular, Courbage et al. (2018) extend Courbage and Rey (2010) by defining formally the nth-order utility premium (see below). They show that welfare is reduced by merging high-order changes in risk instead of facing them separately, e.g., in different corporate entities. Similarly, Ebert et al. (2017) extend Eeckhoudt et al. (2009) by concluding that not only changes in risk are mutually aggravating, but that more severe changes in risk lead to greater mutual aggravation.

In this paper, we extend the results of Eeckhoudt and Schlesinger (2009) in different dimensions. Firstly, we consider changes in risk at any order, instead of comparing a situation of risk with a situation of no risk. This is more representative of real-life situations. Secondly, we do not restrict ourselves to the first derivative of the utility premium with respect to wealth to check whether the utility premium is increasing or decreasing in wealth but we also examine whether the second order derivative points toward a concave or convex shape. This was also addressed by Eeckhoudt and Schlesinger (2006) and by Courbage and Rey (2010), but only for low-order changes in risk. Thirdly, in the more complex case of multiplicative risks, we consider a more general situation, instead of limiting ourselves to the extreme case where all wealth is submitted to the multiplicative risk. The last extension leads, in particular, to challenging results. As expected, it implies the coefficient of partial relative risk aversion as a benchmark, but it also shows that assuming the familiar constant relative risk aversion (CRRA) utility is not sufficient to obtain a simple clear-cut result in all cases.

Compared to Courbage et al. (2018), we address a different question. After having defined the nth-order (additive) utility premium (see below), Courbage et al. (2018)

focus on specific properties of their measure in the case of a mixed risk averse DM facing two risk changes. They show that the measure is superadditive, and the DM prefers to face the risk changes one by one, instead of merging them. Generalizing Courbage and Rey (2010), they also show that their nth-order utility premium is vulnerable to a high order change in background risk. They do not address explicitly the shape of their nth-order utility premium in reaction to a change in wealth − the motivation of this paper − and they consider only additive risks whereas we extend their definition to multiplicative risks in order to also investigate the shape of the nth-order multiplicative utility premium.

Multiplicative risks are implicitly addressed by Eeckhoudt and Schlesinger (2008) and Wong (2019) as they focus on changes in interest rate risk and their impact on precautionary saving. In particular Wong (2019), relying like us on the nth-order additive utility premium defined by Courbage et al. (2018), uses a multiplicative utility premium very similar to a measure defined in the current paper (see equation (3) below). But, these two papers focus on a specific problem − precautionary saving − and none of them address the main motivation of this paper, i.e., the shape of the utility premium.

Understanding this shape is of high importance in the current period of economic and social turmoil brought about by pandemia and global warming. Firstly, the risk of facing losses due to business interruption and layoffs has increased dramatically as a result of emergency measures enacted by governments to slow down the pace of Covid infections. This represents a severe blow to the economy and a loss of welfare for the population. Governments try to compensate these welfare losses with generous subsidies ("Helicopter money"). In terms of utility premium, the welfare losses reflect the worsening of an additive risk. Compensating them with lump sum subsidies will be based on more solid ground if the shape of the additive utility premium is known. If it is decreasing and convex in wealth, it means that wealthier households should be compensated less for their increased risk − but the reduction in compensation should be alleviated progressively when climbing on the wealth ladder. If it is decreasing and concave, the reduction in compensation would increase as wealthier and wealthier households are addressed, leading to a zero subsidy at some stage. Therefore, it is important for public policy purposes to understand the shape of the utility premium for additive risks, not only for low order changes in risk, but also for any change in the risk profile. This paper completes the literature at this level. Secondly, turning to the consequences of global warming, one notes that governments and international organizations are advocating and/or imposing carbon taxes in a desperate attempt to slow down the path of temperature increases. The extent of these tax increases represents a multiplicative risk for firms and households unable to stop abruply their consumption of fossil energy. This leads to welfare losses deserving to be compensated by public authorities in order to ease the transition to a carbon-free economy. In this case, this is the shape of the multiplicative utility premium which is at stake. Will the proposed compensation scheme have to consider the beneficiaries' wealth or not? The answer depends on the shape of the multiplicative utility premium and on whether all wealth is impacted by the risk, or only a share of it. These are the questions hiding behind our mathematical developments in this paper.

The paper is organized as follows. In the next section we start with the general nth-order additive utility premium defined by Courbage et al. (2018) and we show explicitly that it is decreasing and convex in wealth, confirming the intuition derived from results in the latter paper. In section 3, we define the nth-order multiplicative utility premium in the special case where all wealth is exposed to the multiplicative risk and in the general case where only a fraction of wealth is at risk. This leads us to investigate the shape of these premiums with respect to two kind of changes in wealth: a change in the non risky wealth and a change in the wealth exposed to risk. In the first case, the results do not differ from those obtained in the additive risk case. But in the second case, the results are less clear-cut, in particular when the second-order effect is concerned. Turning to the familiar case of a CRRA utility function in section 4, we obtain clear-cut results when all wealth is at risk, but only partial results when a fraction of wealth is at risk. Section 5 concludes and summarizes our results.

2 The nth-order additive utility premium

Consider a decision-maker (DM) endowed with a deterministic wealth level w and also facing an additive risk Y in the current period ($t = 0$). Her total wealth is $\tilde{w}_Y = w + Y$. It is partly at risk. The risk will deteriorate certainly in the next period ($t = 1$) and become X. Using Expected Utility Theory (EUT), the DM's loss of welfare writes as

$$E[u(w + Y)] - E[u(w + X)], \tag{1}$$

where u is is the DM's risk averse utility function: $u' \geq 0$ and $u'' \leq 0$[4].

If Y is a degenerated random variable equal to zero and X is a zero-mean random variable, equation (1) refers to the preference for risk or for certainty defined by Friedman and Savage (1948)[5] : preference for risk if the result of equation (1) is negative, preference for certainty if it is positive. In this particular case, equation (1) was labelled as the "utility premium of Friedman and Savage" by Eeckhoudt and Schlesinger (2009) who also examined some of its properties.

In the general case where Y and X are both risky, we can use stochastic dominance to compare the risk levels between the two random variables. Let's assume that Y dominates X via nth-order stochastic dominance ($X \preceq_{SD-n} Y$)[6]. When the $(n-1)$ moments of Y and X are equal, nth-order stochastic dominance coincides with Ekern's (1980) concept

[4]We assume throughout this article that the support of Y (or X) is defined such that $\tilde{w}_Y$ (or $\tilde{w}_X$ with $\tilde{w}_X = w + X$) is in the domain of u. We also assume that u is n-times differentiable and that its derivative of order $k \geq 1$ has a constant sign, either positive or negative, in the domain of u.

[5]See in particular their footnote 25.

[6]The concept of nth-order stochastic dominance is defined as follows (see for example Jean (1980, 1984)) : Consider Y and X with F and G respectively their two cumulative distribution functions of wealth, defined over a probability support contained within the interval $[a, b]$. Define $F_1 = F$ and $G_1 = G$. Now define $F_{k+1}(z) = \int_a^z F_k(t)dt$ and $G_{k+1}(z) = \int_a^z G_k(t)dt$ for $k \geq 1$. The random variable

of increase in nth-order risk ($X \preceq_{Ekern-n} Y$). Ekern's definition includes the well-known case of mean-preserving increase in risk of Rothschild and Stiglitz (1970), as well as the case of increase in downside risk defined by Menezes et al. (1980). These cases represent, respectively, a second-degree and a third-degree increase in risk.

Given these premises, equation (1) corresponds to the (additive) nth-order utility premium $\omega(w; Y, X)$ defined in Courbage et al. (2018)[7].

Definition 1 (Courbage *et al.* (2018))**.** *Given two independent risks, Y and X such that Y dominates X via nth-order stochastic dominance ($X \preceq_{SD-n} Y$), the function ω defined as $\omega(w; Y, X) = E[u(w+Y)] - E[u(w+X)]$ is named the "nth-order utility premium". It measures the degree of pain due to the aggravating nth-order stochastic dominance risk.*

The nth-order additive utility premium $\omega(w; Y, X)$ is a non monetary measure of the aggravation in terms of risk. It measures the loss of welfare caused by the switch from Y to X when the DM's non risky wealth is w.

Using stochastic dominance properties, first observe that $w(w; Y, X) \geq 0$ $\forall w$ for all utility functions u such that $(-1)^{k+1}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n$. Note that $(-1)^{k+1}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n$ means that all odd derivatives of u are positive and all even derivatives are negative. Following Brockett and Golden (1987) and according to Caballé and Pomansky (1996), an individual with such a utility function is said to be *mixed risk averse* (MRA). Hence, for all n, the nth-order utility premium of a MRA agent is always positive. In other words, such an individual always incurs a pain when facing the passage from risk Y to a more detrimental risk X dominated by nth-order stochastic dominance. If the utility function verifies $(-1)^{k+1}u^{(k)} \geq 0$ $\forall k = 1, \ldots, n$ we will label u as MRA from order 1 to n.

Now, assume that the government wants to compensate the pain due to the risk aggravation by allocating an additional certain revenue Δ at $t = 0$ and $t = 1$. We make the three following assumptions: (i) all agents whom the government wants to help have the same preferences, represented by the utility function u; (ii) all agents face the same risk aggravation, from Y to X; (*iii*) but all agents do not have the same sure wealth w. The government has then to consider two questions before deciding on the monetary amount Δ:

- 1. Is the utility loss due to the aggravating nth-order stochastic dominance risk identical for all agents, whatever their sure wealth w? Or is it bigger or smaller for wealthier DMs? More formally, is the nth-order utility premium an increasing, decreasing or constant function of w?
- 2. What is the sensitivity of this variation to the wealth level? Let's assume that the utility premium decreases with w, as suggested by intuition. For higher w, does

Y dominates X via nth-order stochastic dominance ($X \preceq_{SD-n} Y$) if $F_n(z) \leq G_n(z)$ for all z, and if $F_k(b) \leq G_k(b)$ for $k = 1, 2, .., n$.

[7] See also Courbage and Rey (2010).

the reduction decrease at a lower, constant or bigger rate? Consider, for instance, two low wealth levels w_1 and w_2 ($w_1 < w_2$) and two higher wealth levels w_3 and w_4 ($w_3 < w_4$), with $w_4 - w_3 = w_2 - w_1$. Is $\omega(w_1; Y, X) - \omega(w_2; Y, X)$ bigger than, equal to, or smaller than $\omega(w_3; Y, X) - \omega(w_4; Y, X)$? More formally, is the nth-order utility premium a convex, linear or concave function of w?

We obtain the following answers to these two questions (see proofs in Appendix 1).

Proposition 1.

(a) *The nth-order utility premium $\omega(w; Y, X)$ is decreasing in w for all DMs who are MRA from order 1 to $n + 1$.*

(b) *The nth-order utility premium $\omega(w; Y, X)$ is a convex function of w for all DMs who are MRA from order 1 to $n + 2$.*

In the particular case where stochastic dominance at order n is replaced by Ekern's (1980) increase in risk at order n, we get the following corollary.

Corollary 1.

(a) *In the case where $X \preceq_{Ekern-n} Y$, the nth-order utility premium, $\omega(w; Y, X)$ is decreasing in w for all DMs with a utility function u verifying $(-1)^n u^{(n+1)} \geq 0$.*

(b) *In the case where $X \preceq_{Ekern-n} Y$, the nth-order utility premium, $\omega(w; Y, X)$ is a convex function of w for all DMs with a utility function u verifying $(-1)^{n+1} u^{(n+2)} \geq 0$.*

These two properties are intuitive. When the DM becomes richer, the pain due to the switch towards the nth-order stochastically dominated risk decreases, and the reduction of this pain decreases as he gets richer and richer. They simply extend to any order n the properties already uncovered by Eeckhoudt and Schlesinger (2006) at orders two and four, and by Courbage and Rey (2010) at orders two, three and four. Note that they are also already implicit in Courbage et al. (2018). These authors show that the nth-order additive utility premium is vulnerable to a sure loss and to a background risk − see also Ebert et al. (2017) − as stated in their equations (18) and (19), where $l > 0$ and $E(\tilde{\epsilon}) = 0$:

$$\omega(w - l; Y, X) - \omega(w; Y, X) \geq 0 \Leftrightarrow (-1)^{k+1} u^{(k)} \geq 0 \; \forall k = 1, \ldots, n + 1,$$

$$\omega(w + \tilde{\epsilon}; Y, X) - \omega(w; Y, X) \geq 0 \Leftrightarrow (-1)^{k+1} u^{(k)} \geq 0 \; \forall k = 1, \ldots, n + 2.$$

The first expression reflects item (a) of our Proposition 1 above: the nth-order additive utility premium increases with the introduction of a sure loss reducing wealth. The second expression means that the utility premium is vulnerable to the introduction of a zero-mean background risk, implying that it is convex in wealth, by Jensen's inequality, as stated in item (b) above.

Let us illustrate these results with a few particular cases. First, consider the case used by Friedman and Savage (1948): $Y = 0$ and X is a zero-mean random variable i.e., $X \preceq_{Ekern-2} Y$. Item (a) of Corollary 1 means that the 2^{nd}-order utility premium is a decreasing function of wealth for all prudent DMs ($u''' \geq 0$), or − using the terminology introduced by Eeckhoudt and Schlesinger (2006) and Eeckhoudt et al. (2009) − for all utility functions verifying risk apportionment of order 3.[8] Item (b) of Corollary 1 means that the 2^{nd}-order utility premium is a convex function of w for all temperant DMs ($u^{(4)} \leq 0$), i.e., for all utility functions verifying risk apportionment of order 4.[9]

This is not unexpected. In this additive risk context, the risk faced by the DM is independent of her wealth level w. Now, we know from the theory of risk aversion that the Arrow-Debreu risk premium (i) decreases when the agent gets richer, reflecting DARA (Decreasing Absolute Risk Aversion) and thus prudence; and (ii) increases when the agent becomes exposed to an actuarially neutral background risk, reflecting risk vulnerability and thus temperance. Applying these properties to a utility premium context means:

(i) $\omega(w - l; 0, X) \geq \omega(w; 0, X)$ $\forall l > 0$,

(ii) $\omega(w + \tilde{\epsilon}; 0, X) \geq \omega(w; 0, X)$ $\forall \tilde{\epsilon}$ such as $E(\tilde{\epsilon}) = 0$.

Eeckhoudt and Schlesinger (2006) interpret prudence as the preferred apportionment of risk X and a second harm represented by a sure loss l, and temperance as the preferred apportionment of risk X and a second harm represented by an actuarially neutral risk $\tilde{\epsilon}$, both in a context of equiprobable lotteries. Prudence and temperance reflect a preference for risk disaggregation. It is easy to see that (i) and (ii) above reflect also such preference as they derive respectively from the lottery preferences

(i') $[w - l, w + X; \frac{1}{2}, \frac{1}{2}] \succeq [w, w - l + X; \frac{1}{2}, \frac{1}{2}]$,

(ii') $[w + \tilde{\epsilon}, w + X; \frac{1}{2}, \frac{1}{2}] \succeq [w, w + \tilde{\epsilon} + X; \frac{1}{2}, \frac{1}{2}]$.

In Eeckhoudt et al. (2009b)'s interpretation this means: combining good with bad in the two proposed states of nature is preferred to facing no bad (only good) in one state of nature and a cluster of bads in the other state of nature, bad being a risk for a risk averse DM.

It is expected that these intuitions apply to higher orders. For instance, in the particular case where $X \preceq_{Ekern-3} Y$ featuring a downside risk ($X = [0, -l + \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ and $Y =$

[8]Note that this result was already obtained by Hanson and Menezes (1971) in one of the early papers using the Friedman-Savage utility premium.

[9]In the case where $X \preceq_{SD-2} Y$, note that regularity conditions (u defined over $\mathbb{R}^+$, non-satiation and bounded marginal utility) imply that item (a) of Proposition 1 holds when the DM is only prudent without requiring risk aversion and item (b) of Proposition 1 holds when the DM is only temperant without requiring risk aversion and prudence (see Menegatti 2014, Propositions 2 and 3). Indeed, under these conditions, Menegatti (2014) shows that prudence implies risk aversion and that temperance implies prudence. Thus the MRA assumption is not needed in this case.

$[-l, \tilde{\epsilon}; \frac{1}{2}, \frac{1}{2}]$ with $l > 0$ and $E(\tilde{\epsilon}) = 0$), Corollary 1 means that the 3^{rd}-order utility premium is a decreasing function of w for all temperant DMs and a convex function of w for all edgy DMs ($u^{(5)} \geq 0$), i.e., for all utility functions displaying risk apportionment of order 5. Generalizing these intuitions leads to Proposition 1 and Corollary 1 above.

Our objective, in the rest of this paper, is to check whether the intuitive results of Proposition 1 still apply when risks Y and X interact multiplicatively with wealth.

3 The nth-order multiplicative utility premium

The previous section analyzes the pain generated by the aggravation of an additive risk. However, many real-world problems in economics and finance deal with multiplicative risks. This occurs, for instance, when interest rates, fiscal rates or foreign exchange rates play a role. In the rest of this paper, we consider such a context. The DM faces a multiplicative risk Y in the current period, and the risk will deteriorate surely and become X in the next period. We assume that random variables Y and X are non negative. We denote by W the deterministic DM's wealth level. Two cases may be considered. In the first case, all wealth W is subject to the multiplicative risk. In this case, using EUT as above, the DM's loss of welfare is

$$E[u(WY)] - E[u(WX)]. \tag{2}$$

In the second case, only an amount x with $x > 0$, a share of total wealth, is subject to the multiplicative risk. The rest of wealth is not at risk. We label this certain wealth w, as in the previous section[10]. Then, using EUT, the DM's loss of welfare writes

$$E[u(w + xY)] - E[u(w + xX)]. \tag{3}$$

We proceed by taking equation (3) as the general case[11], keeping equation (2) as a special case.

Definition 2. *Given two independent non negative risks, Y and X such that Y dominates X via nth-order stochastic dominance ($X \preceq_{n-SD} Y$), the function ω_M defined as $\omega_M(w, x; Y, X) = E[u(w+xY)] - E[u(w+xX)]$ is named the "nth-order multiplicative utility premium". It measures the degree of pain due to the aggravating nth-order stochastic dominance multiplicative risk.*

[10] We have $W = w + x$

[11] The same formulation was used in Wong (2019), as he also extends to multiplicative risks the definition of the high-order additive utility premium proposed in Courbage et al. (2018). His paper focuses on precautionary saving in the context of an extended definition of stochastic dominance: (m, n)th-order stochastic dominance.

Note that that the particular case of the 2^{nd}-order multiplicative risk premium assuming $w = 0$, $Y = 1$ (a degenerated random variable) and X such that $E(X) = 1$ is analyzed in Eeckhoudt and Schlesinger (2009). Hence, in their approach, all wealth becomes exposed to a multiplicative risk, a special case of equation (3) above. They show that their 2^{nd}-order multiplicative risk premium is decreasing in wealth if relative prudence $-z\frac{u'''(z)}{u''(z)}$ exceeds 2 (with z the argument of the utility function). Their result is generalized below (Corollary 3a).

Properties of u that guarantee the positive sign of $\omega_M(w, x; Y, X)$ are the same as the ones that guarantee the positive sign of $\omega(w; Y, X)$: $\omega_M(w, x; Y, X) \geq 0 \ \forall (w, x)$ for all utility function u such that $(-1)^{k+1}u^{(k)} \geq 0 \ \forall k = 1, \ldots, n$. This result is easily understood since we can define $xX = \hat{X}$ and $xY = \hat{Y}$ and then $-$ given $x > 0$ $-$ rewrite[12] $\omega_M(w, x; Y, X) = E[u(w + \hat{Y})] - E[u(w + \hat{X})] = \omega(w; \hat{Y}, \hat{X})$ with $\hat{X} \preceq_{SD-n} \hat{Y}$.

Two questions then arise: (1) Is $\omega_M(w, x; Y, X)$ decreasing and convex in w? (2) Is $\omega_M(w, x; Y, X)$ decreasing and convex in x?

For the first question, we readily obtain the following proposition and corollary.

Proposition 2.

(a) The nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ is a decreasing function of w for all DMs who are MRA from order 1 to $n + 1$.

(b) The nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ is a convex function of w for all DMs who are MRA from order 1 to $n + 2$.

Corollary 2.

(a) In the case where $X \preceq_{Ekern-n} Y$, the nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ is a decreasing function of w for all DMs with a utility function u such as $(-1)^n u^{(n+1)} \geq 0$.

(b) In the case where $X \preceq_{Ekern-n} Y$, the nth-order multiplicative utility premium, $\omega_M(w, x; Y, X)$ is a convex function of w for all DMs with a utility function u such as $(-1)^{n+1} u^{(n+2)} \geq 0$.

We note that the study of the relationship between the multiplicative utility premium and riskless wealth w gives the same results as the study of the relationship between the additive utility premium and w for the same reason explaining the positive sign of $\omega_M(w, x; Y, X)$ (see above).

Turning to question (2) above we first note that the answer is not obvious at all, especially for the second element in the question, convexity. To understand the difficulty, we begin with the special case of the 2^{nd}-order multiplicative utility premium, assum-

[12]See proposition 3-11 in Denuit et al. (1998)

ing $w = 0$, $Y = 1$ (a degenerate random variable) and X such that $E(X) = 1$, i.e., $\omega_M(0, x; 1, X) = u(x) - E[u(xX)]$. Considering that the risk is now multiplicative, not additive, and thus proportional to wealth x, what is the intuition for the shape of the multiplicative utility premium as a function of x. Decreasing or increasing ? And convex or concave ? Applying the logic used in the previous section suggests: (j) the multiplicative utility premium is vulnerable to a sure proportional loss; and (jj) the multiplicative utility premium is vulnerable to an independent proportional risk. Item (j) can be rewritten formally as

(j') $\omega_M(0, x(1-l); 1, X) \geq \omega_M(0, x; 1, X)$, where $0 < l < 1$,

that rewrites equivalently as

(j'') $u(x(1-l)) - E[u(x(1-l)X)] > u(x) - E[u(xX)]$, where $0 < l < 1$.

The last expression is the same as the one used by Eeckhoudt et al. (2009a) to reflect the preference for risk disaggregation of a DM facing a sure proportional loss $(1-l)$ and a proportional risk X as defined in the above paragraph. This is their relationship $A_3 \preceq B_3$ – see also Wang and Li (2010). The authors show that (j'') holds if relative prudence exceeds two: $-\frac{xu'''(x)}{u''(x)} > 2$. As shown by Eeckhoudt and Schlesinger (2009), this implies that the multiplicative utility premium $\omega_M(0, x; 1, X)$ is decreasing in x.

We now turn to (jj) to explore whether the same 2^{nd}-order multiplicative utility premium is vulnerable to an independent background risk. Formally, (jj) can be rewritten as

(jj') $\omega_M(0, x\tilde{\epsilon}; 1, X) \geq \omega_M(0, x; 1, X)$, where $E(\tilde{\epsilon}) = 1$ with $\tilde{\epsilon}$ a non negative random variable,

that rewrites equivalently as

(jj'') $E[u(x\tilde{\epsilon})] - E[u(x\tilde{\epsilon}X)] > u(x) - E[u(xX)]$, where $E(\tilde{\epsilon}) = 1$ with $\tilde{\epsilon}$ a non negative random variable.

We remark that this expression means that the multiplicative utility premium $\omega_M(0, x; 1, X)$ is convex. Indeed, let $g(y) = u(y) - E[u(yX)]$ for a given risk X and any y. The convexity of g is defined by $E[g(\tilde{\theta})] > g(E(\tilde{\theta}))$ $\forall\tilde{\theta}$. Using the definition of g, this rewrites as $E[u(\tilde{\theta})] - E[u(\tilde{\theta}X)] > u(E(\tilde{\theta})) - E[u(E(\tilde{\theta})X)]$, $\forall\tilde{\theta}$. Let $\tilde{\theta} = x\tilde{\epsilon}$. We obtain $E(\tilde{\theta}) = x$ since $E(\tilde{\epsilon}) = 1$. Thus, the last expression rewrites equivalently as $E[u(x\tilde{\epsilon}) - E[u(x\tilde{\epsilon}X)] > u(x) - E[u(xX)]$, i.e., (jj''). In their paper, Eeckhoudt et al. (2009a) do not consider (jj''), i.e., the preference for disaggregation of two "harms" $\tilde{\epsilon}$ and X. They limit their analysis to the disaggregation of $(1-l)$ and X. Indeed, contrary to intuition, it is not sufficient that the DMs preferences statisfy $-\frac{xu''''(x)}{u'''(x)} > 3$ to yield (jj''). The analysis is more complex: see Wang and Li (2010), Chiu et al. (2012), and Denuit and Rey (2013). Although dealing with multiplicative risks, these papers do not address this question. Hence, the question remains: Under which conditions do we observe convexity of the multiplicative utility premium ? Equivalently, under which conditions do we observe vulnerability of the multiplicative utility premium to the addition of an independent backgound risk $\tilde{\epsilon}$? These are the questions we address below.

Remark that the risk is now proportional to wealth x. For this reason, it is possible that the DM would prefer aggregate the harms $(1-l)$ and X, and $\tilde{\epsilon}$ and X, instead of disaggregating them according to risk apportionment. In this case, the multiplicative utility premium would be increasing and concave. This possibility is included in the results below.

Before turning to the general relationship between the multiplicative utility premium and the amount of wealth exposed to risk, x, let us recall the definition of the partial risk aversion coefficient introduced by Menezes and Hanson (1970) and Zeckhauser and Keeler (1970):

$$r_p(y; w) = -y\frac{u''(w+y)}{u'(w+y)}.$$

Generalizing this first-order definition to any order k leads to the partial kth-degree risk aversion coefficient used in Chiu et al. (2012)[13]:

$$r_p^{(k)}(y; w) = -y\frac{u^{(k+1)}(w+y)}{u^{(k)}(w+y)}.$$

Using this definition, we obtain the following proposition (see proof in Appendix 2).

Proposition 3.

(a) The nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ *is a decreasing function of* x *for all MRA DMs from order* 1 *to* $n+1$ *if the utility function* u *verifies* $r_p^{(k)}(x\epsilon; w) \geq k$ $\forall(x, w)$ $\forall\epsilon > 0$ $\forall k = 1, \ldots, n$. *The nth-order multiplicative utility premium* $\omega_M(w, x; Y, X)$ *is an increasing function of* x *for all MRA DMs from order* 1 *to* $n+1$ *if the utility function* u *verifies* $r_p^{(k)}(x\epsilon; w) \leq k$ $\forall(x, w)$ $\forall\epsilon > 0$ $\forall k = 1, \ldots, n$.

(b) The nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ *is a convex (concave) function of* x *for all MRA DMs from order* 1 *to* $n+2$ *if the function* c *defined as* $c(\epsilon) = \epsilon^2 u''(w + x\epsilon)$ $\forall\epsilon > 0$ *verifies* $(-1)^{k+1}c^{(k)}(\epsilon) \geq 0\ (\leq 0)$ $\forall\epsilon > 0$ $\forall k = 1, \ldots, n$.

Corollary 3.

(a) In the case where $X \preceq_{Ekern-n} Y$, *the nth-order multiplicative utility premium* $\omega_M(w, x; Y, X)$ *is a decreasing (increasing) function of* x *for all DMs with a utility function* u *verifying* $(-1)^{k+1}u^{(k)} \geq 0$ *for all* $k = n, n+1$ *and* $r_p^{(n)}(x\epsilon; w) \geq n\ (\leq n)$ $\forall(x, w)$ $\forall\epsilon > 0$.

[13] Note that the relative risk aversion coefficient of degre k corresponds to the special case of the partial kth-degree risk aversion coefficient where $w = 0$ and all wealth is at risk: $r_p^{(k)}(y; 0) = r_r^{(k)}(y) = -y\frac{u^{(k+1)}(y)}{u^{(k)}(y)}$.

(*b*) *In the case where* $X \preceq_{Ekern-n} Y$, *the nth-order multiplicative utility premium* $\omega_M(w,x;Y,X)$ *is a convex (concave) function of x for all DMs with a utility function u verifying* $(-1)^{k+1}u^{(k)} \geq 0$ *for all* $k = n, n+1, n+2$ *and* $(-1)^{n+1}c^{(n)}(\epsilon) \geq 0\ (\leq 0)\ \forall \epsilon > 0$, *where the function c is defined in Proposition 3(b).*

Comparing Proposition 3(a) and Corollary 3(a) with Proposition 1(a) and Corollary 1(a), we observe that the multiplicative utility premium behaves, with respect to x, the same way as the additive utility premium, with respect to w (it is decreasing), if the kth-order partial relative risk aversion coefficient exceeds the kth threshold. This reminds of several results in the literature where a similar result is obtained when dealing with multiplicative risks: see Eeckhoudt et al. (2009a), Wang and Li (2010), Chiu et al. (2012) and Denuit and Rey (2013).

The case of the second-order shape (convexity) addressed in part (b) is however more complex as it involves a new function, $c(\epsilon)$.

The first derivative of the function c, $c^{(1)}$, writes as $c^{(1)} = 2\epsilon u''(w+x\epsilon)+(\epsilon)^2 x u'''(w+x\epsilon)$. Then $c^{(1)} \geq 0$ is equivalent to $r_p^{(2)}(x\epsilon; w) \geq 2$. The kth-order derivative of the function c, $c^{(k)}$, for all $k \geq 2$, writes as (see proof in Appendix 3):

$$c^{(k)}(\epsilon) = k(k-1)x^{k-2}u^{(k)}(w+x\epsilon) + 2k\epsilon(x)^{k-1}u^{(k+1)}(w+x\epsilon) + \epsilon^2 x^k u^{(k+2)}(w+x\epsilon). \quad (4)$$

Using partial relative index coefficients, we obtain, for $k \geq 2$ (see proof in Appendix 4):

$$Sgn\{c^{(k)}(\epsilon)\} = (-1)^{k+1} Sgn\{k(k-1) + r_p^{(k)}(x\epsilon; w)(-2k + r_p^{(k+1)}(x\epsilon; w))\}. \quad (5)$$

We are now in a position to illustrate these results. Consider first the case where $X \preceq_{SD-1} Y$: $X = a$ and $Y = b$ with $0 < a < b$. The multiplicative utility premium of order 1 writes as $\omega_M(w,x;Y,X) = u(w+xb) - u(w+xa)$. Following Proposition 3, $\frac{d\omega_M(w,x;Y,X)}{dx} \leq 0$ for all utility function u such as $u' \geq 0$ and $u'' \leq 0$ if $r_p^{(1)} \geq 1$, i.e if $-y\frac{u''(w+y)}{u'(w+y)} \geq 1$. Item ($b$) of Proposition 3 means that $\frac{d^2\omega_M(w,x;Y,X)}{dx^2} \geq 0$ for all u such as $u' \geq 0$, $u'' \leq 0$ and $u''' \geq 0$ if $c^{(1)} \geq 0$ that is equivalent to $r_p^{(2)} \geq 2$. Consider now the case where $X \preceq_{Ekern-2} Y$: $Y = 1$ and X such that $E(X) = 1$. The 2^{nd}-order multiplicative utility premium writes as $\omega_M(w,x;Y,X) = u(w+x) - E[u(w+xX)]$. Following Corollary 3, $\frac{d\omega_M(w,x;Y,X)}{dx} \leq 0$ if u verifies $u'' \leq 0$ and $u''' \geq 0$ and if $r_p^{(2)} \geq 2$. Item (b) of Corollary 3 means that $\frac{d^2\omega_M(w,x;Y,X)}{dx^2} \geq 0$ for all u such as $u'' \leq 0$, $u''' \geq 0$ and $u^{(4)} \leq 0$ if $c'' \leq 0$, with $c'' = 2u''(w+x\epsilon) + 4\epsilon x u'''(w+x\epsilon) + \epsilon^2 x^2 u^{(4)}(w+x\epsilon)$. The expression of c'' can be rewritten using partial relative risk aversion coefficients. We obtain

$$Sgn\{c''\} = Sgn\{-2 + r_p^{(2)}(x\epsilon; w)[4 - r_p^{(3)}(x\epsilon; w)]\} \quad (6)$$

We immadiately observe that, without additional assumptions, $c'' \leq 0$ is not necessarily verified. So the convexity of the 2^{nd}-order multiplicative utility premium is not a trivial property.

Note, however, that if $r_p^{(3)}$ is sufficiently high (more precisely if $r_p^{(3)} \geq 4$), $Sgn\{c''(\epsilon)\}$ is negative (convexity is obtained). More generally, considering Eq. (5), condition $r_p^{(k+1)} \geq 2k$ $\forall k = 1, \dots, n$ is a sufficient condition to obtain $(-1)^{k+1}c^{(k)}(\epsilon) \geq 0 \quad \forall \epsilon > 0$ $\forall k = 1, \dots, n$, i.e., to obtain the convexity of the nth-order multiplicative utility premium as a function of the amount exposed to risk. This leads to the following corollary.

Corollary 4.

A sufficient condition for the convexity of the nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ *as a function of* x *for all MRA DMs from order* 1 *to* $n + 2$ *is* $r_p^{(k+1)} \geq 2k$ $\forall k = 1, \dots, n$.

The condition $r_p^{(k+1)} \geq 2k$ appears to be new in the literature where $r_p^{(k)}$ is usually compared to k as in our Proposition 3a above[14].

The next section examines the case of a specific utility function.

4 Case of the CRRA utility function

Trying to provide more substance to the results of the previous section, we now turn towards the special case of a familar utility function. The Constant Relative Risk Aversion (CRRA) utility function writes as $u(z) = \frac{1}{1-\gamma} z^{1-\gamma}$ with $\gamma > 0$ and $\gamma \neq 1$. This function is MRA since $(-1)^{1+k}u^{(k)} > 0$ for all $k \geq 1$. With this function, the coefficient of relative risk aversion $r_r(z)$ is constant and equal to γ and the kth degree of this coefficient is constant and equal to $r_r^{(k)}(z) = \gamma + k - 1$. However, the partial risk aversion coefficient is not constant. For total wealth $w + y$, with y the wealth exposed to risk, it is $r_p(y; w) = \frac{y}{w+y}\gamma$. It varies with the share of wealth exposed to risk. At order k, it is

$$r_p^{(k)}(y; w) = \frac{y}{w + y}(\gamma + k - 1). \tag{7}$$

To analyze the shape of the multiplicative utility premium in this case, we proceed in two steps: (1) All wealth is subject to a mutiplicative risk; (2) Only a part of wealth is subject to this risk.

[14]Note that in a context of correlated risks, Denuit and Rey (2014) already introduced a new benchmark comparing $r_p^{(k)}$ to $k - 1$ to capture the sensitivity of the marginal expected utility to a correlation parameter. This second-order condition ($r_p^{(2)}$ compared to 1) appeared also later in a paper by Gollier (2015) addressing the impact of inequalities and econonomic convergence on the efficient discount rate (defined as a ratio of marginal expected utilities).

4.1 All wealth is at risk

This case corresponds to the multiplicative utility premium defined in equation (2), or to the more general multiplicative utility premium defined in equation (3), assuming $w = 0$. In this case, with a CRRA utility function, the kth-degree partial relative risk aversion coefficient is reduced to the kth-degree relative risk aversion coefficient (see above). The conditions of Proposition 3a, $r_p^{(k)} \geq k$ $(\leq k)$ become $r_r^{(k)} \geq k$ $(\leq k)$. As $r_r^{(k)} = \gamma + k - 1$ and as $\gamma \neq 1$, this means $\gamma > 1$ (< 1), i.e., a relative risk aversion coefficient larger (smaller) than one. Concerning the condition of Proposition 3b, we obtain in this case that equation (5) rewrites as

$$Sgn\{c^{(k)}\} = (-1)^{k+1} Sgn\{\gamma(\gamma - 1)\}. \tag{8}$$

If $\gamma > 1$, this means that the sign of $c^{(k)}$ is the same as the sign of $u^{(k)}$ positive when k is odd and negative if k is even and thus $(-1)^{k+1} c^{(k)} \geq 0$ for all $k \geq 1$. When $\gamma < 1$, the opposite result obtains. If all wealth is at risk, we obtain thus the following proposition:

Proposition 4.

If all wealth is at risk and if u is CRRA with relative risk aversion γ, the nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ is a decreasing and convex function of x when $\gamma > 1$; it is an increasing and concave function of x when $\gamma < 1$[15].

The result is quite clear, given that only relative risk aversion is at stake in this case, and it is constant. Note, in addition, that the shape of the multiplicative utility premium as a function of the amount at risk x mirrors its shape as a function of riskless wealth w (if any) only if risk aversion is relatively high (greater than one).

This corresponds to the case of a DM who prefers to disaggregate risks. In contrast, if risk aversion is lower ($\gamma < 1$), meaning that the DM prefers to aggregate risks, the multiplicative utility premium is in this case increasing and concave. The result derives from the fact that relative risk aversion is here constant for any k and reads $r_r^{(k)} = \gamma + k - 1$. Thus, comparing $r_r^{(k)}$ to k means comparing γ to 1 (see Loubergé et al. (2020)). In addition, signing $c^{(k)}$ means also comparing γ to 1 (see equation (8) above).

Matters are not so simple, from a mathematical point of view, when total wealth is split in two parts: a risky part and a non risky part. The non constant partial risk aversion featured in equation (7) re-enters the scene in this case.

[15]Note that Eeckhoudt and Schlesinger (2009) already obtained the result for the direction of the movement (decreasing when $\gamma > 1$, increasing when $\gamma < 1$) in the specific case $Y = 1$ and $E(X) = 1$.

4.2 Only a fraction of wealth is at risk

Still using a CRRA utility function, but leaving aside the assumption that all wealth is at risk, brings us back to the comparison between: on one hand, the partial risk aversion coefficient (7) - approppriately adjusted to take multiplicative risks into account - and on the other hand, the values of k or n, depending on whether stochastic dominance is used (Proposition 3), or whether Ekern's increases in risk are used (Corollary 3). For instance, considering Proposition 3a, we have $\frac{\partial \omega_M}{\partial x} \leq 0$ for all MRA DMs from order 1 to $n+1$ if $r_p^{(k)}(x\epsilon; w) \geq k \ \forall (x, w) \ \forall \epsilon > 0 \ \forall k = 1, \ldots, n$. Using a CRRA utility function, the condition is written $\frac{x\epsilon}{w+x\epsilon}(\gamma + k - 1) \geq k$. Defining $h = \frac{x\epsilon}{w+x\epsilon}$, $h < 1$, the condition becomes : $h(\gamma + k - 1) \geq k$. It depends on h, the share of total wealth exposed to the multiplicative risk, and not only on γ, relative risk aversion. This is a source of indetermination. Consider the simplest possible case, first-order stochastic dominance, where $k = n = 1$, $X = a$, $Y = b$ with $0 < a < b$. Using Corollary 3, we get the following results:

ω_M is decreasing in x if $r_p^{(1)} \geq 1$, i.e, $h \geq \frac{1}{\gamma}$; it is increasing in x if $r_p^{(1)} \leq 1$, i.e, $h \leq \frac{1}{\gamma}$.

ω_M is convex in x if $r_p^{(2)} \geq 2$, i.e, $h \geq \frac{2}{\gamma+1}$; it is concave in x if $r_p^{(2)} \leq 2$, i.e, $h \leq \frac{2}{\gamma+1}$.

Hence two values play a role in this case: $\frac{1}{\gamma}$ and $\frac{2}{\gamma+1}$. Note that:

- $\gamma < 1 \Rightarrow 1 < \frac{2}{\gamma+1} < \frac{1}{\gamma}$. As $h < 1$, this means that ω_M is increasing and concave in this case.
- $\gamma > 1 \Rightarrow \frac{1}{\gamma} < \frac{2}{\gamma+1} < 1$. With $h < 1$, several sub-cases are then possible:
  - ω_M is an increasing and concave function of x when $h \in [0, \frac{1}{\gamma}]$,
  - ω_M is a decreasing and concave function of x when $h \in [\frac{1}{\gamma}, \frac{2}{\gamma+1}]$,
  - ω_M is a decreasing and convex function of x when $h \in [\frac{2}{\gamma+1}, 1]$.

We observe that these results comply with Proposition 4, where $h = 1$. Illustrating the general cases proves still more difficult, especially for the second-order conditions. It is possible, however, to determine the first-order impact of x on the multiplicative utility premium as follows, if $\gamma < 1$.

Proposition 5.

If a share $h < 1$ of total wealth is exposed to a multiplicative risk and if u is CRRA with relative risk aversion γ, the nth-order multiplicative utility premium $\omega_M(w, x; Y, X)$ is an increasing function of x when $\gamma < 1$.

Proof:

Define $\alpha(k) = \frac{k}{\gamma+k-1}$. Using CRRA utility with relative risk aversion γ, Proposition 3a indicates that

$\omega_M(w, x; Y, X)$ decreases with x if $r_p^{(k)} \geq k$, i.e, if $h \geq \alpha(k)$, where h is defined above, $h \leq 1$.

$\omega_M(w, x; Y, X)$ increases with x if $r_p^{(k)} \leq k$, i.e., if $h \leq \alpha(k)$.

Note, in addition, that:

$\gamma < 1 \Rightarrow \alpha(k) > 1$ and $\alpha'(k) < 0 \; \forall k \geq 1$,

$\gamma > 1 \Rightarrow \alpha(k) < 1$ and $\alpha'(k) > 0 \; \forall k \geq 1$.

Hence, $\omega_M(w, x; Y, X)$ increases with x if $\gamma < 1$, all $k \geq 1$. *QED*

Note that we recover in this case ($\gamma < 1$), the first-order relationship between the multiplicative utility premium and the amount x exposed to risk observed in Proposition 4: it is positive. If $\gamma > 1$, as both h and $\alpha(k)$ are less than 1, and as $\alpha'(k) > 0$, the result depends on three parameters: h, γ and n. For instance, if $X \preceq_{Ekern-3} Y$ (third-order increase in risk) and $\gamma = 2$, we get $\alpha(n) = 0.75$. Thus the multiplicative utility premium $\omega_M(w, x; Y, X)$ increases with x if the share of wealth exposed to multiplicative risk h is less than 0.75, and it decreases if $h \geq 0.75$. A larger aversion to risk, for instance $\gamma = 2.5$, will reduce this threshold ratio to 0.66. A fourth-order increase in risk ($n = 4$) will drive it to a higher $h = 0.80$. Unfortunately, it appears impossible to provide a clear result leading to a simple proposition.

5 Conclusion

Our objective in this paper is to improve our knowledge about the shape of the utility premium introduced informally by Friedman and Savage (1948) and defined more formally by Eeckhoudt and Schlesinger (2009). This premium measures the loss of welfare in an expected utility framework when a risk averse DM faces an aggravation of risk. It was extended by Courbage et al. (2018) to risk deteriorations at any order n, using stochastic dominance or Ekern's (1980) definitions for increases in risk. All this literature focusses on additive risks. Starting from there, we first show that the nth-order additive utility premium is a decreasing and convex function of non-risky wealth (Proposition and Corollary 1). But our main motivation is to investigate whether this clear and intuitive shape is preserved when the aggravation bears on multiplicative risks. Addressing multiplicative risks yields a first complication. One need make a distinction between risk-free wealth and the amount of wealth exposed to multiplicative risks. We observe that the shape of the nth order mutiplicative utility premium as a function of risk-free wealth does not deviate from the relationship observed when the additive utility premium is the object of study: it is decreasing and convex (Proposition and Corollary 2). However, different shapes are possible when we focus on the relationship between the mutiplicative utility premium and the amount of wealth exposed to risk. Depending on conditions where the coefficient of partial relative risk aversion plays a leading role, the multiplicative utility premium at any order n may be decreasing or increasing, and convex or concave (Proposition 3 and Corollary 3). Trying to lend more substance to our results, we finally turn towards a familiar special case, the case of a CRRA utility function. Unfortunately, the loss of generality is not rewarded with more clear-cut results. Results are complete and quite clear only when all wealth is exposed to risk. Depending on whether the CRRA coefficient is larger or smaller than unity, the nth order multiplicative utility premium is then decreasing and convex, or increasing and concave (Proposition 4). If only a share of wealth is exposed to multiplicative risks, the results are partial and limited to the case where the CCRA coefficient is less than one (Proposition 5).

References

Arrow, K.J., 1965. *Aspects of the Theory of Risk-Bearing*, Yrjo Jahnssen Foundation, Helsinki.

Brockett, P.L., and Golden, L.L., 1987. A class of utility functions containing all the common utility functions, *Management Science*, 33, 955-964.

Caballé, J., and Pomansky, A., 1996. Mixed risk aversion, *Journal of Economic Theory*, 71, 485-513.

Chiu, H., Eeckhoudt. L., and Rey, B., 2012. On relative and partial risk attitudes: Theory and implications, *Economic Theory*, 50, 151-167.

Courbage, C., Loubergé, H. and Rey, B., 2018. On the properties of high-order non monetary measures for risk, *Geneva Risk and Insurance Review*, 43, 77-94.

Courbage, C., and Rey, B., 2010. On non monetary measures in the face of risk and the signs of the derivatives, *Bulletin of Economic Research*, 69, 295-304.

Crainich, D. and Eeckhoudt, L., 2008. On the intensity of downside risk aversion, *Journal of Risk and Uncertainty*, 36, 267-276.

Denuit, M., Lefevre, C., and Shaked, M., 1998. The s-convex orders among real random variables, with applications, *Mathematical Inequalities and Application*, 1(4), 585613.

Denuit, M. and Rey, B., 2013. Another look at risk apportionment, *Journal of Mathematical Economics*, 49, 335-343.

Denuit, M. and Rey, B., 2014. Benchmark values for higher order coefficients of relative risk aversion, *Theory and Decision*, 76, 81-94.

Ebert, S., Nocetti, D.C. and Schlesinger, H., 2017. Greater mutual aggravation, *Management Science*, 64, 2809-2811.

Eeckhoudt, L., Etner, J. and Schroyen, F. 2009a. The values of relative risk aversion and prudence: A context-free interpretation, *Mathematical Social Sciences*, 58, 1-7.

Eeckhoudt, L. and Schlesinger, H., 2006. Putting risk in its proper place, *American Economic Review*, 96, 280-289.

Eeckhoudt, L. and Schlesinger H., 2009. On the utility premium of Friedman and Savage, *Economics Letters*, 105, 46-48.

Eeckhoudt, L., Schlesinger, H. and Tsetlin, E., 2009b. Apportioning risks via stochastic dominance, *Journal of Economic Theory*, 113, 1-31.

Ekern, S., 1980. Increasing Nth-degree risk, *Economics Letters*, 6, 329-333.

Friedman, M. and Savage L.J., 1948. The utility analysis of choices involving risk, *Journal of Political Economy*, 56, 279-304.

Gollier,C., 2015. Discounting, inequalities and economic convergence, *Journal of Environmental Economics and Management*, 69, 53-61.

Hanson, D.L. and Menezes, C., 1971. On a neglected aspect of the theory of risk aversion, *Western Economic Journal*, 9, 211-217.

Heinzel, C., 2019. Comparing utility premia under additive and multiplicative risks, Working paper, INRA, Rennes.

Huang, J. and Stapleton, R., 2015. The utility premium of Friedman and Savage, comparative risk aversion, and comparative prudence, *Economics Letters*, 134, 34-36.

Jean, W.H., 1980. The geometric mean and stochastic dominance, *Journal of Finance*, 35, 151-158.

Jean, W.H., 1984. The harmonic mean and other necessary conditions for stochastic dominance, *Journal of Finance*, 39, 527-534.

Jindapon, P. and Neilson, W., 2007. Higher-order generalizations of Arrow-Pratt and Ross risk aversion: A comparative statics approach, *Journal of Economic Theory*, 136, 719-728.

Jindapon, P., Liu L. and Neilson, W., 2021. Comparative risk apportionment, *Economic Theory Bulletin*, 9, 91-112.

Li, J. and Liu, L., 2014. The monetary utility premium and interpersonal comparisons, *Economics Letters*, 125, 257-260.

Liu, L. and Meyer, J., 2013. Substituting one risk increase for another A method for measuring risk aversion, *Journal of Economic Theory*, 148, 2706-2718.

Liu, L. and Neilson, W., 2019. Alternative approaches to comparative nth-degree risk aversion, *Management Science*, 65, 3824-3834.

Loubergé, H., Malevergne, Y., Rey, B., 2020. New Results for additive and multiplicative risk apportionment, *Journal of Mathematical Economics*, 90, 140-151.

Menegatti, M. (2014). New results on the relationship among risk aversion, prudence and temperance. *European Journal of Operational Research*, 232, 613-617.

Menezes, C., Geiss, C. and Tressler J., 1980. Increasing downside risk, *American Economic Review*, 70, 921-932.

Menezes, C., and Hanson, D.L., 1970. On the theory of risk aversion, *International Economic Review*, 1, 481-487.

Pratt, J., 1964. Risk aversion in the small and in the large, *Econometrica*, 32, 122-136.

Rothschild, M. and Stiglitz, J., 1970. Increasing risk I: A definition, *Journal of Economic Theory*, 2, 225-243.

von Neumann, J. and Morgenstern, O., 1947. *Theory of Games and Economic Behavior*, Princeton University Press, Princeton, 2nd edition.

Wang, J., and Li, J., 2010. Multiplicative risk apportionment, *Mathematical Social Sciences*, 60, 79-81.

Wong, K.P. (2018). Comparative higher-order risk aversion and higher-order prudence, *Economics Letters*, 169, 38-42.

Wong, K.P. (2019). An interpretation of the condition for precautionary saving: the case of greater higher-order interest rate risk, *Journal of Economics*, 126, 275-286.

Zeckhauser, R., and Keeler, E., 1970. Another type of risk aversion, *Econometrica* , 38, 661-665.

Appendix 1

$w(x;Y,X) = E[u(x+Y)] - E[u(x+X)]$.

$\frac{dw(x;Y,X)}{dx} = E[u'(x+Y)] - E[u'(x+X)]$.

Let's define the function a as follows: $a(\epsilon) = u'(x+\epsilon)\ \forall \epsilon$ for a given x. The previous expression rewrites as

$\frac{dw(x;Y,X)}{dx} = E[a(Y)] - E[a(X)]$. Using SD properties, we obtain $\frac{dw(x;Y,X)}{dx} \leq 0$ if the function a verifies $(-1)^{k+1}a^{(k)} \leq 0\ \forall k = 1,\ldots,n$ since $X \preceq_{SD-n} Y$.

$a'(\epsilon) = u''(x+\epsilon)$,

$a''(\epsilon) = u'''(x+\epsilon)$,

$a'''(\epsilon) = u^{(4)}(x+\epsilon)$,

...

$a^{(k)}(\epsilon) = u^{(k+1)}(x+\epsilon)$.

We obtain $(-1)^{k+1}a^{(k)} \leq 0\ \forall k = 1,\ldots,n$ for all DMs who are MRA from 1 to $n+1$.

$\frac{d^2w(x;Y,X)}{dx^2} = E[u''(x+Y)] - E[u''(x+X)]$.

Let's define the function b as follows: $b(\epsilon) = u''(x+\epsilon)\ \forall \epsilon$ for a given x. The previous expression rewrites as

$\frac{d^2w(x;Y,X)}{dx^2} = E[b(Y)] - E[b(X)]$. Using SD properties, we obtain $\frac{d^2w(x;Y,X)}{dx^2} \geq 0$ if the function b verifies $(-1)^{k+1}b^{(k)} \geq 0\ \forall k = 1,\ldots,n$ since $X \preceq_{SD-n} Y$.

$b'(\epsilon) = u'''(x+\epsilon)$,

$b''(\epsilon) = u^{(4)}(x+\epsilon)$,

$b'''(\epsilon) = u^{(5)}(x+\epsilon)$,

...

$b^{(k)}(\epsilon) = u^{(k+2)}(x+\epsilon)$.

We obtain $(-1)^{k+1}b^{(k)} \geq 0\ \forall k = 1,\ldots,n$ for all DMs who are MRA from 1 to $n+2$. ■

Appendix 2

$w_M(w,x;Y,X) = E[u(w+xY)] - E[u(w+xX)]$.

$\frac{dw_M(w,x;Y,X)}{dx} = E[Yu'(w+xY)] - E[Xu'(w+xX)]$.

Let's define the function A as follows: $A(\epsilon) = \epsilon u'(w+x\epsilon)\ \forall \epsilon > 0$ for given w and $x > 0$.

Using this notation, the previous expression rewrites as

$\frac{dw_M(w,x;Y,X)}{dx} = E[A(Y)] - E[A(X)]$. Using SD properties, we obtain $\frac{dw_M(w,x;Y,X)}{dx} \leq 0$ if the function A verifies $(-1)^{k+1}A^{(k)} \leq 0\ \forall k = 1, \ldots, n$ since $X \preceq_{SD-n} Y$.

$A'(\epsilon) = u'(x+\epsilon) + x\epsilon u''(x+\epsilon),$

$A''(\epsilon) = 2xu''(x+\epsilon) + x^2\epsilon u'''(x+\epsilon),$

$A'''(\epsilon) = 3x^2u'''(x+\epsilon) + x^3\epsilon u^{(4)}(x+\epsilon),$

$\ldots$

$A^{(k)}(\epsilon) = kx^{k-1}u^{(k)}(x+\epsilon) + x^k\epsilon u^{(k+1)}(x+\epsilon).$

Assuming u such as $(-1)^{k+1}u^{(k)} \geq 0\ \forall k = 1, \ldots, n+1$, $(-1)^{k+1}A^{(k)} \leq 0\ \forall k = 1, \ldots, n$ is equivalent to $r_p^{(k)}(x\epsilon; w)) \geq k\ \forall k = 1, \ldots, n$.

We obtain then $\frac{dw_M(w,x;Y,X)}{dx} \leq 0 (\geq 0)$ if $r_p^{(k)}(x\epsilon; w)) \geq k (\leq k)\ \forall k = 1, \ldots, n$.

$\frac{d^2w_M(w,x;Y,X)}{dx^2} = E[Y^2u''(w+xY)] - E[X^2u''(w+xX)].$

Let's define the function c as follows: $c(\epsilon) = \epsilon^2u''(w+x\epsilon)\ \forall \epsilon > 0$ for given w and $x > 0$. Using this notation, the previous expression rewrites as

$\frac{d^2w_M(w,x;Y,X)}{dx^2} = E[c(Y)] - E[c(X)].$

Using SD properties, we obtain $\frac{d^2w_M(w,x;Y,X)}{dx^2} \geq 0 (\leq 0)$ if the function c verifies $(-1)^{k+1}c^{(k)} \geq 0 (\leq 0)\ \forall \epsilon > 0$, $\forall k = 1, \ldots, n$ since $X \preceq_{SD-n} Y$. ■

Appendix 3

$c(\epsilon) = \epsilon^2u''(w+x\epsilon)$

$c'(\epsilon) = 2\epsilon u''(w+x\epsilon) + \epsilon^2xu'''(w+x\epsilon)$

$c''(\epsilon) = 2u''(w+x\epsilon) + 4\epsilon xu'''(w+x\epsilon) + \epsilon^2x^2u^{(4)}(w+x\epsilon)$

$c'''(\epsilon) = 6xu'''(w+x\epsilon) + 6\epsilon x^2u^{(4)}(w+x\epsilon) + \epsilon^2x^3u^{(5)}(w+x\epsilon)$

$c^{(4)}(\epsilon) = 12x^2u^{(4)}(w+x\epsilon) + 8\epsilon x^3u^{(5)}(w+x\epsilon) + \epsilon^2x^4u^{(6)}(w+x\epsilon)$

$\ldots$

$$c^{(k)}(\epsilon) = \alpha_kx^{k-2}u^{(k)}(w+x\epsilon) + \beta_k\epsilon x^{k-1}u^{(k+1)}(w+x\epsilon) + \epsilon^2x^ku^{(k+2)}(w+x\epsilon)\ \forall k \geq 2 \quad (a)$$

We must determine the value of parameters α_k and β_k. Derivating $c^{(k)}$, we obtain the expression of $c^{(k+1)}\ \forall k \geq 2$ that is (in that follows, we denote by (.) the argument $(w+x\epsilon)$ in order to simplify mathematical expressions):

$$c^{(k+1)}(\epsilon) = (\alpha_k + \beta_k)x^{k-1}u^{(k+1)}(.) + (\beta_k + 2)\epsilon x^ku^{(k+2)}(.) + \epsilon^2x^{k+1}u^{(k+3)}(.) \quad (b)$$

Rewriting now Eq. (a) at the order $k+1$, we obtain, $\forall k \geq 2$:

$$c^{(k+1)}(\epsilon) = \alpha_{k+1} x^{k-1} u^{(k+1)}(.) + \beta_{k+1} \epsilon x^k u^{(k+2)}(.) + \epsilon^2 x^{k+1} u^{(k+3)}(.) \qquad (c)$$

Using (b) and (c), we obtain:

$$\alpha_{k+1} = \alpha_k + \beta_k \ \forall k \geq 2 \qquad (d)$$

$$\beta_{k+1} = \beta_k + 2 \ \forall k \geq 2 \qquad (e)$$

Let's begin to determine the value of β_k $\forall k \geq 2$.

Using the expression of $c^{(2)}$, we read $\beta_2 = 4$, and using (e), we obtain:

$\beta_3 = \beta_2 + 2 = 4 + 2$

$\beta_4 = \beta_3 + 2 = (4 + 2) + 2$

$\beta_5 = \beta_4 + 2 = (4 + 2 + 2) + 2$

$\beta_6 = \beta_5 + 2 = (4 + 2 + 2 + 2) + 2$

...

$\beta_k = \beta_{k-1} + 2 = 4 + 2(k-2) = 2k$

Using the expression of $c^{(2)}$, we read $\alpha_2 = 2$, and using (d), we obtain:

$\alpha_3 = \alpha_2 + \beta_2 = 2 + \beta_2$

$\alpha_4 = \alpha_3 + \beta_3 = (2 + \beta_2) + \beta_3$

$\alpha_5 = \alpha_4 + \beta_4 = (2 + \beta_2 + \beta_3) + \beta_4$

$\alpha_6 = \alpha_5 + \beta_5 = (2 + \beta_2 + \beta_3 + \beta_4) + \beta_5$

...

$\alpha_k = 2 + \sum_{j=2}^{k-1} \beta_j = 2 + 2\sum_{j=2}^{k-1} j = 2 + 2\frac{(k-2)(k+1)}{2} = k(k-1).$

Using expressions of α_k and β_k, (a) becomes:

$c^{(k)}(\epsilon) = k(k-1)x^{k-2}u^{(k)}(w+x\epsilon) + 2k\epsilon x^{k-1}u^{(k+1)}(w+x\epsilon) + \epsilon^2 x^k u^{(k+2)}(w+x\epsilon)\ \forall k \geq 2.$ ■

Appendix 4

In that follows, we denote by (.) the argument $(w+x\epsilon)$ in order to simplify mathematical expressions.

$c^{(k)}(\epsilon) = k(k-1)x^{k-2}u^{(k)}(.) + 2k\epsilon x^{k-1}u^{(k+1)}(.) + \epsilon^2 x^k u^{(k+2)}(.)\ \forall k \geq 2.$

For all MRA DMs and for all k odd, we have $u^{(k)} \geq 0$, $u^{(k+1)} \leq 0$ and $u^{(k+2)} \geq 0$. We obtain then, for all $k \geq 2$ and k odd,

$Sgn\{c^{(k)}(\epsilon)\} = Sgn\{k(k-1)u^{(k)}(.) + 2k\epsilon x u^{(k+1)}(.) + \epsilon^2 x^2 u^{(k+2)}(.)\}$ (since $x^{k-2} > 0$) that is equivalent to

$$Sgn\{c^{(k)}(\epsilon)\} = Sgn\{k(k-1) - 2k\Big(-\epsilon x \frac{u^{(k+1)}}{u^{(k)}}\Big) + \Big(-\epsilon x \frac{u^{(k+2)}}{u^{(k+1)}}\Big)\Big(-\epsilon x \frac{u^{(k+1)}}{u^{(k)}}\Big)\}$$

$$= Sgn\{k(k-1) + r_p^{(k)}(x\epsilon; w)(-2k + r_p^{(k+1)}(x\epsilon; w))\}.$$

Similarly, for all MRA DMs and for all k even, we have $u^{(k)} \leq 0$, $u^{(k+1)} \geq 0$ and $u^{(k+2)} \leq 0$. We obtain then, for all $k \geq 2$ and k even,

$$Sgn\{c^{(k)}(\epsilon)\} = -Sgn\{k(k-1) + r_p^{(k)}(x\epsilon; w)(-2k + r_p^{(k+1)}(x\epsilon; w))\}.$$

We conclude then, for all $k \geq 2$:

$$Sgn\{c^{(k)}(\epsilon)\} = (-1)^{(k+1)} Sgn\{k(k-1) + r_p^{(k)}(x\epsilon; w)(-2k + r_p^{(k+1)}(x\epsilon; w))\}.$$ ■