What do financial markets say about the exchange rate?*

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Abstract

Not very much.

We study various financial market structures disciplined by the absence of equilibrium arbitrage opportunities. In circumstances when financial markets are informative about the exchange rate, the constraints they impose on the exchange rate behavior yield counterfactual implications. In contrast, financial market structures that are consistent with observed empirical properties of the exchange rate impose few constrains on its equilibrium behavior, and hence financial markets contain nearly no information about the exchange rate.

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Introduction

In this paper, we investigate whether financial markets are informative about exchange rates. We completely characterize how the absence of arbitrage imposes restrictions on exchange rate dynamics for a wide class of economies. Euler equations imply how expected currency depreciation rates and exchange rate risk relate to stochastic discount factors (SDFs) of domestic and foreign households. The market structure (completeness, integration, intermediation) as well as the nature of asset risks across countries shape these relations. We develop evidence about the latter, establishing a financial disconnect: financial asset returns are only weakly related to exchange rates. Taking empirics and theory together, we conclude that, depending on market structure, implications for exchange rates are either sharp but counterfactual or very limited.

We consider an environment with a domestic and a foreign representative household, each of which is trading a given set of assets in its own currency. These assets could be the same, have some overlap or be completely distinct across households. Each household has an SDF, and its Euler equation prices the assets that it trades. Our main assumption is lack of arbitrage in international markets, that is, for portfolios of domestic and foreign assets. While simple, this condition captures the essence of many theories of financial markets. For example, it arises when one or both households can invest in all assets, or when an intermediary is able to access all assets.

Restrictions on the behavior of the exchange rate come from the structure of the domestic and the foreign set of assets, and their relation with the exchange rate. We introduce a taxonomy of shocks affecting asset returns to characterize these restrictions. Specifically, we delineate globally and locally traded shocks. Globally traded shocks can be replicated separately by portfolios of domestic assets in domestic currency and portfolios of foreign assets in foreign currency. Local shocks are the remaining sources of variation in asset returns. The depreciation rate can be exposed to these two types of shocks as well as experience variation that is unspanned by asset returns.

In this setting, two results characterize entirely the restrictions on the behavior of the exchange rate. The first one disciplines shocks to the depreciation rate — exchange rate risk — while the second constrains the expected depreciation rate.

The first result states that the projection of the log depreciation rate on global shocks coincides with the projection of the difference of the logs of foreign and domestic SDFs on the same shocks,

$$E_t(\Delta s_{t+1}|\boldsymbol{\epsilon}^G_{t+1}) = E_t(m^*_{t+1} - m_{t+1}|\boldsymbol{\epsilon}^G_{t+1}).$$

This general condition nests in particular the well-known complete market relation $\Delta s_{t+1} = m^*_{t+1} - m_{t+1}$. The latter relation is the source of many international macroeconomics puzzles, such as the cyclicality puzzle of Backus and Smith (1993) and the volatility puzzle of Brandt, Cochrane, and Santa-Clara (2006). Our general result offers a path to overcome these puzzles by imposing weaker restrictions.

The second result characterizes departures of the expected depreciation rate from uncovered interest parity (UIP). The important requirement for this restriction in our setting is that the depreciation rate is spanned by a combination of domestic and foreign assets in their respective currencies. If this condition holds, we show that UIP deviations consist of two components. One is the standard complete market result of an FX risk premium. The other one reflects differences in how foreign and domestic investors price risk. Just like with the first result, this offers flexibility in matching the expected depreciation rate, a quantity which the complete markets framework struggles to replicate without aggravating the volatility and cyclicality puzzles.

Our two results highlight the importance of relations between foreign and domestic assets: presence of globally traded shocks for the first one, spanning of the depreciation rate for the second one. These relations can arise in a market structure where some assets are traded in common by both households or, even absent any integration, if different assets have related risks. As the next step of our analysis, we investigate specific implications of the two results for a variety of such settings.

We first study the role of different forms of market integration. We consider situations in which markets are not only incomplete, but also in which domestic and foreign investors have access to different sets of assets. As long as one investor can trade the risk-free bond of the other country, the exchange rate becomes spanned by asset returns — specifically by the carry trade — and the FX risk premium coincides with complete markets. However, such an economy is still relatively flexible in terms of cyclicality and volatility of exchange rates. The critical feature that leads to puzzles with these quantities is when both investors can access the risk-free bond of the other country, an economy studied in Lustig and Verdelhan (2015). In this case, the exchange rate itself is a globally traded shock — both investors can engage in the carry trade — and hence becomes tightly connected to SDFs. If both investors can also trade the same risky assets, this can only bring the exchange rate closer to complete markets. In contrast, if all assets but one risk-free bond are traded by both investors, restrictions on the exchange rate can remain weak.

Second, we study the role of the nature of risks in the financial markets of each economy. To do so, we remove any direct form of market integration, and concentrate on a setting in which intermediaries trade and enforce the absence of arbitrage on international markets. In these economies, the exchange rate can exhibit significant risk that is not spanned by asset returns. This unspanned risk can justify an FX risk premium independent from those determined by local SDFs. In addition, exchange rate risk is connected to the household SDFs only through the presence of globally traded shocks. For example if the domestic and foreign economy are driven by the same global cycle, this global shock will drive both SDFs and the exchange rate, and they will be connected. However, if the two economies are each driven by their own set of shocks, the connection is broken; SDFs and the exchange rate can be arbitrarily related. As such, intermediated economies can lead to realistic exchange rates, but exactly when financial markets impose no discipline.

Motivated by the intermediated model, we turn to the data to quantify the relations between asset returns and exchange rates. Equivalently, we ask how much an econometrician, even in a complete market economy, can hope to learn about the exchange rate from observing other asset returns. We study all G10 countries relative to the U.S. between 1988 and 2022. In terms of asset returns, we use various stock indices (market, value, growth, industries) and government bonds of different maturities. First, we find consistent evidence that asset returns do not span exchange rates. Regressing depreciation rates on all asset returns of the home and foreign country lead to R^2s below 50%, statistically and economically far from the perfect relation necessary to discipline the FX risk premium. Still, if global shocks are important, exchange rate shocks could be determined by the difference of SDFs. Here again, the evidence does not suggest these restrictions are empirically relevant. Using canonical correlation analysis, we find that the most correlated portfolios across country have at most a correlation of 0.8, already a tenuous notion of globally traded shocks. And, even if one is willing to consider those portfolios as globally traded, they only explain a very small fraction of the variation in exchange rates.

Overall, these results suggest that financial markets are not very informative about exchange rates. In market structures that impose discipline on exchange rates, the same puzzles as in the complete market setting arise. Conversely, market structures that accommodate realistic exchange rates do not pin down much about them. One interpretation of these results is negative: the enterprise of focusing on macro Euler equations or on data on other financial assets is unlikely to lead to sharp conclusions about exchange rates. But there is also a more positive take: some theories, in particular centered on intermediated markets, can yield realistic exchange rates. One just has to dig deeper into what determines the activity and trading choices of these intermediaries to get at exchange rate determination.

1 Setup

We are interested in restrictions on the behavior of exchange rates coming from properties of other asset returns. To answer this question, we introduce a general framework and derive two sets of restrictions, on the risk (variance-covariance) of exchange rate depreciation and the expected exchange rate depreciation.

There are two countries, which we refer to as Home and Foreign. Denote S_t the nominal exchange rate at date t equal to the number of units of home currency for one unit of foreign currency. All along the paper, lowercase letters represent logs.

For example, s_t is the log exchange rate. We also denote with a tilde demeaned variables, that is, shocks. For example, $\widetilde{\Delta s}_{t+1} = \Delta s_{t+1} - E_t \Delta s_{t+1}$.

1.1 Assets

We consider various sets of assets that investors can have access to: domestic, foreign, or combinations of the two. Within these asset sets, investors can form arbitrary portfolios. Specifically, we start from a collection of assets with log returns $\boldsymbol{r}_{t+1} = (r_{1,t+1}, \ldots, r_{N,t+1})$. The corresponding set of assets contains all feasible portfolios, $\{r_{p,t+1} | \exists \boldsymbol{w}_t \in \mathbb{R}^N : \boldsymbol{w}_t' \boldsymbol{\iota} = 1, r_{p,t+1} = \log(\boldsymbol{w}_t' \exp(\boldsymbol{r}_{t+1}))\}$.

To maintain tractability, we follow Campbell and Viceira (2002) and approximate the log portfolio excess returns relative to a risk-free rate r_{ft}:

$$\begin{aligned} r_{p,t+1} - r_{ft} &= \log\left(\boldsymbol{w}_t' e^{\boldsymbol{r}_{t+1} - r_{ft}\boldsymbol{\iota}}\right) \\ &\approx \boldsymbol{w}_t'(\boldsymbol{r}_{t+1} - r_{ft}\boldsymbol{\iota}) + \frac{1}{2}\boldsymbol{w}_t' \operatorname{diag}(\boldsymbol{\Sigma}_t) - \frac{1}{2}\boldsymbol{w}_t' \boldsymbol{\Sigma}_t \boldsymbol{w}_t, \end{aligned} \tag{1}$$

where $\boldsymbol{\Sigma}_t$ is the variance-covariance matrix of log returns. This approximation allows us to represent portfolios returns as linear combination of log returns. Importantly, it is stable by recombination, leading to the same result when applied in two steps or all at once for a portfolio of portfolios.

We adapt this framework to an international setting by considering two sets of returns: one in domestic currency H, another in foreign currency F. As an example, H may include a domestic sovereign bond, or foreign equity index converted to domestic currency.

We denote the returns on the assets accessible by the domestic investors $\boldsymbol{r}_{t+1} = (r_{1,t+1}, \ldots, r_{N,t+1})$, and assume this collection includes a risk-free asset with return r_{ft} in home currency known at time t. Furthermore, we assume that asset returns are log-normal, that is $\boldsymbol{r}_{t+1}$ are multivariate normal, $MVN(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t)$. Similarly, the returns of the assets accessible by the foreign investor are $\boldsymbol{r}^*_{t+1}$ in foreign currency, log-normal of size N^*, and contain a foreign-currency risk-free rate of r^*_{ft} .

Global and local shocks. Intuitively, returns are affected by a collection of shocks, some of which are local to each economy, $\boldsymbol{\epsilon}_{t+1}$ or $\boldsymbol{\epsilon}^*_{t+1}$, while others are common to both, i.e., global shocks, $\boldsymbol{\epsilon}^G_{t+1}$. This notion corresponds to

$$\tilde{\boldsymbol{r}}_{t+1} = \boldsymbol{P}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{P}^G\boldsymbol{\epsilon}^G_{t+1}, \tag{2}$$

$$\tilde{\boldsymbol{r}}^*_{t+1} = \boldsymbol{P}^*\boldsymbol{\epsilon}^*_{t+1} + \boldsymbol{P}^{\star G}\boldsymbol{\epsilon}^G_{t+1}. \tag{3}$$

To make this representation operational, we specify how to construct the local and global shocks.

Definition 1. *Local and global shocks are a collection* $(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1}, \boldsymbol{\epsilon}^G_{t+1})$ *such that:*

1. *Shocks to asset returns in each country can be represented as a combination of global shocks and local shocks of that country (equations (2) and (3)).*

2. *Global shocks can be constructed as innovations to portfolios of assets in each country:* $\boldsymbol{\epsilon}^G_{t+1} = \boldsymbol{A}^G\tilde{\boldsymbol{r}}_{t+1} = \boldsymbol{A}^{\star G}\tilde{\boldsymbol{r}}^*_{t+1}$

3. *Local shocks can be replicated by portfolios of assets in their respective countries,* $\boldsymbol{\epsilon}_{t+1} = \boldsymbol{A}\tilde{\boldsymbol{r}}_{t+1}$ *and* $\boldsymbol{\epsilon}^*_{t+1} = \boldsymbol{A}^*\tilde{\boldsymbol{r}}^*_{t+1}$*, such that* $(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1}) \perp \boldsymbol{\epsilon}^G_{t+1}$.

These shocks are unique up to rotation and rescaling and form bases in the respective economies. Importantly, one can go from asset returns to the shocks and from the shocks to asset returns. That is $(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^G_{t+1})$ is the basis in H, and $(\boldsymbol{\epsilon}^*_{t+1}, \boldsymbol{\epsilon}^G_{t+1})$ is the basis in F. In practice, the notion of global shock in Definition 1 is quite restrictive as it requires perfectly correlated portfolios in H and F. In section 2.3 we discuss how a notion of approximate global shocks, from strongly but not perfectly correlated portfolios, can also be relevant.

1.2 Stochastic discount factors.

We specify valuation mechanisms for each set of assets, H and F, separately.

Assumption 1. *The domestic (log) stochastic discount factor (SDF) m_{t+1} prices all assets in H, i.e., it satisfies the Euler equation*

$$\forall r_{t+1} \in H : \quad E_t \left[\exp(m_{t+1} + r_{t+1})\right] = 1. \tag{4}$$

Similarly, the foreign log SDF m^*_{t+1} prices all assets in F :

$$\forall r^*_{t+1} \in F : \quad E_t \left[\exp(m^*_{t+1} + r^*_{t+1})\right] = 1. \tag{5}$$

For now, we do not take a stand on the origins of these discount factors. In some applications, the discount factors represent optimal decisions of domestic and foreign households. For example with CRRA utility, we would have $m_{t+1} = -\gamma \log(C_{t+1}/C_t)$, with C_t being aggregate domestic consumption, and γ a coefficient of risk aversion. In other applications, the discount factors are simply a representation of the risk-return relation among assets. For example, the SDF could be constructed from asset returns as $m_{t+1} = \boldsymbol{\lambda}_t' \boldsymbol{r}_{t+1}$, with $\boldsymbol{\lambda}_t \in \mathbb{R}^N$, in the spirit of Hansen and Jagannathan (1991).

We focus on situations with log-normal SDFs. The Euler equations imply that expected excess returns are proportional to the covariance with the stochastic discount factors. In our log-normal setting, this corresponds to:

$$\forall r_{t+1} \in H: \quad E_t(r_{t+1}) + \frac{1}{2} var_t(r_{t+1}) = r_{ft} - cov_t(m_{t+1}, r_{t+1}), \tag{6}$$

$$\forall r^*_{t+1} \in F: \quad E_t(r^*_{t+1}) + \frac{1}{2} var_t(r^*_{t+1}) = r^*_{ft} - cov_t(m^*_{t+1}, r^*_{t+1}). \tag{7}$$

Our assumptions so far ensure that each of the domestic and foreign set of asset returns have standard and tractable properties. Importantly, note that none of them involves explicitly the exchange rate. Next, we turn to the connection between domestic and foreign asset returns.

1.3 Exchange rate depreciation

Recall that s_t is the log of the nominal exchange rate, and therefore Δs_{t+1} is the nominal depreciation. Specifically, $\Delta s_{t+1} > 0$ means that home currency loses its purchasing power of foreign currency, that is more units of home currency are need to buy one unit of foreign currency. Conversely, $\Delta s_{t+1} < 0$ corresponds to the home currency appreciation. Financial markets are concerned with exchange rate depreciations and appreciations, not the level of the exchange rate, as foreign currency asset returns are increased by the exchange rate depreciation when expressed in home currency terms.

The financial markets cares about both the expected exchange rate depreciation, $E_t \Delta s_{t+1}$, and the exchange rate depreciation risk, $\widetilde{\Delta s}_{t+1} = \Delta s_{t+1} - E_t \Delta s_{t+1}$, as they affect expected return and risk of foreign currency-assets. To streamline the discussion, we represent the depreciation rate generically as follows:

$$\Delta s_{t+1} = E_t(\Delta s_{t+1}) + v_{t+1} + u_{t+1}, \tag{8}$$

where the innovation v_t represents variation in the depreciation rate spanned by financial assets, $v_{t+1} = E(\widetilde{\Delta s}_{t+1} | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$, and u_{t+1} is the unspanned innovation, $cov_t(v_{t+1}, u_{t+1}) = 0$.

We can use Definition 1 to represent v_{t+1} in terms of global and local shocks:

$$v_{t+1} = \boldsymbol{S}^G \boldsymbol{\epsilon}^G_{t+1} + \boldsymbol{S} \boldsymbol{\epsilon}_{t+1} + \boldsymbol{S}^* \boldsymbol{\epsilon}^*_{t+1} \tag{9}$$

for some conformable matrices $\boldsymbol{S}^G$, $\boldsymbol{S}$ and $\boldsymbol{S}^*$ and $u_{t+1} \perp (\boldsymbol{\epsilon}^G_{t+1}, \boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})$.

1.4 Connecting domestic and foreign markets

We are interested in how international financial trade restricts the properties of exchange rates. We focus on a simple implication of this activity: the absence of arbitrage. Lack of arbitrage in international markets arises in a very large class of models. It is always satisfied in theories with frictionless asset markets in which all investors can purchase all assets (e.g., Colacito and Croce, 2011). It is also a feature of many theories with frictions in which some, but not all investors can access all asset markets. In particular theories focusing on the role of intermediaries often assume that representative households in each country are limited in their access to markets, while financial intermediaries can access all assets and would not be limited in exploiting arbitrage opportunities (e.g., Gabaix and Maggiori, 2015, Itskhoki and Mukhin, 2021).

We consider the set of international portfolios I: combinations of positions in domestic and foreign assets ultimately converted to the domestic currency.[1] Following our notations, international portfolios are generated by the set of assets $\check{\boldsymbol{r}}_{t+1} = (\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1} + \Delta s_{t+1}) = (r_{1,t+1}, \ldots, r_{N,t+1}, r^*_{1,t+1} + \Delta s_{t+1}, \ldots, r^*_{N^*,t+1} + \Delta s_{t+1})$. That is, assets that are contained in H and F with returns converted in domestic currency are jointly contained in I.

Assumption 2. *There are no arbitrage opportunities in the set of international portfolios I.*

In our log-normal setting, the main implication no arbitrage is that if a portfolio has no risk, it must earn the risk-free rate of return:

$$\forall r_{p,t+1} \in I, \; var_t(r_{p,t+1}) = 0 \quad \Rightarrow \quad E_t(r_{p,t+1}) = r_{f,t}. \tag{10}$$

Discussion of assumptions. What do Assumptions 1 and 2 involve? Assumption 1 is merely a definition of the sets H and F. If there exists a domestic asset that does not satisfy the Euler equation (6), it is not part of the set H, and thus our propositions below have less information to work with. Theories of convenience yield, financial frictions and segmented markets are isomorphic from the point of view of Assumption 1 in that they all limit the scope of sets H and F. We discuss this further below.

[1] Our conclusions are unchanged if we focus on international arbitrage in foreign currency.

Assumption 2 requires that there is no arbitrage in the set of assets I, which at least includes assets in H and F converted into domestic currency. Set I can also be arbitrarily richer than this without changing our results below. However, in contrast to the first assumption, Assumption 2 is restrictive. The propositions below rely on no arbitrage within the set of assets I, and thus would not generally hold if pure arbitrage strategy within the set of assets I are either made infeasible or costly due to e.g. financing, leverage or regulatory constraints. As we will see, in some contexts, Assumption 2 has limited content. We strengthen this assumption below from exact no arbitrage to the lack of large profitable trading opportunities.

We discuss below various example with integrated versus intermediated financial markets. In the case of integrated markets, the sets H and F overlap in terms of physical assets. For example, a home-currency risk free bond is available to agents abroad as one of the risky assets after converting into foreign currency. In such situation, it is the agents with SDFs m_{t+1} and $m^{\star}_{t+1}$, who price assets in H and F respectively, can themselves be engaged in international arbitrage, that is ensure no arbitrage opportunities within the set I.

In contrast, the sets H and F may have no overlap, and then the absence of international arbitrage must be ensured by intermediaries who have access to assets in I. In this case, we may introduce an additional SDF m^{I}_{t+1} of intermediaries that prices all assets $\check{r}_{t+1}$ in I, e.g. such that $m^{I}_{t+1} = \boldsymbol{\lambda}^{I\prime}_{t} \check{\boldsymbol{r}}_{t+1}$ for some $\boldsymbol{\lambda}^{I}_{t} \in \mathbb{R}^{N+N^{\star}}$. However, specifying such SDF is not required by our propositions that rely on the lack of exact arbitrage. Furthermore, for the same reason our propositions do not require us to specify which of the two cases — integrated or intermediated — we are in.

Lastly, we use the approximation in (1) and the log-normality assumption to derive simple analytical expressions. Our results hold more generally without relying on log-normality or the approximation in (1), but this comes at the cost of more complex higher order expressions, as we describe in the appendix.

1.5 Two international portfolios

We consider arbitrage trading strategies with two international portfolios.

Carry trade. One zero-cost portfolio, often referred to as carry, entails taking long and short positions in related assets:

$$R_{\text{carry},t+1} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t. \tag{11}$$

Traditionally, the traded assets are taken to be domestic and foreign risk-free (one-period) bonds. But carry does not have to be limited by that. For instance, Lustig, Stathopoulos, and Verdelhan (2013) consider long-term bonds. More generally, one could use any pair of assets that are close to each other, e.g., $corr_t(r_{t+1}, r^*_{t+1}) \approx 1$.

The key characteristic of the carry trade is that it exposes the arbitrageur to currency risk:

Lemma 1. *The conversion from foreign to home returns in the carry portfolio introduces exposure to currency risk,* $\tilde{r}_{\text{carry},t+1} = \tilde{r}_{t+1} - \tilde{r}^*_{t+1} + \widetilde{\Delta s}_{t+1}$.

Proof. We map the zero-cost portfolio (11) into the log approximation of a funded portfolio in equation (1) by adding a position in the risk-free asset:

$$R_{p,t+1} \equiv R_{\text{carry},t+1} + R_{f,t} = R_{t+1} - R^*_{t+1} \cdot S_{t+1}/S_t + R_{f,t}.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to USD, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the domestic risk-free asset with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the log gross return relative to the risk-free rate $R_{p,t+1}/R_{f,t}$:

$$\begin{aligned} r_{\text{carry},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r^*_{t+1} - \Delta s_{t+1} + cov_t(r_{t+1} - r^*_{t+1} - \Delta s_{t+1}, r^*_{t+1} + \Delta s_{t+1}). \end{aligned} \tag{12}$$

Thus, the shocks to the exchange rate have an impact on the portfolio performance. ■

Differential carry. That carry is exposed to currency risk prompts us to consider another zero-cost portfolio, labeled as differential carry, which is long one unit of the domestic asset, and short one unit of the foreign asset, financed at the respective risk-free rates:

$$R_{\text{diff},t+1} = (R_{t+1} - R_{ft}) - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t. \tag{13}$$

Intuitively, this portfolio does not introduce additional currency exposure because, in contrast to carry, only the foreign excess return is converted to USD. We demonstrate this formally in the following lemma.

Lemma 2. *The conversion from foreign to US returns in the diff portfolio does not* *introduce additional exposure to currency risk,* $\tilde{r}_{\text{diff},t+1} = \tilde{r}_{t+1} - \tilde{r}^*_{t+1}$.

Proof. We map the zero-cost portfolio (13) into a funded portfolio to use the approximation of equation (1):

$$R_{p,t+1} \equiv R_{diff,t+1} + R_{f,t} = R_{t+1} - (R^*_{t+1} - R^*_{ft}) \cdot S_{t+1}/S_t.$$

The portfolio $R_{p,t+1}$ corresponds to the weights $w_1 = 1$ in the domestic risky asset R_{t+1}, $w_2 = -1$ in the foreign risky asset converted to USD, $R^*_{t+1} \cdot S_{t+1}/S_t$, and $w_3 = 1$ in the foreign risk-free asset converted to USD, $R^*_{ft} \cdot S_{t+1}/S_t$, with $\boldsymbol{w}_t = (w_1, w_2, w_3)'$. These weights lead to an expression for the relative log return:

$$\begin{aligned} r_{\text{diff},t+1} &\equiv r_{p,t+1} - r_{ft} \\ &= r_{t+1} - r_{ft} - (r^*_{t+1} - r^*_{ft}) - cov_t(r^*_{t+1}, \Delta s_{t+1}) + cov_t(r^*_{t+1}, r_{t+1} - r^*_{t+1}). \end{aligned} \tag{14}$$

Thus, only the covariance of the foreign return with the exchange rate has a material impact on portfolio performance, not the shocks to the exchange rate. ■

The disappearance of exchange rate risk for the diff returns is in part due to our portfolio approximation. In Appendix Section B, we confirm that this approximation is very tight empirically. We compare the excess returns on various stock portfolios and sovereign bonds in their origin currency and in converted currency. The correlation between the two monthly series is always around 99.9%.

2 The general asset market view of exchange rates

In this section, we characterize the restrictions on the behavior of the exchange rate imposed by the absence of international arbitrage. We show that Assumptions 1 and 2 impose two sets of necessary restrictions on the depreciation rate: one on the shocks to the depreciation rate $\widetilde{\Delta s}_{t+1} = v_{t+1} + u_{t+1}$, and another on the expected depreciation rate $E_t \Delta s_{t+1}$. Appendix XX shows that these restrictions are sufficient as well.

We demonstrate that in the complete-markets setting these two sets of restrictions lead to the well-known asset market view of exchange rates and the puzzles that come with it. Sections 3 and 4 spell out the implications of these restrictions for a much larger set of market structures and revisit the puzzles in light of these results.

2.1 Exchange rate shocks

Risks that are present in both domestic and foreign assets — what we call global shocks — must be priced in a consistent way once adjusting for the conversion of currency. The following proposition formalizes this intuition.

Proposition 1. *Under Assumptions 1 and 2,* $\forall r_{t+1} \in H, r^*_{t+1} \in F : \tilde{r}_{t+1} = \tilde{r}^*_{t+1}$

$$E(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \widetilde{r}_{t+1}) = E(\widetilde{\Delta s}_{t+1} |, \widetilde{r}_{t+1}). \tag{15}$$

In words, if a domestic and a foreign portfolio returns are perfectly correlated, then the projection of the difference of SDF shocks on these returns must coincide with the projection of the depreciation rate shock on these returns. Perhaps surprisingly, the proposition relies on the apples-to-oranges comparison of shocks to returns in different currencies. Lemma 2 explains why that is a natural requirement and sets the stage for the proposition's proof.

Proof. Consider the log diff portfolio in equation (14). By Lemma 2, the portfolio has no exposure to the asset return when $\tilde{r}_{t+1} = \tilde{r}^*_{t+1}$, i.e., the shocks to foreign and domestic return perfectly offset each other. We see immediately that the portfolio has no risk, and as such must have expected returns equal to the risk-free rate.

Using the foreign and domestic Euler equations (6) and (7) to represent the domestic and foreign expected returns, we obtain

$$cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{t+1}) = 0. \tag{16}$$

This equation is equivalent to

$$cov(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} - \widetilde{\Delta s}_{t+1}, \widetilde{r}_{t+1}) = 0, \tag{17}$$

which under log-normality implies equation (15). ■

Because Proposition 1 must hold for any risk that is both in the domestic and foreign set of portfolios, it must also hold in terms of multivariate projections on all such returns $\{\tilde{r}_{t+1}\}$ spanned by both H and F. By definition, such intersection of H and F is the global shock, $\boldsymbol{\epsilon}^G_{t+1}$. We thus have:

Corollary (to Proposition 1)**.** *The projection of the depreciation rate on global shocks* *coincides with the projection of the difference in the SDFs on global shocks:*

$$E(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1}|\boldsymbol{\epsilon}^G_{t+1}) = E(\widetilde{\Delta s}_{t+1}|\boldsymbol{\epsilon}^G_{t+1}). \tag{18}$$

As we highlighted earlier, this condition is also sufficient. That means that despite the different types of shocks affecting the depreciation rate, which are enumerated in equations (8) and (9), only the global ones carry information about the relation between SDFs and the depreciation rate. Proposition 1 does not impose any restrictions associated with either local shocks $(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^\star_{t+1})$ or unspanned shocks $\boldsymbol{u}_{t+1}$.

Complete markets: the cyclicality and volatility puzzles. As an example, consider the case of complete financial markets. Financial markets are complete when investors have access to the full set of Arrow-Debreu securities in both markets. In this setting, one can construct returns such that $\tilde{r}_{t+1} = \tilde{r}^*_{t+1}$ for any desired value of $\tilde{r}_{t+1}$. Select $\tilde{m}^*_{t+1} - \tilde{m}_{t+1} - \widetilde{\Delta s}_{t+1}$ as such value. Then Proposition 1 implies

$$\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} = \widetilde{\Delta s}_{t+1}. \tag{19}$$

Innovations to the depreciation rate must equal innovations to the difference of stochastic discount factors, completely pinning down exchange rate shocks.

This result leads to two puzzles about the behavior of the exchange rate. First, consider the variance of the depreciation rate:

$$\begin{aligned} var_t(\Delta s_{t+1}) &= var_t(m^\star_{t+1} - m_{t+1}) \\ &= var_t(m^\star_{t+1}) + var_t(m_{t+1}) - 2cov_t(m_{t+1}, m^\star_{t+1}). \end{aligned} \tag{20}$$

Brandt, Cochrane, and Santa-Clara (2006) argue that this equation creates a volatility puzzle, with the exchange rate being not volatile enough. Typical observed Sharpe ratios on domestic assets imply highly volatile SDFs, much more so than exchange rate depreciation. The mild correlation of macro quantities across countries suggests that the SDFs are not correlated enough for the last term of equation (20) to offset this high variance and obtain realistic exchange rate risk.

Further, changes in exchange rates must be perfectly correlated with relative marginal utilities of the domestic and foreign households, that is, the home currency depreciates in relatively good times for home investors. As first noticed in Backus and Smith (1993), this implication is counterfactual to various measures of good times.

2.2 Expected depreciation rate

We turn to restrictions on the behavior of the expected depreciation rate. Unlike for the shocks to the exchange rate, no arbitrage constrains expected depreciation if and only if there is an exact arbitrage trade involving the exchange rate.

Proposition 2. *If the exchange rate is spanned by asset returns,* $\widetilde{\Delta s}_{t+1} = E(\widetilde{\Delta s}_{t+1}|\widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$*, then there exists* $r_{t+1} \in H$ *and* $r^*_{t+1} \in F$ *such that* $\widetilde{r}_{t+1} =$

$\widetilde{r}^*_{t+1} + \widetilde{\Delta s}_{t+1}$ *and we have:*

$$E_t(\Delta s_{t+1}) = \underbrace{r_{ft} - r^*_{ft}}_{UIP} - \underbrace{cov_t(m_{t+1}, \Delta s_{t+1})}_{\text{Exchange rate risk premium}} - \underbrace{\frac{1}{2} var_t(\Delta s_{t+1})}_{\text{Siegel paradox}}$$
$$+ \underbrace{cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{t+1})}_{\text{Deviation from integrated markets}} \quad (21)$$
$$= r_{ft} - r^*_{ft} - cov_t(m^*_{t+1}, \Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1})$$
$$+ cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{t+1}).$$

The first line of the expression represents the standard expression for the (log) currency risk premium in frictionless markets. If there is no risk premium, uncovered interest parity (UIP) implies that the expected depreciation rate offsets differences in interest rates across the two countries, which is the first term. The second term represents the risk premium. The third term reflects the convexity adjustment due to the log transformation. The presence of the second line is novel to the literature. It reflects the potential lack of integration, that is, the inability of the domestic investor to value the foreign assets. The expression in lines three and four is an alternative way to express the same result using the relation $\widetilde{r}_{t+1} = \widetilde{r}^*_{t+1} + \widetilde{\Delta s}_{t+1}$. The convexity adjustment term has a different sign in accordance with the Siegel (1972) paradox.

Proof. Note that the spanning condition is equivalent to the existence of $r_{t+1} \in H$ and $r^*_{t+1} \in F$ such that $\widetilde{r}_{t+1} = \widetilde{r}^*_{t+1} + \widetilde{\Delta s}_{t+1}$.[2] In other words, spanning of the exchange rate is equivalent to saying that there is a pair of domestic and foreign assets with identical shocks when converting to the same currency. The absence of arbitrage implies that a portfolio long the domestic asset and short the converted

[2] Because H and F contain risk-free assets, it is possible that $\widetilde{r}_{t+1} = 0$ or $\widetilde{r}^*_{t+1} = 0$.

foreign asset, aka the carry portfolio, must have 0 expected excess returns.

Consider then the log approximation of the carry portfolio in equation (12). This corresponds, using equations (6) and (7), to:

$$\begin{aligned} r_{ft} - cov_t(m_{t+1}, r_{t+1}) = r^*_{ft} &- cov_t(m^*_{t+1}, r^*_{t+1}) \\ &+ E_t(\Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) + cov_t(r^*_{t+1}, \Delta s_{t+1}). \end{aligned} \tag{22}$$

Using the spanning relation, we can eliminate r^*_{t+1}:

$$\begin{aligned} r_{ft} - cov_t(m_{t+1}, r_{t+1}) = r^*_{ft} &- cov_t(m^*_{t+1}, r_{t+1}) + cov_t(m^*_{t+1}, \Delta s_{t+1}) \\ &+ E_t(\Delta s_{t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) + cov_t(r_{t+1}, \Delta s_{t+1}). \end{aligned} \tag{23}$$

Isolating the expected depreciation rate gives the first line of equation (21). Equivalently, we can express everything as a function of r^*_{t+1}. ■

The proposition relies on the depreciation rate being spanned by financial assets, which corresponds to $u_{t+1} = 0$ in our decomposition, that is $\widetilde{\Delta s}_{t+1} = v_{t+1} = E(\widetilde{\Delta s}_{t+1} | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^*_{t+1})$. If that is not the case, $E_t(\Delta s_{t+1})$ is unconstrained.

Complete markets: the currency risk premium puzzle. Let us revisit the case of complete markets. In this setting $\tilde{\boldsymbol{r}}_{t+1}$ and $\tilde{\boldsymbol{r}}^*_{t+1}$ span $\widetilde{\Delta s}_{t+1}$. In addition, the market integration term is equal to 0 because of equation (19). Therefore, Proposition 2 implies currency risk premium from the domestic and foreign perspectives:

$$E_t(\Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) = r_{ft} - r_{ft}^* - cov_t(m_{t+1}, \Delta s_{t+1}), \tag{24}$$

$$E_t(-\Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) = r_{ft}^* - r_{ft} - cov_t(m_{t+1}, -\Delta s_{t+1}). \tag{25}$$

This relation leads to the currency risk premium puzzle. The complete-market setting does generate deviations from uncovered interest parity (UIP) via a risk premium for currency risk. Standard international models struggle with generating the empirically observed magnitude and cyclicality of the deviations simultaneously with addressing the first two puzzles.

Using the Euler equations, we can express $r_{ft} - r_{ft}^*$ in equation (24) in terms of SDFs. As a result,

$$E_t(\Delta s_{t+1}) = E_t(m_{t+1} - m_{t+1}^*).$$

The mean depreciation rate must equal the mean of the difference of stochastic discount factors. Combining this equation with equation (19) we obtain the classic "asset market view" result for exchange rates

$$m_{t+1}^* - m_{t+1} = \Delta s_{t+1}, \tag{26}$$

which completely pins down the depreciation rate.

Interestingly, our derivation highlights that this result does not hinge on the classic notion of market completeness. It is enough to be able to span $\tilde{m}_{t+1}^* - \tilde{m}_{t+1}$ and $\widetilde{\Delta s}_{t+1}$ in each country to apply Propositions 1 and 2 and obtain the complete-market result of equation (26). Such situations can arise in two cases.

First, the set of assets in each country is dense enough for the required spanning to hold. We can think of this situation as a limiting case of projecting $\tilde{m}^*_{t+1} - \tilde{m}_{t+1}$ and $\widetilde{\Delta s}_{t+1}$ on more and more rich set of assets until the R^2 of the projection converges to 1.

Second, the set of shocks in the economy that drive $m^\star_{t+1} - m_{t+1}$ and Δs_{t+1} is sparse enough that there exist assets in both countries that allow to trade both the exchange rate and the SDFs, even is the set of assets is not very dense. This situation may occur in models where all equilibrium objects are driven by a few global macro shocks $\boldsymbol{\epsilon}^G$, such as productivity, or monetary policy.

The value of our approach is that it allows to draw implications for the behavior of the exchange rate away from the case of complete markets. In the following sections, we study these implications for a variety of market structures. A first path for Propositions 1 and 2 to be relevant is when some assets are present in both H and F. We focus on these settings with some market integration in Section 3. The other path is when the structure of shocks in the home and foreign economy is such that potential arbitrage trades emerge between the two economies. We study these economies with financial intermediaries in Section 4. Before this, we show briefly how our results are robust to small deviations from the case of perfectly correlated trades.

2.3 Limiting quasi-arbitrage

In many empirically relevant cases, there are no pairs of assets that perfectly offset each other or perfectly span the exchange rate. In this section we offer one path to discipline the behavior of the exchange rate when we are close to pure arbitrage strategies (perfect spanning) and there is an upper bound on how profitable such strategies could be in equilibrium. Specifically, we consider a stronger set of restrictions than no arbitrage (Assumption 2), which limit the profitability of international trades.

Assumption 3. *(No quasi-arbitrage) There is an upper bound B on Sharpe ratios in international markets:*

$$\forall r_{p,t+1} \in I, \left| E_t(r_{p,t+1}) + \frac{1}{2} var_t(r_{p,t+1}) - r_{f,t} \right| \leq B\sqrt{var_t(r_{p,t+1})} \tag{27}$$

It is immediate to notice that Assumption 3 implies Assumption 2. In addition this assumption restricts the Sharpe ratio of trades in international markets. Such bounds have a long tradition in finance, going back to Ross (1976), Cochrane and Saa-Requejo (2000), Kozak, Nagel, and Santosh (2020). Intuitively, they can be motivated by the view that if trades that are too profitable emerged in equilibrium, new financial institutions would step in to take advantage of them. We assume that a similar bound applies to the domestic and foreign SDFs, which implies $\sqrt{var_t(m_{t+1})} < B$ and $\sqrt{var_t(m^*_{t+1})} < B$.

We revisit the two propositions of the previous sections under this condition. We focus on some simple implications of the Sharpe ratio bound; Appendix Section A.1 provides a more complete treatment.

Proposition 3. *Under Assumption 3,* $\forall r_{t+1} \in H, r^*_{t+1} \in F$

$$\left| cov_t \left(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, \frac{r^*_{t+1}}{\sqrt{var_t(r^*_{t+1})}} \right) \right| \leq 2B\sqrt{\frac{1}{\rho^2} - 1}, \tag{28}$$

where $\rho = corr_t(r_{t+1}, r^*_{t+1})$.

Proposition 3 gives a precise notion of how strong the relation between domestic and foreign assets must be to obtain the projection equation of Proposition 1. As the two assets become perfectly correlated ($\rho = 1$), one can construct a risk-free diff portfolio and the projection equation obtains exactly. Away from this perfect situation, there is residual risk in the diff portfolio, which allows some deviations from the projection result. When we turn to the data in Section 5, we will look for approximate global shocks: portfolios in F with a very high correlation with portfolios in H.

Similarly, we can revisit Proposition 2 for when the exchange rate is not exactly spanned.

Proposition 4. *Under Assumption 3,* $\forall r_{t+1} \in H, r^*_{t+1} \in F$ *such that* $\Delta s_{t+1} = r_{t+1} - r^*_{t+1} + \epsilon_{t+1}$ *with* ϵ_{t+1} *uncorrelated with* r_{t+1} *and* r^*_{t+1}*, then*

$$\Big| E_t[\Delta s_{t+1}] + \frac{1}{2} var_t(\Delta s_{t+1}) \\ - \left[r_{f,t} - r^*_{f,t} - cov_t(m_{t+1}, \Delta s_{t+1}) + cov_t(m^*_{t+1} - m_{t+1} - \Delta s_{t+1}, r^*_{t+1}) \right] \Big| \\ \leq 2B \sqrt{var_t(\Delta s_{t+1})} \sqrt{1 - R^2}, \tag{29}$$

where R^2 *is the R-squared of a regression of* Δs_{t+1} *on* r_{t+1} *and* r^*_{t+1}.

We see immediately that as we converge to spanning the exchange rate ($R^2 \to 1$), this condition converges to Proposition 2. Intuitively, when spanning is not perfect, there is risk remaining in the carry trade that attempts to hedge exchange rate risk. This risk allows potential deviations from the standard risk premium formulation. Equation (29) gives a quantitative sense to how close one is from perfect spanning.

For example, obtaining a bound on risk premium 10 times tighter than for the case of no spanning whatsoever ($R^2 = 0$) necessits a R^2 of 99%.

The loosest version of the bound occurs for the standard carry trade using only risk-free assets. In this case, $r_{t+1} = r_{ft}$, $r^*_{t+1} = r^*_{ft}$, and $\widetilde{\epsilon}_{t+1} = \widetilde{\Delta s}_{t+1}$.

Corollary (to Proposition 4)**.** *Under Assumption 3, deviations from uncovered interest parity are bounded by:*

$$\left| E_t(\Delta s_{t+1}) + \frac{1}{2} var_t(\Delta s_{t+1}) - (r_{f,t} - r^*_{f,t}) \right| \leq B\sqrt{var_t(\Delta s_{t+1})}. \tag{30}$$

This condition bounds the Sharpe ratio of the classic carry trade independently of any spanning condition.

3 Integrated markets

In this section, we consider various forms of market integration, that is market structures in which at least some assets can be traded in common by domestic and foreign household. In this interpretation of the model, the absence of international arbitrage is a consequence of one of the households having access to these commonly traded assets.

We characterize how, depending on the form of integration, various aspects of the exchange rate dynamics are pinned down. In particular, the critical point at which the exchange rate puzzles of complete markets start to occur is with bilateral integration of risk-free asset, because in this case both households can trade exchange rate risk.

3.1 Asymmetric integration: one risk-free bond

Consider a setting where only the foreign risk-free bond is tradeable by domestic and foreign households. Such a cases often arises in the context of sovereign bonds of emerging economies (H), which restrict participation in their market to the investors of their domicile, but these investors are not prevented from trading US bonds (F). This case corresponds to $H = \{r_{ft}, r^{*}_{ft} + \Delta s_{t+1}\}$ and $F = \{r^{\star}_{ft}\}$.

Proposition 1 requires exposure to a set of common risks, which does not apply in this case. In contrast, Proposition 2 applies precisely because the domestic household has access to the carry trade (based on risk-free bonds). Specifically equation (21) implies the same risk premium from the domestic perspective as in complete markets, equation (24). However, its foreign counterpart in equation (25) does not hold because the foreign household does not have access to the carry trade.

In this setting, the FX risk premium puzzle is unchanged, though only present from the domestic perspective. No-arbitrage requirement does not impose any constraints on the shocks to the depreciation rate. Therefore, there is full flexibility to match the volatility and cyclicality puzzles.

3.2 Symmetric integration: two risk-free bonds

Opening bilateral trade in risk-free bonds immediately leads to more stringent restrictions on the behavior of the exchange rate. Assume now that domestic and foreign investors can invest in the risk-free asset of the other country. Lustig and Verdelhan (2015) focus on this setting, which corresponds to $H = \{r_{ft}, r^{\star}_{ft} + \Delta s_{t+1}\}$ and $F = \{r^{\star}_{ft}, r_{ft} - \Delta s_{t+1}\}$.

In this case, shocks to the depreciation rate $\widetilde{\Delta s}_{t+1}$ are present both in H and F, and are the only shocks therein. As such, Proposition 1 applies with respect to this shock, which leads to

$$\widetilde{\Delta s}_{t+1} = E(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \Delta s_{t+1}), \tag{31}$$

because the projection of the depreciation rate on itself is the depreciation rate. Here $u_{t+1} = 0$ mechanically because the asset spanning the depreciation rate is the depreciation rate itself, or, more precisely, the carry return on the strategy based on risk-free assets. Therefore, the implication of Proposition 2 still coincides with equation (24).

Interestingly, the combination of equations (24) and (33) implies that a simplified version of equation (21) from Proposition 2 holds from the foreign perspective as well:

$$E_t(\Delta s_{t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) = r_{ft} - r^*_{ft} - cov_t(m^*_{t+1}, \Delta s_{t+1}). \tag{32}$$

As a result, if two of three equations (24), (32), and (31) hold, then the third one holds as well. This conclusion parallels in a more limited way the complete-market case, where the knowledge of two variables out of m_{t+1}, m^*_{t+1} and Δs_{t+1} implies the third one.

Equations (24) and (31) are also equivalent to the ones in Proposition 1 of Lustig and Verdelhan (2015). Therefore, we concur with these authors that one can make only limited progress on addressing the three exchange rate puzzles within such a market structure. Specifically,

$$var_t(\Delta s_{t+1}) = var_t(m^\star_{t+1} - m_{t+1} | \Delta s_{t+1}) \leq var_t(m^\star_{t+1} - m_{t+1}),$$

which potentially alleviates the volatility puzzle. Next, because the currency risk premium is controlled by exactly the same equation (24) as in the complete-markets case, partial integration with two risk-free bonds does not help in resolving the premium puzzle. As regards the cyclicality puzzle, the projection of the SDF difference on the depreciation rate has a coefficient of one just like in the complete-markets case, so in this respect partial integration is not helpful. Having said that, the correlation between relative consumption growth rates in the domestic and foreign economies and depreciation rate is less than perfect:

$$\begin{aligned} corr_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1}) &= \frac{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}{\sqrt{var_t(m^*_{t+1} - m_{t+1}) \cdot var_t(\Delta s_{t+1})}} \\ &\leq \frac{cov_t(m^*_{t+1} - m_{t+1}, \Delta s_{t+1})}{var_t(\Delta s_{t+1})} = 1. \end{aligned}$$

Next, we show that the restrictions of this setting continue to hold as more assets, either domestic or foreign, are bilaterally traded. In contrast, richer forms of asymmetric integration do not necessarily lead to such constraints. Thus, we conclude that bilateral trading in risk-free bonds imposes the critical restrictions on the depreciation rate. That is because trading these bonds amounts to the ability for both households to trade in exchange rate itself, which leads to the projection equation (31).

3.3 Symmetric integration: many assets

We allow for a broader set of assets to be traded by both domestic and foreign households. This implies that $H = \text{span}(\boldsymbol{r}_{t+1}) = \text{span}(\boldsymbol{r}^*_{t+1} + \Delta s_{t+1})$ and $F = \text{span}(\boldsymbol{r}^*_{t+1}) = \text{span}(\boldsymbol{r}_{t+1} - \Delta s_{t+1})$. Because these sets include the risk-free bonds, the exchange rate is sill spanned, $u_{t+1} = 0$. Proposition 2 implies the same risk-premium result as in the complete-markets case, equations (24) and (25) .

In this setting, the domestic household has access to assets with returns $\boldsymbol{r}^*_{t+1} + \Delta s_{t+1}$, where the first element is $r^*_{ft} + \Delta s_{t+1}$. Therefore, this household can trade $\boldsymbol{r}^*_{t+1} - r^*_{ft}$ by going long risky assets and shorting the risk-free asset. Thus, the domestic household can trade the same risks as the foreign one. The household can also isolate currency risk from all other risks by trading this way. The same logic applies to the foreign household's ability to trade domestic risks. Proposition 1 then applies to all traded risks:

$$\widetilde{\Delta s}_{t+1} = E(\widetilde{\Delta s}_{t+1} | \Delta s_{t+1}, \boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1}) = E(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \Delta s_{t+1}, \boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1}). \tag{33}$$

Therefore, projection of $m^*_{t+1} - m_{t+1}$ on the depreciation rate and asset returns has a loading of one on the depreciation rate and zero on all other assets. While more stringent than the condition with only risk-free assets (equation (31)), this relation does not weaken the conclusions about cyclicality and volatility of exchange rate. If anything, they bring the behavior of the depreciation rate closer to the complete market case, as the risky returns span more and more states of the world.

3.4 Symmetric integration for all assets but one risk-free bond

Consider the setup of the previous section with the only exception that F does not contain $r_{ft} - \Delta s_{t+1}$, just like in section 3.1. Just like in the previous section, the domestic household can trade foreign risks and isolate currency risk from all other risks. The foreign household, however, can no longer separate out the currency risk because $r_{ft} - \Delta s_{t+1}$ is inaccessible.

To see this consider the case of one risky asset in each country: $H = (r_{ft}, r_{1,t+1}, r^*_{ft} + \Delta s_{t+1}, r^*_{1,t+1} + \Delta s_{t+1})$, $F = (r^*_{ft}, r^*_{1t+1}, r_{1,t+1} - \Delta s_{t+1})$. The foreign household can trade the risks in $r^*_{1,t+1} - (r_{1,t+1} - \Delta s_{t+1})$. These risk are accessible to the domestic household as well. But there is no trade that can isolate the currency risk for the foreign household.

Proposition 1 then implies:

$$\begin{aligned}
&E(\widetilde{\Delta s}_{t+1} | \{r^*_{j,t+1}\}_{j\in J^*}, \{r_{j,t+1} - \Delta s_{t+1}\}_{j\in J}) \\
&= E(\widetilde{m}^*_{t+1} - \widetilde{m}_{t+1} | \{r^*_{j,t+1}\}_{j\in J^*}, \{r_{j,t+1} - \Delta s_{t+1}\}_{j\in J}),
\end{aligned} \tag{34}$$

where J and J^* are sets of domestic and foreign risky assets, respectively. Even though the depreciation rate appears in the projection, there is no reason to believe that the projected depreciation rate would be close to the actual one, in general.

This result reinforces the importance of bilateral trading of risk-free assets for the emergence of the cyclicality and volatility puzzles. In particular, we see that it is not so much the amount of integration that matters — here many assets are commonly traded — but whether both domestic and foreign households can gain exposure to the exchange rate risk.

4 Intermediated markets

In this section, we remove all assumptions about integration: domestic investors trade domestic assets while foreign investors trade foreign assets. It might seem that such a setting would remove any constraint on the dynamics of the exchange rate. But this is not necessarily the case: we maintain our assumption of the absence of arbitrage opportunities in international markets. Intuitively, this implies that a financial institution having access to both the domestic and foreign asset markets should not be able to earn arbitrage profits. Such a condition often arises in models where international financial trade is operated by financial intermediaries. While their decisions might be affected by various frictions, it is often assumed that they could enter in arbitrage trade. For example, a risk-based constraint such as Value-at-Risk in Basel requirements does not penalize risk-free trades.

In such settings, restrictions on the behavior of the exchange rate might come from relation between the risks of domestic and foreign assets. We consider various configurations where it is and is not the case.

4.1 No global shocks

A first polar case occurs when the two economies are spanned by a distinct set of shocks. While these shocks might be correlated, there is no redundancy between domestic and foreign returns. This corresponds to the condition:

$$\text{rank}(var_t(\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1})) = \text{rank}(var_t(\boldsymbol{r}_{t+1})) + \text{rank}(var_t(\boldsymbol{r}^*_{t+1})). \tag{35}$$

In such a situation, it is impossible to construct a pair $r_{t+1} \in H$, $r^*_{t+1} \in F$ such that $\tilde{r}_{t+1} = \tilde{r}^*_{t+1}$, i.e., $\boldsymbol{\epsilon}^G_{t+1}$ does not exist. Therefore, Proposition 1 does not apply.

Shocks to the exchange rate can have an arbitrary variance and correlation with the asset space: both u_{t+1} and v_{t+1} can be arbitrary large. As a result, there is nothing connecting $m^*_{t+1} - m_{t+1}$ to Δs_{t+1}. The cyclicality and volatility puzzles do not arise in this setting.

While condition (35) is about the correlation structure of returns in each of the countries, it connects naturally with the structure of shocks driving the home and foreign economies. To see this, consider the case of the two economies being in autarky on the real side (for example if the two countries consume different goods). Suppose, all shocks to firm productivity and output in each country are driven by vectors of shocks $\boldsymbol{\epsilon}_{t+1}$ and $\boldsymbol{\epsilon}^*_{t+1}$ satisfying $\text{rank}(var_t(\boldsymbol{\epsilon}_{t+1}, \boldsymbol{\epsilon}^*_{t+1})) = \text{rank}(var_t(\boldsymbol{\epsilon}_{t+1})) + \text{rank}(var_t(\boldsymbol{\epsilon}^*_{t+1}))$. The lack of global real shocks implies the lack of global financial shocks, i.e., condition (35) holds.

4.2 Global shocks only

Naturally, it is plausible that there are the same shocks affecting returns in each of the two countries. The second polar case is when all shocks between the two economies are global, that is:

$$\text{rank}(var_t(\boldsymbol{r}_{t+1}, \boldsymbol{r}^*_{t+1})) = \text{rank}(var_t(\boldsymbol{r}_{t+1})) = \text{rank}(var_t(\boldsymbol{r}^\star_{t+1})), \tag{36}$$

i.e., there is no $\boldsymbol{\epsilon}_{t+1}$ or $\boldsymbol{\epsilon}^\star_{t+1}$. Such a situation would occur in a setting in which the two economies are driven by the same set of shocks, although potentially with different exposure to these shocks. For example, all variation could be driven by a global financial cycle, with the U.S. more sensitive than other countries to this cycle.

In this case, Proposition 1 is applicable for any return because the same risks are present in both economies. This implies that

$$E(\widetilde{m}^\star_{t+1} - \widetilde{m}_{t+1} | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^\star_{t+1}) = E(\widetilde{\Delta s}_{t+1} | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^\star_{t+1}).$$

Furthermore, it is natural to assume that each economy's stochastic discount factor is spanned by the same global shocks. Such spanning occurs in many theories In this case, the restriction boils down to:

$$\widetilde{m}^\star_{t+1} - \widetilde{m}_{t+1} = E(\widetilde{\Delta s}_{t+1} | \widetilde{\boldsymbol{r}}_{t+1}, \widetilde{\boldsymbol{r}}^\star_{t+1}) = v_{t+1}. \tag{37}$$

The projection of the exchange rate on asset returns, v_{t+1} is exactly equal to the difference between stochastic discount factors. While this condition is reminiscent of the projection relation with integrated risk-free asset markets, equation (31), the two are very different because the projection is towards asset returns instead of towards the exchange rate. Now it is a regression of the exchange rate depreciation on the difference of log SDFs which yields a coefficient of 1. The unspanned component

u_{t+1} is unbounded, and we have:

$$var_t(\Delta s_{t+1}) = var_t(m^{\star}_{t+1} - m_{t+1}) + var_t(u_{t+1}) \geq var_t(m^{\star}_{t+1} - m_{t+1})$$

This result deepens the volatility puzzle. If economies are entirely driven by global shocks, exchange rate volatility can only be larger than in the complete market case. In terms of cyclicality, the conclusions are mixed. The presence of unspanned volatility can weaken the correlation of the exchange rate with volatility. However, because of the projection result (37), one cannot weaken the exposure of the depreciation rate to the difference in the SDFs.

4.3 Global and local shocks

We can characterize the set of admissible exchange rate processes using the representation in equations (2) and (3). Suppose that the domestic and foreign stochastic discount factors can be written as combination of the same shocks:

$$\tilde{m}_{t+1} = \boldsymbol{M}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{M}^G\boldsymbol{\epsilon}^G_{t+1} \tag{38}$$

$$\tilde{m}^{\star}_{t+1} = \boldsymbol{M}^{\star}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{M}^{\star G}\boldsymbol{\epsilon}^G_{t+1}. \tag{39}$$

An exchange rate process satisfies the implications of Proposition 1 if and only if its spanned innovation v_{t+1} in equation (9) is such that

$$\boldsymbol{S}^G = \boldsymbol{M}^{\star G} - \boldsymbol{M}^G. \tag{40}$$

In other words, Proposition 1 constrains the projection of the exchange rate on global shocks to coincide with the projection of the difference of stochastic discount factors on global shocks. It does not impose any constraints on either local-shock loadings or the presence of unspanned risk.

This result offers a clear path towards a resolution of the puzzles on exchange rate shocks. If the volatility of the global component of the difference in SDFs, $(\boldsymbol{M}^{\star G} - \boldsymbol{M}^{G})\epsilon_{t+1}^{G}$ is small enough, there is enough room for the exchange rate to load on local shocks or unspanned shocks while respecting a realistic level of exchange rate volatility. These additional sources of variation can weaken the correlation of the exchange rate with the difference of SDFs. This correlation can even be negative, depending on the loadings on local shocks $\boldsymbol{S}$ and $\boldsymbol{S}^{\star}$. As such, the cyclicality puzzle could be satisfied as well.

To illustrate this possible reversal, consider the following simple example with one global shock and one local shock for the home country, i.e., $\epsilon^{\star} = 0$. Intuitively, this correponds to situation where there is a global cycle, and a U.S.-specific cycle. In this case,

$$cov_t(m_{t+1}^{\star} - m_{t+1}, \Delta s_{t+1}) = (M^{\star G} - M^{G}) S^{G} var_t(\epsilon_{t+1}^{G}) - M S var_t(\epsilon_{t+1}).$$

If the U.S. suffers more than the rest of the world in a global recession, $M^{\star G} < M^{G}$, equation (40) implies that the dollar must appreciate, $S^{G} < 0$. Such a narrative coincides broadly with the "exorbitant duty" articulated in Gourinchas and Rey (2022). However, there is no restriction on the behavior of the exchange rate during a U.S.-specific recession. In particular, the dollar might depreciate during such events, $S > 0$. On balance, if S is large relative to S^{G}, then unconditionally, the exchange rate will be negatively correlated with the difference of SDFs, solving the cyclicality puzzle.

4.4 The FX risk premium

So far, in this section, we have focused on implications for the shocks to the exchange rate. What does this setting say about expected depreciation? To make progress in both the spanned and unspanned cases, we use Proposition 4, which relies on no quasi-arbitrage. We focus on a pair of returns that gets as close as possible to spanning: $\tilde{r}_{t+1} = (M^{\star G} - M^G)\epsilon^G_{t+1} + S\epsilon_{t+1}$ and $\tilde{r}^{\star}_{t+1} = -S^{\star}\epsilon^{\star}_{t+1}$. With these returns, we have $\widetilde{\Delta s}_{t+1} = \tilde{r}_{t+1} - \tilde{r}^{\star}_{t+1} + u_{t+1}$. Substituting in with the local and global shocks Proposition 4 implies:

$$
\begin{aligned}
&\bigg| E_t(\Delta s_{t+1}) - \left[r_{ft} - r^{\star}_{ft} - cov_t(m_{t+1}, \Delta s_{t+1}) - \frac{1}{2} var_t(\Delta s_{t+1}) \right] \\
&\quad + (M^{\star} - S^{\star}) S^{\star} var_t(\epsilon^{\star}) - (M + S) S^{\star} cov_t(\epsilon, \epsilon^{\star}) \bigg| \\
&\quad\quad \leq 2B\sqrt{var_t(u_{t+1})}
\end{aligned}
\tag{41}
$$

When the exchange rate is spanned, $u_{t+1} = 0$, the only path to rationalize deviations of the FX risk premium from the complete markets formula is a specific pattern of conditional correlation of the exchange rate and SDFs along local shocks. This pattern can be very stringent: in the example of the previous section without foreign local shocks, the deivation from integrated markets term is always 0.

Unspanned risks offer a more flexible path to solve the FX risk premium puzzle. The presence of these risks simply restricts the currency risk premium to a range instead of a specific value. The size of this range is such that the traditional carry trade with risk-free assets does not generate Sharpe ratios that are too large.

5 Empirical Analysis

In this section we investigate whether a broad collection of asset returns is informative about properties of the exchange rate. We limit the asset set in each country to sovereign bonds and various stock portfolios of that country. There are two interpretations of this choice. First, we ask the empirical question, irrespective of market structure, of how much one can hope to learn about the behavior of exchange rates from knowledge of the price of other assets in their origin currency. Second, staying close to the model of Section 4, we are quantifying the restrictions imposed on the behavior of the exchange rate in economies in which only intermediaries participate in international markets.

We first demonstrate that exchange rates appear to have a large component u_{t+1} unrelated to the returns of other traded assets. Then, we provide methods to characterize global shocks. Both of these exercises lead to the conclusion that, for the data we consider, other assets do not impose strong restrictions on the behavior of exchange rates.

5.1 Data

We consider countries corresponding to G10 currencies between 2/1988 and 12/2022. We consider Germany as the representative country for the euro. Prior to the introduction of the euro, we use the German Deutschemark and splice these series together beginning in 1999. Our analysis focuses at the monthly frequency. We obtain exchange rates from WM/Reuters. Government bond yields are from each country's central bank websites. Monthly bond returns are computed from bond yields using a second-order Taylor approximation. We obtain equity indices from MSCI. For each country, 10 different industry indices and 3 different style equity indices (Large + Mid Cap, Value, Growth) are sourced. Risk-free rates are calculated by dividing the 1-year yield by 12.

5.2 Is the exchange rate spanned?

Motivated by Proposition 2, we ask whether the depreciation rate is spanned by combination of domestic and foreign asset returns. We implement regressions of the form:

$$\Delta s_{t+1} = \alpha + \beta' \boldsymbol{r}_{t+1} + \beta^{\star\prime} \boldsymbol{r}^{\star}_{t+1} + u_{t+1}. \tag{42}$$

Here the residual u_{t+1} is a direct estimate of the unspanned component of the depreciation rate in equation (8).

We report the adjusted R^2 of these regressions. Exact spanning corresponds to an R^2 of 1. Furthermore, Proposition 4 highlights that R^2 is an appropriate measure of economic distance to the case of perfect spanning.

Table 1 reports the results. We always report the results for the combination of assets in the United States and another country. Each column in the table corresponds to a given country. Each row reflects a particular combination of assets used in the regression. Broadly speaking, we consider bonds and equities separately and in combination. Within each asset class, we zoom in on various individual contributions.

Major asset classes do not span exchange rates. When looking at all assets together, the R^2s range from 25% for Switzerland to 45% for Canada (in each case combined with the U.S.). Most of the explanatory power comes from the equity side. For example in the case of Canada, the combination of market, value, growth and industry returns explain 42% of variation in the depreciation rate. While the market alone gets to some substantial amount of variation — 27% for Canada —, the addition of industry returns is particularly informative. Consistent with the evidence in Chernov and Creal (2023), bond returns only explain a modest amount of variation in exchange rates: between 0.2% and 7% for the 10-year bond alone, and between 7.2% and 14% for the combination of bonds at all maturities.

We refer to the observation that asset returns do not span changes in exchange rates as the *financial exchange rate disconnect*. While the R^2s we obtain from regressions on asset returns are meaningfully larger than their counterpart with real quantities, these magnitudes are much too small for leading to meaningful theoretical implications. Taking the strictest definition of absence of arbitrage, only a value of 1 leads to the relevance of Proposition 2. Even with the looser approach of Proposition 4, even the largest numbers we measure only imply a bound for the expected depreciation that is $\sqrt{1-0.45} = 67\%$ of the bound with an R^2 of 0, not much tighter. Thus, observing the properties of returns on other assets is not informative about the expected currency depreciation rates.

The flipside of this conclusion is that the unspanned component of the depreciation rates, u_{t+1}, is large. In the context of models of intermediated markets analyzed in section 4, that offers more flexibility in capturing realistic properties of exchange rates and addressing the currency puzzles.

5.3 Identifying global shocks

In this section we quantify the importance of global shocks ϵ_{t+1}^{G}, which play the key role in Proposition 1. We do so using two empirical approaches. The undirected approach uses canonical correlation analysis (CCA) to identify these shocks from the asset return data. The directed approach starts from candidates for global shocks such as global macro and financial variables proposed in the literature.

5.3.1 Undirected approach

The CCA procedure finds a US and a foreign portfolio of asset returns consisting of $\boldsymbol{r}_{t+1}$ and $\boldsymbol{r}_{t+1}^{\star}$, respectively, such that they have the highest correlation possible in sample. Next, conditional on finding this pair, the procedure looks for the next maximally correlated pair of portfolios that are orthogonal to their first pair. And so on.

According to Definition 1, global shocks would manifest themselves as innovations to portfolios with perfect correlation. In that case, Proposition 1 implies that projections of the depreciation rate and the difference in the SDFs on the global shocks coincide. In the data, even the largest correlation could be less than 1. According to Proposition 3, proximity of this correlation to 1 measures the proximity of the depreciation rate to the prediction of Proposition 1.

Table 2 reports the results. Each column represents a foreign country. For a given country, each row reports the canonical correlation between the assets of that country and the US assets, reported in order of importance, starting from the largest.

The values of the largest correlations range from 64% for New Zealand to 90% for Canada. In some cases lower ranked correlations are similar to the largest one, like for Canada or the UK. In other cases, the magnitude of correlation drops off quickly, e.g., for New Zealand or Norway.

Strictly speaking, the evidence suggests that there are no global shocks amongst the assets that we consider. Proposition 1 then implies that we cannot connect variation in depreciation rates to that of the difference in the SDFs. Now consider Proposition 3, which explicitly allows for less than perfect correlation. We can see from equation (28) that a correlation value of 1 takes one back to Proposition 1. Consider now a seemingly small departure from 1 to 0.9 as in the case of Canada. The expression in equation (28) implies that this value moves the constant scaling $2B$ from 0 to 0.5. As a result, the upper bound on the departure between the depreciation rate and the SDF difference is equal to the maximal Sharpe ratio. This bound will grow as correlation drops further, indicating that there is not much one can say about the connection between the shocks to exchange rates and those to the SDFs.

We can be more generous with interpreting the evidence in Table 2 and assign a value of 1 to each estimated correlation that is above a certain threshold. We consider the value of 60% as such a threshold. We denote the matrix of foreign portfolio weights by $\boldsymbol{w}^{\star}$; if there is only one global shock, this is a vector. We ask how much variation in the depreciation rate is explained by global shocks. We implement regressions of the form:

$$\Delta s_{t+1} = \alpha + \beta^{G\prime}(\boldsymbol{w}^{\star\prime}\boldsymbol{r}^{\star}_{t+1}) + \varepsilon_{t+1}. \tag{43}$$

The R^2 of such a regression is the fraction variance in exchange rate explained by global shocks. Because we assume that the corresponding domestic portfolio is perfectly correlated with its foreign counterpart, we do not include it in the regression. The regression residual is a direct estimate of the contribution of local and unspanned shocks to the depreciation rate, $\varepsilon_{t+1} = \boldsymbol{S}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{S}^{\star}\boldsymbol{\epsilon}^{\star}_{t+1} + u_{t+1}$.

Combining with the results of regression (42), we can decompose variation in the depreciation rate into the contribution of global, local, and unspanned shocks. Specifically, we have $var(\boldsymbol{S}^{G}\boldsymbol{\epsilon}^{G}_{t+1}) = var(\beta^{G\prime}(\boldsymbol{w}^{\star\prime}\boldsymbol{r}^{\star}_{t+1}))$ for global shocks, and $var(\boldsymbol{S}\boldsymbol{\epsilon}_{t+1} + \boldsymbol{S}^{\star}\boldsymbol{\epsilon}^{\star}_{t+1}) = var(\varepsilon_{t+1}) - var(u_{t+1})$ for local shocks. Figure 1 reports these quantities as fraction of the variation in depreciation rate; the contributions mechanically add up to 1.

For all currencies, at least half of the variation in exchange rates is unspanned by asset returns — the financial disconnect we have already noted. Global shocks contribute very little to variation in the depreciation rates. The contribution is of the order of a few percentage points, with the exception of Australia and Canada with contributions around 25%. These estimates should be seen as an upper bound on the role of global shocks; remember that estimated global shocks include any pair of portfolios with correlation above 60%, far from the strict Definition 1.

5.3.2 Directed approach

Given a candidate variable X_{t+1} for a global shocks, we look for r_{t+1} and $r^{\star}_{t+1}$ which are maximally correlated with X_{t+1}. *To be completed.*

Just like the financial disconnect leads to weak restrictions about the expected depreciation rate, the small role of global shocks implies weak restrictions about exchange rate risks. SDFs estimated from other asset returns can at most explain 25% of shocks to exchange rates. Again, the flipside of this conclusion is that models based on intermediation have the flexibility to replicate the empirical properties of the exchange rate. For example, in the economy of Section 4.3, we saw that having sizable exposure to local shocks relative to global shocks is necessary to resolve the cyclicality puzzle. The exposure of the depreciation rate to local shocks is always at least of similar magnitude as to global shocks, and often much larger.

6 Conclusion

In this paper, we propose a general framework to understand how much financial markets determine the behavior of exchange rates. Our theory accommodates many settings: complete or incomplete markets, arbitrary forms of market integration, or situations in which international financial trade happens through intermediaries. We characterize restrictions on the behavior of exchange rates due to the absence of arbitrage. These restrictions can be simply summarized by two conditions that share the simplicity of the complete market result while having richer implications.

We use these results to study many different market structures. We find that in settings in which financial markets are informative about the exchange rate, they lead to the same counterfactual implications as in complete markets. In contrast, some structures, such as intermediated markets, do not impose much restrictions on exchange rates at all. This lack of structure is driven by two properties of the data. First, there is a financial exchange rate disconnect: depreciation rates are not that correlated to asset returns. Second, few shocks are globally traded, and they explain even less of the variation in exchange rates.

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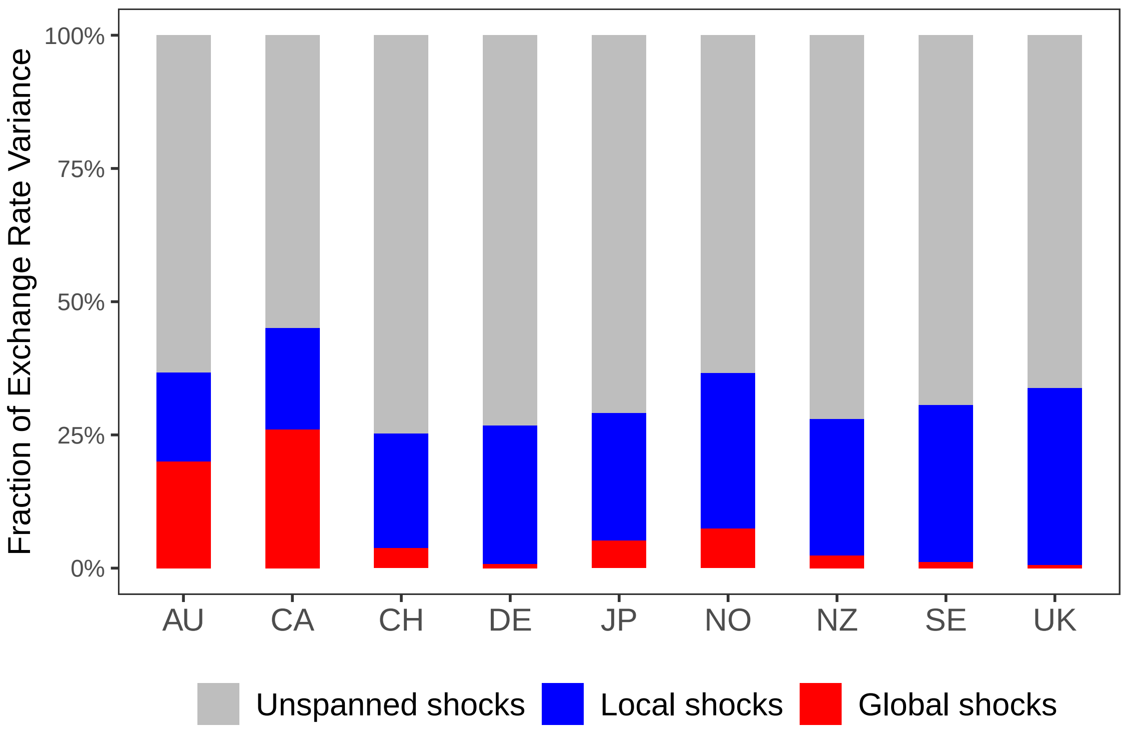
Figure 1: Decomposition of exchange rate innovations



Notes: The figure reports the fraction of variance in exchange rates explained by globally traded shocks, local shocks, and shocks that are not spanned by asset returns. Each bar is a different country's currency relative to the U.S. dollar.

Table 1: Spanning of depreciation rates by asset returns – R^2

Dependent Variable	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Bonds									
10Y	0.25	0.33	7.49	5.36	4.73	1.05	4.79	4.01	0.92
All Maturities	7.23	7.89	15.72	10.15	13.66	5.67	13.95	11.52	13.65
Stocks									
Mkt	21.67	26.56	6.96	4.44	11.24	16.56	16.20	12.34	12.71
Mkt + Value/Growth	21.60	27.98	6.75	5.06	12.47	17.16	15.91	12.71	13.68
Mkt + Value/Growth + Ind.	35.07	41.61	18.55	22.78	29.41	24.53	24.00	19.61	26.88
Bond + Equity	36.74	45.05	26.79	29.13	36.64	27.95	30.62	25.28	33.80
N	419	395	419	419	406	419	414	419	419

Notes: The table reports the adjusted R^2 of a regression of the depreciation rate on various subsets of asset returns, as in equation (42). Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is a different country's currency relative to the U.S. dollar. The first row uses only 10-year bonds, while the second entertains maturities between 2 and 10 years, obtained from various central banks. The next three row successively add various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. The final row considers all assets simultaneously.

Table 2: Maximally correlated shocks across asset markets

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Rank 1	75.27	89.82	83.07	75.01	79.47	64.31	78.33	82.95	85.87
Rank 2	65.0	85.06	74.17	64.43	63.49	53.95	65.72	62.62	78.7
Rank 3	61.16	83.44	66.7	58.71	57.14	41.73	59.57	60.41	73.55
Rank 4	57.04	78.79	64.9	51.31	45.86	35.98	55.55	56.12	68.02
Rank 5	51.01	76.82	52.8	46.81	41.74	31.44	49.63	52.32	65.85
Rank 6	41.67	70.79	44.19	46.62	33.59	25.33	38.94	46.83	62.21
Rank 7	34.19	62.84	42.3	41.94	26.88	22.99	38.2	41.16	55.83
Rank 8	31.57	56.2	36.66	39.57	25.8	14.58	33.82	35.18	51.39
N	419	395	419	419	406	419	414	419	419

*N*otes: The table reports the correlation in % between the maximally correlated portfolios of asset returns between the U.S. and each country. The successive pairs of portfolio are orthogonal to each other, and obtained by canonical correlation analysis. Domestic asset returns are in domestic currency; foreign asset returns are in foreign currency. Each column is for a different country's assets relative to the U.S. assets. The assets include government bonds of maturities between 2 and 10 years (obtained from various central banks) and various stock portfolios: the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios (from MSCI).

Internet Appendix

A Approximate versions of Propositions 1 and 2

A.1 Limiting quasi-arbitrage

We derive the implications of Assumption 3.

Revisiting Proposition 1.

Proposition 5. *Under Assumption 3,* $\forall r_{t+1} \in H, r^\star_{t+1} \in F$, *if* $\tilde{r}_{t+1} = \tilde{r}^\star_{t+1} + \epsilon_{t+1}$ *with* ϵ_{t+1} *uncorrelated with* $r^\star_{t+1}$ *then*

$$\left|cov_t(m^\star_{t+1} - m_{t+1} - \Delta s_{t+1}, r^\star_{t+1}) - cov_t(m_{t+1}, \epsilon_{t+1})\right| \leq B\sqrt{var_t(r^\star_{t+1})}\sqrt{\frac{1-R^2}{R^2}}, \tag{44}$$

where R^2 *is the R-squared of a regression of* r_{t+1} *on* $r^\star_{t+1}$.

Proof. Consider again the diff portfolio, with returns given by equation (14). We see that $var(r_{\text{diff},t+1}) = var(\epsilon_{t+1})$. Plugging this result into assumption 3 gives the proposition immediately. ■

Proposition 5 gives a precise notion of how strong the relation between the two assets must be to obtain the projection equation of Proposition 1. In particular, we see that as the two assets become perfectly correlated and the R^2 of regressing one on the other converges to 1, we converge towards the projection equation exactly.

A less desirable feature of this result for some applications is that it relies on knowledge of $cov_t(m_{t+1}, \epsilon_{t+1})$, which can be evaluated only in a fully articulated model. We can formulate some closely related bounds that avoid this property.

Corollary (to Proposition 3)**.** $\forall r_{t+1} \in H, r^\star_{t+1} \in F$, *if* $\tilde{r}_{t+1} = \tilde{r}^\star_{t+1} + \epsilon_{t+1}$ *with* ϵ_{t+1} *uncorrelated with* $r^\star_{t+1}$*:*

(i) A necessary condition for the diff portfolio to satisfy assumption 3 is:

$$\left|cov_t(m^\star_{t+1} - m_{t+1} - \Delta s_{t+1}, r^\star_{t+1})\right| \leq 2B\sqrt{var(\epsilon_{t+1})}. \tag{45}$$

(ii) A sufficient condition for the diff portfolio to satisfy assumption 3 is:

$$\left|cov_t(m^\star_{t+1} - m_{t+1} - \Delta s_{t+1}, r^\star_{t+1})\right| \leq (B - \sqrt{var(m_{t+1})})\sqrt{var(\epsilon_{t+1})}. \tag{46}$$

Finally, we can derive Proposition 3. Start from a pair $r_{t+1} \in H$ and $r^\star_{t+1} \in F$ with $corr_t(r_{t+1}, r^\star_{t+1} = \rho$. We can rescale the assets and obtain:

$$\frac{1}{\rho}\frac{r_{t+1}}{\sqrt{var_t(r_{t+1})}} = \frac{r^\star_{t+1}}{\sqrt{var_t(r^\star_{t+1})}} + \epsilon_{t+1},$$

with ϵ_{t+1} uncorrelated with $r^\star_{t+1}$. Applying part (i) of the Corrolary immediately leads to Proposition 3.

Revisiting Proposition 2.

Proposition 6. *Under Assumption 3, $\forall r_{t+1} \in H, r^\star_{t+1} \in F$ such that $\Delta s_{t+1} = r_{t+1} - r^\star_{t+1} + \epsilon_{t+1}$ with ϵ_{t+1} uncorrelated with r_{t+1} and $r^\star_{t+1}$, then*

$$\begin{aligned}
&\Big| E_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1}) \\
&\qquad - \left[r_{f,t} - r^\star_{f,t} - cov_t(m_{t+1}, \Delta s_{t+1} - \epsilon_{t+1}) + cov_t(m^\star_{t+1} - m_{t+1} - \Delta s_{t+1}, r^\star_{t+1})\right] \Big| \\
&= \Big| E_t[\Delta s_{t+1}] + \frac{1}{2}var_t(\Delta s_{t+1} - \epsilon_{t+1}) + \frac{1}{2}var_t(\epsilon_{t+1}) \\
&\qquad - \left[r_{f,t} - r^\star_{f,t} - cov_t(m_{t+1}, \Delta s_{t+1} - \epsilon_{t+1}) + cov_t(m^\star_{t+1} - m_{t+1} - \Delta s_{t+1}, r^\star_{t+1})\right] \Big| \\
&= \Big| E_t[\Delta s_{t+1}] - \frac{1}{2}var_t(\Delta s_{t+1} - \epsilon_{t+1}) + \frac{1}{2}var_t(\epsilon_{t+1}) \\
&\qquad - \left[r_{f,t} - r^\star_{f,t} - cov_t(m^\star_{t+1}, \Delta s_{t+1} - \epsilon_{t+1}) + cov_t(m^\star_{t+1} - m_{t+1} - \Delta s_{t+1}, r_{t+1})\right] \Big| \\
&\qquad\quad \leq B\sqrt{var_t(\Delta s_{t+1})}\sqrt{1 - R^2},
\end{aligned} \tag{47}$$

where R^2 is the R-squared of a regression of Δs_{t+1} on r_{t+1} and $r^\star_{t+1}$.

Proof. Consider the carry portfolio given by equation (12). Plugging in the assumption about ϵ_{t+1}, we see that $var(r_{carry,t+1}) = var(\epsilon_{t+1})$. We use the Euler equations to compute the risk premium of the portfolio:

$$\begin{aligned}
E\left[r_{\text{carry},t+1}\right] - r_{ft} + \frac{1}{2}var(r_{\text{carry},t+1}) \\
&= E(r_{t+1}) - E(r^{\star}_{t+1}) - E(\Delta s_{t+1}) - cov(\epsilon_{t+1}, r_{t+1} + \epsilon_{t+1}) + \frac{1}{2}var(\epsilon_{t+1}) \\
&= -E\left(\Delta s_{t+1}\right) - \frac{1}{2}var(\Delta s_{t+1}) + r_{ft} - r^{\star}_{ft} - cov(m_{t+1}, \Delta s_{t+1} - \epsilon_{t+1}) \\
&\quad + cov(m^{\star}_{t+1} - m_{t+1} - \Delta s_{t+1}, r^{\star}_{t+1}). \qquad (48)
\end{aligned}$$

Using the Sharpe ratio bound of Assumption 3 gives the first formulation of condition (47) immediately. Similar calculations lead to the other formulations in (47). ■

As before, we see immediately that as we converge to spanning the exchange rate ($R^2 \to 1$), this condition converges to Proposition 2. We also notice that the Siegel paradox sign flip only affect the variance of the spanned risk $\Delta s_{t+1} - \epsilon_{t+1} = r_{t+1} - r^{\star}_{t+1}$, not that of the unspanned risk ϵ_{t+1}.

We obtain Proposition 4 by noticing that the term $cov(m_{t+1}, \epsilon_{t+1})$ can be bounded as well using the Sharpe ratio bound.

B Testing the portfolio approximation

We report the correlation (in %) between the excess return on various stock portfolios —Table 3— and bonds of different maturities —Table 5— in their origin currency and converted to U.S. dollars. Tables 4 and 6 start from the U.S. version of these portfolios and converts them to foreign currency. These correlations are pervasively extremely high, almost all over 99.9%.

Table 3: Correlation between excess returns in different currencies: foreign stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
Market	99.88	99.91	99.93	99.96	99.88	99.89	99.91	99.94	99.94
Value	99.92	99.94	99.93	99.96	99.89	99.85	99.92	99.93	99.94
Growth	99.82	99.88	99.93	99.96	99.9	99.93	99.92	99.95	99.94
Oil, Gas, Coal	99.89	99.93	NA	99.96	99.92	99.92	99.93	NA	99.96
Basic Material	99.84	99.94	99.94	99.95	99.88	99.91	99.91	99.96	99.91
Consumer Discretionary	99.91	99.95	99.93	99.96	99.92	99.94	99.94	99.93	99.96
Consumer Products, Services	99.88	99.96	99.97	99.95	NA	NA	99.94	99.93	99.98
Industrials	99.90	99.91	99.94	99.95	99.89	99.92	99.92	99.94	99.94
Health Care	99.91	99.97	99.96	99.96	NA	99.91	99.93	99.96	99.97
Financials	99.92	99.95	99.94	99.96	99.89	99.93	99.91	99.93	99.92
TeleCom	99.92	99.95	99.96	99.96	99.92	99.84	99.93	99.94	99.96
Technology	99.91	99.88	99.96	99.96	99.86	NA	99.94	99.95	99.95
Utilities	99.93	99.91	99.94	99.97	NA	99.93	NA	99.95	99.97

*N*otes: The table reports the correlation (in %) between the excess return on various stock indices expressed in their home currency and converted to U.S. dollar. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table 4: Correlation between excess return in different currencies: U.S. stocks

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US Market	99.88	99.94	99.95	99.96	99.87	99.90	99.92	99.94	99.94
US Value	99.90	99.95	99.96	99.96	99.87	99.91	99.92	99.95	99.95
US Growth	99.87	99.93	99.94	99.96	99.88	99.90	99.92	99.94	99.94
US Oil, Gas, Coal	99.90	99.96	99.97	99.98	99.92	99.92	99.94	99.96	99.96
US Basic Material	99.81	99.9	99.92	99.95	99.85	99.88	99.9	99.93	99.93
US Consumer Discretionary	99.91	99.95	99.95	99.96	99.9	99.91	99.92	99.95	99.95
US Consumer Products, Services	99.93	99.97	99.97	99.97	99.92	99.93	99.94	99.96	99.96
US Industrials	99.86	99.93	99.94	99.96	99.84	99.90	99.90	99.94	99.94
US Health Care	99.90	99.96	99.95	99.96	99.88	99.93	99.93	99.95	99.96
US Financials	99.91	99.95	99.95	99.94	99.87	99.93	99.91	99.92	99.94
US TeleCom	99.87	99.93	99.95	99.95	99.9	99.91	99.93	99.96	99.95
US Technology	99.88	99.93	99.94	99.96	99.89	99.91	99.92	99.94	99.94
US Utilities	99.84	99.92	99.94	99.96	99.85	99.88	99.91	99.96	99.94

Notes: The table reports the correlation (in %) between the excess return on various stock indices expressed in the U.S. dollars and converted to foreign currency. The portfolios include the market (a combination of large and mid-cap stocks), value and growth portfolios, and 10 industry portfolios, all from MSCI. Each column corresponds to a different country.

Table 5: Correlation between excess return in different currencies: foreign bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
2Y Bond	99.86	99.97	99.92	99.97	NA	99.85	99.91	99.91	99.95
3Y Bond	99.86	99.97	99.92	99.97	99.91	NA	NA	99.93	99.96
4Y Bond	NA	99.97	99.93	99.97	NA	NA	NA	99.94	99.96
5Y Bond	99.87	99.97	99.93	99.97	99.91	99.85	99.91	99.93	99.96
6Y Bond	NA	99.96	99.93	99.97	NA	NA	NA	99.92	99.96
7Y Bond	NA	99.96	99.93	99.96	NA	NA	99.91	99.91	99.96
8Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.90	99.96
9Y Bond	NA	99.96	99.92	99.96	NA	NA	NA	99.89	99.96
10Y Bond	99.87	99.96	99.93	99.96	99.91	99.88	99.91	99.88	99.96

*N*otes: The table reports the correlation (in %) between the excess return on government bonds of different maturity expressed in their home currency and converted to U.S. dollars. Bond returns are constructed from yields obtained from each country's central bank. Each column corresponds to a different country.

Table 6: Correlation between excess return in different currencies: U.S. bonds

	AU	CA	DE	JP	NO	NZ	SE	CH	UK
US 2Y Bond	99.9	99.95	99.95	99.97	99.91	99.93	99.95	99.93	99.96
US 3Y Bond	99.91	99.96	99.95	99.97	99.92	99.93	99.95	99.92	99.96
US 4Y Bond	99.92	99.96	99.94	99.96	99.92	99.94	99.95	99.91	99.96
US 5Y Bond	99.91	99.97	99.93	99.96	99.91	99.94	99.95	99.89	99.95
US 6Y Bond	99.91	99.97	99.93	99.96	99.89	99.94	99.94	99.88	99.95
US 7Y Bond	99.9	99.96	99.92	99.96	99.88	99.94	99.94	99.86	99.95
US 8Y Bond	99.89	99.96	99.91	99.96	99.86	99.93	99.93	99.85	99.95
US 9Y Bond	99.88	99.96	99.9	99.96	99.85	99.93	99.93	99.84	99.95
US 10Y Bond	99.88	99.96	99.9	99.96	99.84	99.93	99.92	99.83	99.94

Notes: The table reports the correlation (in %) between the excess return on U.S. government bonds of different maturity expressed in U.S.. dollars and converted to foreign currency. Bond returns are constructed from yields obtained from the Federal Reserve. Each column corresponds to a different country.