New Control Variates for Lévy Processes and Asian Options

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Outline

- Control variates for Lévy process models
  - Control variate framework
  - Option pricing examples

- Variance reduction for Asian options
  - A unified framework for non-Gaussian models
  - The proposed method is a combination of
    - Control Variate (CV)
    - Conditional Monte Carlo (CMC)
Monte Carlo (MC) Method: General Principles

- Estimation of an unknown parameter: $\mu = E[Y]$
- Generation of iid sample $Y_1, Y_2, \ldots, Y_n$
- The estimator: $\hat{\mu} = \frac{\sum_{i=1}^{n} Y_i}{n}$
- To quantify the error $\hat{\mu} - \mu$:
  - Central Limit Theorem: $\frac{\hat{\mu} - \mu}{s_n/\sqrt{n}} \Rightarrow N(0, 1)$ as $n \to \infty$.
  - Probabilistic error bound: $\Phi^{-1}(1 - \alpha/2) \cdot s/\sqrt{n}$
  - To get smaller error bound:
    - Increase the sample size $n$ (Larger computational time) $O(1/\sqrt{n})$
    - Decrease the variance $s^2$ (Variance reduction techniques)
Problem definition

- Lévy process \( \{L(t), t \geq 0\} \)
  - Stationary and independent increments, and \( L(0) = 0 \)

- Functional of \( L \):
  - \( q(L(t_1), \ldots, L(t_d)) \)
  - Time grid \( 0 = t_0 < t_1 < t_2 < \ldots < t_d \) with \( t_j = j \Delta t \)
  - \( t_j = j \Delta t \Rightarrow \) increments \( L(t_i) - L(t_{i-1}) \) are iid

- Estimation of \( E[q(L(t_1), \ldots, L(t_d))] \) by simulation.

- A new variance reduction method
Problem definition

- In the literature, there exist variance reduction methods suggested for Lévy processes
  - They are often special to the 'process type' or 'problem type'

- A new control variate (CV) method

- It can be applied for any Lévy process for which the probability density function (PDF) of the increments is available in closed form

- Numerical examples: path-dependent options
Control Variate Method

- Estimator: \( Y = q(L) - c^T (V - E[V]) \)
  - \( V = (V_1, \ldots, V_m)^T \) set of CVs with known \( E[V] \)
  - \( c = (c_1, \ldots, c_m)^T \) the coefficient vector (optimal \( c^* \) by linear regression)

- Successful if strong linear dependence: \( VRF = 1/(1 - R^2) \).

- Our CV framework:
  - Special CV, tailored to \( q() \)
  - General CVs, selected from a *basket* of CVs (not tailored to \( q() \))
Functional of a Brownian Motion (BM).

Brownian motion \( \{W(t), t \geq 0\} \) with parameters \( \{\mu, \sigma\} \):
- \( W(t) = \mu t + \sigma B(t) \)
- \( B(t) \) is a standard BM

Functional \( \zeta(W(t_1), \ldots, W(t_d)) \)
- Similar to the original function: \( \zeta \sim q \).

**Known expectation:** \( E[\zeta(W)] \) is available in closed form


**Special CV**

- similarity of paths: \((W(t_1), \ldots, W(t_d)) \sim (L(t_1), \ldots, L(t_d))\) and similarity of functions: \(\xi \sim q\)

  \[\Rightarrow \text{Large correlation between } q(L) \text{ and } \xi(W)\]

- For similar paths,
  - \(\mu = E[L(1)] \text{ and } \sigma = \sqrt{\text{Var}(L(1))}\)
  - Using CRN (common random numbers) for path simulation

- Comonotonic increments lead to maximal correlation
  - \(U \sim U(0,1)\)
  - \(L(t_i) - L(t_{i-1}) \leftarrow F_{L}^{-1}(U)\)
  - \(W(t_i) - W(t_{i-1}) \leftarrow F_{BM}^{-1}(U)\)
Inverse CDFs:
- $F_{BM}^{-1}(U)$, Inverse CDF of normal distribution.
- $F_{L}^{-1}(U)$, non-tractable.

Approximation of $F_{L}^{-1}(U)$ by numerical inversion algorithm of Derflinger et al. (2010)

It requires only PDF (probability density function)

For many Lévy processes, PDF is available in closed form (while CDF and the inverse CDF are not).
General CVs

- Simple path characteristics of $L$ and $W$ (e.g. average, maximum)

- They are not tailored to $q()$

- We call them as 'general CVs' since they are applicable to any $q()$, whereas $\zeta(W)$ is called 'special CV' as it is designed considering the special properties of $q()$.

- Let $\gamma(W,L)$ be a function of the paths of $W$ and $L$ that evaluates the set of path characteristics.
Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

1: for $i = 1$ to $n$ do
2:  for $j = 1$ to $d$ do
3:     Generate uniform variate $U \sim U(0,1)$.
4:     Set $X_j \leftarrow F_L^{-1}(U)$ and $Z_j \leftarrow F_{BM}^{-1}(U)$.
5:     Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
6:  end for
7:  Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - E[\zeta(W)]) - c_2^T (\gamma(W,L) - E[\gamma(W,L)])$.
8: end for
Algorithm

**Require:** special CV function $\zeta()$, general CV function $\gamma()$

1: for $i = 1$ to $n$ do
2:     for $j = 1$ to $d$ do
3:         Generate uniform variate $U \sim U(0,1)$.
4:         Set $X_j \leftarrow F_L^{-1}(U)$ and $Z_j \leftarrow F_{BM}^{-1}(U)$.
5:         Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
6:     end for
7:     Set $Y_i \leftarrow q(L) - c_1 \left( \zeta(W) - E[\zeta(W)] \right) - c_2^T \left( \gamma(W,L) - E[\gamma(W,L)] \right)$.
8: end for
Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

1: for $i = 1$ to $n$ do
2: \hspace{1em} for $j = 1$ to $d$ do
3: \hspace{2em} Generate uniform variate $U \sim U(0, 1)$.
4: \hspace{2em} Set $X_j \leftarrow F_{-1}^{-1}(U)$ and $Z_j \leftarrow F_{BM}^{-1}(U)$.
5: \hspace{2em} Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
6: \hspace{1em} end for
7: Set $Y_i \leftarrow q(L) - c_1 \left( \zeta(W) - E[\zeta(W)] \right) - c_2^T \left( \gamma(W, L) - E[\gamma(W, L)] \right)$.
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Algorithm

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7:   Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - E[\zeta(W)]) - c_2^T (\gamma(W, L) - E[\gamma(W, L)])$.
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Algorithm

Require: special CV function $\zeta()$, general CV function $\gamma()$

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5:     Set $L(t_j) \leftarrow L(t_{j-1}) + X_j$ and $W(t_j) \leftarrow W(t_{j-1}) + Z_j$
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7:   Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - E[\zeta(W)]) - c_2^T (\gamma(W, L) - E[\gamma(W, L)])$.
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6:     end for
7:     Set $Y_i \leftarrow q(L) - c_1 (\zeta(W) - E[\zeta(W)]) - c_2^T (\gamma(W, L) - E[\gamma(W, L)])$.
8: end for
CV Selection

- In algorithm, the user has to provide the CV functions $\zeta()$ and $\gamma()$.

- The selection of special CV $\zeta()$ depends on the problem type, as it is tailored to $q()$.

- Our approach for the selection of general CVs:
  - A large *basket* of CV candidates
  - Stepwise backward linear regression.
    - The $t$-statistics of regression coefficients: $t = \frac{\hat{\beta}}{s.e.(\hat{\beta})}$
    - Check the significance: $t \in (-5, 5)$?

- pilot simulation run
CV Selection

- **Stepwise backward regression**
  1. Start with a full regression model
  2. Remove the CV with the smallest absolute t-statistic from the model, if its value is smaller than 5
  3. Recompute the t-statistics of the remaining CVs for the new regression model
  4. Steps 2-3 are repeated until all absolute t values > 5
  5. Use the remaining CVs for the main simulation

- Why not use all CVs in the basket?
  - Simulation or evaluation of expectation of some CVs can be expensive.
  - Backward regression automatically eliminates the CV if it is not useful.
Complexity

- A single regression with \( k \) covariates requires \( O(n_p k^2) \) operations
  - \( k \) number of CVs
  - \( n_p \) sample size of pilot simulation

- The worst case: All CVs are useless \( O(n_p k^3) \)

- Since \( n_p < n \), no substantial increase in the computational time
Basket of CVs

- Path characteristics of which the expectation is available in closed form.

- Not exhaustive and depends on our knowledge of the closed form solutions.

- No CV that require a numerical method to evaluate the expectation.

Our notation:
- CVL: path characteristics of \( L \) (internal CVs)
- CVW: path characteristics of \( W \) (external CVs)
### Table: Basket of CVs.

<table>
<thead>
<tr>
<th>Label</th>
<th>CV</th>
<th>Label</th>
<th>CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVL1</td>
<td>$L(t_d)$</td>
<td>CVW1</td>
<td>$W(t_d)$</td>
</tr>
<tr>
<td>CVL2</td>
<td>$\exp(L(t_d))$</td>
<td>CVW2</td>
<td>$\exp(W(t_d))$</td>
</tr>
<tr>
<td>CVL3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} L(t_i)$</td>
<td>CVW3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} W(t_i)$</td>
</tr>
<tr>
<td>CVL4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} L(t_i)\right)$</td>
<td>CVW4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} W(t_i)\right)$</td>
</tr>
<tr>
<td>CVL5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(L(t_i))$</td>
<td>CVW5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(W(t_i))$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CVW6</td>
<td>$\max_{0 \leq i \leq d} W(t_i)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CVW7</td>
<td>$\exp\left(\max_{0 \leq i \leq d} W(t_i)\right)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CVW8</td>
<td>$\sup_{0 \leq u \leq t_d} W(u)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CVW9</td>
<td>$\exp\left(\sup_{0 \leq u \leq t_d} W(u)\right)$</td>
</tr>
</tbody>
</table>
Basket of CVs

- All CVs in the basket are easy to simulate

- A bit more difficult CVs: $\sup_{0 \leq u \leq t_d} W(u)$ and $e^{\sup_{0 \leq u \leq t_d} W(u)}$

- Simulation of $\sup_{0 \leq u \leq t_d} W(u)$ conditional on $(W(t_1), \ldots, W(t_d))$

- 

$$\sup_{0 \leq u \leq t_d} W(u) = \max_{1 \leq i \leq d} \left( \sup_{t_{i-1} \leq u \leq t_i} W(u) \right).$$

- generate the maxima of $d$ Brownian bridges
Basket of CVs

- CDF of the maximum of a Brownian bridge

\[
P\left(\sup_{0 \leq u \leq t} W(u) \leq x \mid W(t) = y\right) = 1 - \exp\left(- \frac{2x(x - y)}{\sigma^2 t}\right),
\]

- Inversion

\[x = 0.5 \left(y + \sqrt{y^2 - 2\sigma^2 t \log U}\right),\]

where \(U \sim U(0, 1)\) is a uniform random number

- \(\mathbb{E}\left[\sup_{0 \leq u \leq t_d} W(u) \mid W(t_1), \ldots, W(t_d)\right]\) as alternative to \(\sup_{0 \leq u \leq t_d} W(u)\)

- requires numerical integration, not efficient
CVs in the basket are strongly correlated with each other.

*Multicollinearity:* It inflates the standard errors of the estimates of the regression coefficients

It can be a problem for the accuracy of the estimates of the \( t \) statistics, when the sample size is too small.

\( n_p = 10^4 \) is generally sufficient to get relatively stable estimates of the \( t \) values.
## Basket of CVs: Expectation formulas

**Table:** Expectation formulas for the CVs depending on the terminal value and the averages.

<table>
<thead>
<tr>
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<th>CV</th>
<th>Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVL1</td>
<td>$L(t_d)$</td>
<td>$d \mathbb{E}[X]$</td>
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<tr>
<td>CVL2</td>
<td>$\exp(L(t_d))$</td>
<td>$M_{\Delta t}(1)^d$</td>
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<td>$\frac{1}{d} \sum_{i=1}^{d} L(t_i)$</td>
<td>$\mathbb{E}<a href="d+1">X</a>/2$</td>
</tr>
<tr>
<td>CVL4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} L(t_i)\right)$</td>
<td>$\prod_{i=1}^{d} M_{\Delta t}(i/d)$</td>
</tr>
<tr>
<td>CVL5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(L(t_i))$</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} M_{\Delta t}(1)^i$</td>
</tr>
<tr>
<td>CVW1</td>
<td>$W(t_d)$</td>
<td>$d \mu \Delta t$</td>
</tr>
<tr>
<td>CVW2</td>
<td>$\exp(W(t_d))$</td>
<td>$e^{(d(\mu \Delta t + \sigma^2 \Delta t/2))}$</td>
</tr>
<tr>
<td>CVW3</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} W(t_i)$</td>
<td>$\mu \Delta t(d+1)/2$</td>
</tr>
<tr>
<td>CVW4</td>
<td>$\exp\left(\frac{1}{d} \sum_{i=1}^{d} W(t_i)\right)$</td>
<td>$\exp(\tilde{\mu} + \tilde{\sigma}^2/2)$</td>
</tr>
<tr>
<td>CVW5</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} \exp(W(t_i))$</td>
<td>$\frac{1}{d} \sum_{i=1}^{d} e^{(i(\mu \Delta t + \sigma^2 \Delta t/2))}$</td>
</tr>
</tbody>
</table>
## Basket of CVs: Expectation formulas

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<th>CV</th>
<th>Expectation</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVW6</td>
<td>( \max_{0 \leq i \leq d} W(t_i) )</td>
<td>by Spitzer’s identity</td>
</tr>
<tr>
<td>CVW7</td>
<td>( \exp(\max_{0 \leq i \leq d} W(t_i)) )</td>
<td>by Öhgren (2001)</td>
</tr>
<tr>
<td>CVW8</td>
<td>( \sup_{0 \leq u \leq t_d} W(u) )</td>
<td>e.g. Shreve (2004)</td>
</tr>
<tr>
<td>CVW9</td>
<td>( \exp(\sup_{0 \leq u \leq t_d} W(u)) )</td>
<td>e.g. Shreve (2004)</td>
</tr>
</tbody>
</table>
A Simple Example

- We use only general CVs in the basket *without* a special CV

\[ q(L) = \exp(\max_{0 \leq i \leq d} L(t_i)) \]

- \( L \) is a generalized hyperbolic (GH) process
  - \( \Delta t = 1/250 \)
  - \( \lambda = 1.5, \alpha = 189.3, \beta = -5.71, \delta = 0.0062, \mu = 0.001 \)
  - the increment distribution is close to normal but has a higher kurtosis

- Variance Reduction Factors (VRFs)
  - For \( d = 5 \), VRF = 560
  - For \( d = 50 \), VRF = 395
Examples from Option Pricing

- Underlying stock: \( S(t) = S(0)e^{L(t)} \),
  - Non-normal logreturns with high kurtosis.

- Payoff of path dependent options: \( \psi(S(t_1), \ldots, S(t_d)) \)

- Price
  \[ e^{-rt_d}E[\psi(S(t_1), \ldots, S(t_d))], \]

- \( q(L) = \psi(S(0)e^{L}) \)
**Examples from Option Pricing**

- **Special CV:** a similar option with analytically available price under geometric Brownian Motion (GBM)
  - $\zeta()$ corresponds to $\psi_{CV}()$ payoff function of new option.
  - $\{\tilde{S}(t), t \geq 0\}$ stock price under GBM:
    \[
    \tilde{S}(t) = \tilde{S}(0)e^{W(t)} = \tilde{S}(0)\exp((r - \sigma^2/2)t + \sigma B(t)).
    \]
  - We set $\sigma = \sqrt{\text{Var}(L(1))}$ and $\tilde{S}(0) = S(0)$

- **General CVs:** Use the basket
Option Examples

- **Asian Option:** \( \psi_A(S) = \left( \frac{\sum_{i=1}^{d} S(t_i)}{d} - K \right)^+ \)

  Special CV: \( \psi_G(\tilde{S}) = \left( (\prod_{i=1}^{d} \tilde{S}(t_i))^{1/d} - K \right)^+ \).

- **Lookback Option:** \( \psi_L(S) = (\max_{0 \leq i \leq d} S(t_i) - K)^+ \),

  Special CV: \( \psi_{LC}(\tilde{S}) = (\sup_{0 \leq u \leq t_d} \tilde{S}(u) - K)^+ \).
**Numerical Results**

Table: Results for Asian and lookback options under GH process with $T = 1, \Delta t = 1/250, r = 0.05, S(0) = 100, n = 10^4$. Error: 95% error bound.

<table>
<thead>
<tr>
<th>Option</th>
<th>$K$</th>
<th>Price</th>
<th>Error</th>
<th>VRF-A</th>
<th>VRF-G</th>
<th>VRF-S</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asian</td>
<td>90</td>
<td>12.239</td>
<td>0.004</td>
<td>1,743</td>
<td>185</td>
<td>78</td>
</tr>
<tr>
<td></td>
<td>100</td>
<td>4.912</td>
<td>0.005</td>
<td>530</td>
<td>51</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>1.240</td>
<td>0.006</td>
<td>121</td>
<td>13</td>
<td>40</td>
</tr>
<tr>
<td>Lookback</td>
<td>110</td>
<td>7.534</td>
<td>0.012</td>
<td>294</td>
<td>57</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>120</td>
<td>3.297</td>
<td>0.012</td>
<td>160</td>
<td>35</td>
<td>44</td>
</tr>
<tr>
<td></td>
<td>130</td>
<td>1.266</td>
<td>0.011</td>
<td>79</td>
<td>17</td>
<td>32</td>
</tr>
</tbody>
</table>

- VRF-A: VRF obtained by using all (significant) CVs,
- VRF-G: VRF obtained by using only general CVs (CVLs and CVWs),
- VRF-S: VRF obtained by using only special CV
Numerical Results

- Efficiency factor: \( EF = \frac{\sigma^2_N t_N}{\sigma^2_{CV} t_{CV}} \)
  - \( t_N \) and \( t_{CV} \) are the CPU times of naive simulation and CV method.

- In naive simulation, we used the subordination (the standard method in the literature).

- Asian option: \( t_N/t_{CV} = 1 \)

- Lookback option: \( t_N/t_{CV} = 0.7 \)

- Time of the pilot simulation run is between 30% and 50% of the main simulation
Success of the method

- Proximity of increment (log-return) distribution to the normal distribution.

- Shape depends on $\Delta t$
  - $\Delta t \to \infty$, gets close to normal
  - $\Delta t \to 0$, very high kurtosis

- In option pricing,
  - $\Delta t = 1/4$, quarterly monitoring
  - $\Delta t = 1/12$, monthly monitoring
  - $\Delta t = 1/50$, weekly monitoring
  - $\Delta t = 1/250$, **daily monitoring**
  - $\Delta t \to 0$, continuous monitoring (not possible in practice)
### Asian option example for variance gamma (VG) process

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\Delta t$</th>
<th>Price</th>
<th>Error</th>
<th>VRF</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>1/4</td>
<td>31.562</td>
<td>0.004</td>
<td>2,966</td>
</tr>
<tr>
<td></td>
<td>1/12</td>
<td>31.156</td>
<td>0.004</td>
<td>2,582</td>
</tr>
<tr>
<td></td>
<td>1/50</td>
<td>31.002</td>
<td>0.006</td>
<td>894</td>
</tr>
<tr>
<td></td>
<td>1/250</td>
<td>30.975</td>
<td>0.019</td>
<td>87</td>
</tr>
<tr>
<td>100</td>
<td>1/4</td>
<td>5.903</td>
<td>0.003</td>
<td>1,949</td>
</tr>
<tr>
<td></td>
<td>1/12</td>
<td>5.229</td>
<td>0.003</td>
<td>1,815</td>
</tr>
<tr>
<td></td>
<td>1/50</td>
<td>4.972</td>
<td>0.005</td>
<td>795</td>
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<tr>
<td></td>
<td>1/250</td>
<td>4.912</td>
<td>0.015</td>
<td>72</td>
</tr>
<tr>
<td>130</td>
<td>1/4</td>
<td>0.082</td>
<td>0.002</td>
<td>114</td>
</tr>
<tr>
<td></td>
<td>1/12</td>
<td>0.034</td>
<td>0.001</td>
<td>101</td>
</tr>
<tr>
<td></td>
<td>1/50</td>
<td>0.025</td>
<td>0.001</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td>1/250</td>
<td>0.020</td>
<td>0.002</td>
<td>15</td>
</tr>
</tbody>
</table>

**Table:** Using 'special CV' for Asian VG options with $T = 1$ and different $\Delta t$'s; $n = 10,000$; Error: 95% error bound; VRF: variance reduction factor.
Conclusions

- A general control variate framework for the functionals of Lévy processes.

- The method exploits the strong correlation between the original Lévy process and an auxiliary Brownian motion
  - Numerical inversion of CDFs

- In the CV framework,
  - special control variates tailored to the functionals
  - general control variates selected from a large basket of control variate candidates

- In the application to path dependent options, we observe moderate to large variance reductions
Asian options

- Stock price process \( \{S(t), t \geq 0\} \)

Arithmetic average call option

\[
P_A(S) = \left( \frac{1}{d} \sum_{i=1}^{d} S(t_i) - K \right)^+
\]

- Time grid \(0 = t_0 < t_1 < t_2 < \ldots < t_d = T\) with \(t_j = j\Delta t\)

- Option price: \(e^{-rT}E[P_A(S)]\)

Geometric Brownian motion (GBM)

\[
S(t) = S(0) \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) t + \sigma B(t) \right\}, \quad t \geq 0
\]

- No closed form solution for the price
Simulation of Asian options

- Efficient numerical methods under GBM
  - PDE based finite difference methods, e.g. Večeř (2001)
  - Approximations, e.g. Curran (1994); Lord (2006)

- Monte Carlo simulation
  - Advantage: Probabilistic error bound
  - Disadvantage: Slow convergence rate

- Variance reduction method
CVs for Asian options

- Classical CV method of Kemna and Vorst (1990)
  - Arithmetic and geometric averages:
    \[ A = \frac{1}{d} \sum_{i=1}^{d} S(t_i) \quad \text{and} \quad G = \left( \prod_{i=1}^{d} S(t_i) \right)^{1/d} \]
  - If \( S(t_i) \)'s are close to each other, then \( A \sim G \)
  - \( P_A = (A - K)^+ \sim (G - K)^+ = P_G \)

- \( E[P_G] \) is available in closed form under GBM

- Very successful, if \( \sigma \) and \( T \) are small
CVs for Asian options

- Lower bound \( \mathbb{E}[(A - K) \mathbf{1}_{\{G>K\}}] \) suggested by Curran (1994):

\[
(A - K)^+ = (A - K)^+ \mathbf{1}_{\{G\leq K\}} + (A - K)^+ \mathbf{1}_{\{G > K\}} \\
= (A - K)^+ \mathbf{1}_{\{G\leq K\}} + (A - K) \mathbf{1}_{\{G > K\}},
\]

- New CV by Dingeç and Hörmann (2013)

\[
Y_{CV} = P_A - c (W - \mathbb{E}[W]),
\]

where \( W = (A - K) \mathbf{1}_{\{G > K\}} \).

- \( \mathbb{E}[W] \) is available in closed form under GBM
CVs for Asian options

- If we set $c = 1$, then $Y_{CV} = (A - K)^+ 1_{G \leq K} + E[W]$

- Conditional Monte Carlo (CMC) for $Y = (A - K)^+ 1_{G \leq K}$
  - New estimator as conditional expectation: $E[Y|Z] = \int Y dF(G)$
  - All variance due to $G$ is removed

- Algorithm in Dingeç and Hörmann (2013)
  - New CV + CMC + additional CVs
  - Larger VRF than the classical CV
  - Special to GBM
Non-Gaussian models

- Under GBM, log-returns are iid normals

- Observed facts
  - Non-normality of log-returns,
    Higher kurtosis, heavier tails than normal
  - Volatility clustering
    - Large absolute log-returns are followed by large absolute log-returns
    - Non-linear dependency between log-returns

- Alternative models to GBM
  - Lévy process, (i.i.d. log-returns)
  - Stochastic volatility models
  - Regime switching models
A unified framework for non-Gaussian models

- Three models
  - Generalized hyperbolic (GH) Lévy process (Prause, 1999)
  - Heston stochastic volatility (SV) model (Heston, 1993)
  - Regime switching (RS) model (Hardy, 2001)

- A unified framework
  - Stock price process: $S(t) = S(0)e^{X(t)}$
  - Log-returns: $\Delta X_i = X(t_i) - X(t_{i-1})$
Unified Framework

The unified representation

$$\Delta X_i = \Gamma_i + \Lambda_i Z_i, \quad i = 1, \ldots, d,$$

- $\Gamma_i, \Lambda_i$’s are modulated by stochastic process $\{V(t), t \geq 0\}$
- $\Gamma = f_m(V)$ and $\Lambda = f_v(V)$.
- $Z_i$’s are i.i.d. standard normal variables independent of $V(t)$.
- $(\Delta X_1, \ldots, \Delta X_d | V)$ is multivariate normal

The variance process $V(t)$

- GH Lévy: GIG process (subordinator)
- Heston: CIR process
- Regime switching: Discrete Time Markov Chain (DTMC)
Typical Control Variate Methods for the Unified Framework

- The standard CV approach mentioned in Glasserman (2004)

\[ Y_{CV} = P_A - c(\tilde{P}_G - \mathbb{E}[\tilde{P}_G]). \]

where \( \tilde{P}_G \) denote the payoff of the geometric average option under GBM
  - Using common \( Z \) to introduce correlation

- A more elaborate CV approach by Zhang (2011)

\[ Y_{CV} = P_A - c(P_G - \mathbb{E}[P_G|V]). \]

- They do not reduce the variance due to \( V \)
CV of Dingeç and Hörmann (2013): \( W = (A - K) 1_{\{G>K\}} \)

It will reduce the variance coming from both random variables (\( Z \) and \( V \))

Evaluation of the expectation \( \mu_W = E[W] \)

- Lemmens et al. (2010) use \( \mu_W \) as lower bound for the price under Lévy processes
- Our new observation: Formulas of Lemmens et al. (2010) can be used for any model allowing the computation of joint characteristic function (JCF) of the log-return vector \( \Delta X = (\Delta X_1, \ldots, \Delta X_d) \).
Expectation of the CV

- Formula of Lemmens et al. (2010) (after simplifications)

\[
\mu_W = \frac{g(0)}{2} - \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-i\omega L} g(\omega) - g(0)}{\omega} d\omega,
\]

- \( g(\omega) = \frac{S(0)}{d} \sum_{j=1}^{d} \phi_{\bar{X},X_j}(\omega, -i) - K \phi_{\bar{X}}(\omega). \)

\( \phi_{\bar{X}}(\omega) \): CF of \( \bar{X} = \sum_{j=1}^{d} X(t_j)/d \)

\( \phi_{\bar{X},X_j}(\omega_1, \omega_2) \): bivariate CF of \( \bar{X} \) and \( X(t_j) \)

- Both CFs can be evaluated, if \( \phi_{\Delta X}(u) = E[e^{iu^T \Delta X}] \) is available
Expectation of the CV

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Expectation of the CV

- Formulas for JCF: $\varphi_{\Delta X}(u) = E[e^{iu^T\Delta X}]$

- Lévy process: only requires CF of i.i.d. increment

- Heston model: given by Rockinger and Semenova (2005) for affine jump diffusion models

- Regime switching model: it is possible to derive a simple recursion
Improving the CV Method by CMC

- By setting $c = 1$, we get

$$Y_{CV} = Y + \mu W \quad \text{with} \quad Y = (A - K)^+ 1_{\{G \leq K\}}.$$  

- Conditional Monte Carlo (CMC) for $Y = (A - K)^+ 1_{\{G \leq K\}}$
  
  ▶ Simulation output as a function of two random inputs $Y = q(V, Z)$
  
  ▶ The idea: Simulation of standard multinormal vector $Z$ in a specific direction $\vartheta \in \mathbb{R}^d$, $||\vartheta|| = 1$ by the formula

$$Z = \vartheta \Xi + (I_d - \vartheta \vartheta^T)Z', \quad \Xi \sim N(0, 1), \quad Z' \sim N(0, I_d), \quad (1)$$

where $I_d$ is $d \times d$ identity matrix.

  ▶ Select the direction depending on $V$,

$$\vartheta_i(V) = \frac{(d - i + 1)\Lambda_i}{\sqrt{\sum_{j=1}^{d} (d - j + 1)^2 \Lambda_i^2}}, \quad i = 1, \ldots, d, \quad (2)$$

where $\Lambda = f_V(V)$. 

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\[
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**Conditional Monte Carlo (CMC) for $Y = (A - K)^+ 1_{\{G \leq K\}}$**

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Conditional Monte Carlo

- $Y = q(V, \Xi, Z')$

- Use $E[Y|Z', V]$ as an estimator.

\[
E[Y|Z' = z, V = v] = \frac{1}{d} \sum_{i=1}^{d} s_i(z, v) e^{a_i(v)^2/2} [\Phi(k(v) - a_i(v)) - \Phi(b(z, v) - a_i(v))] \\
- K [\Phi(k(v)) - \Phi(b(z, v))].
\]

- $\Phi()$: CDF of std. normal dist.

- $b(z, v)$ is the root of equation, $A(x) - K = 0$, which is found by Newton’s method
Algorithm

1: Compute $\mu_W$
2: \textbf{for} $i = 1$ \textbf{to} $n$ \textbf{do}
3: \hspace{1em} Simulate a variance path $V$
4: \hspace{1em} Simulate $Z' \sim N(0,I_d)$
5: \hspace{1em} Compute $E[Y|Z',V]$
6: \hspace{1em} Set $Y_i \leftarrow e^{-rT}(E[Y|Z',V] + \mu_W)$
7: \textbf{end for}
8: \textbf{return} $\bar{Y}$ and the error bound $\Phi^{-1}(1 - \alpha/2) s/\sqrt{n}$.

- Up to 10 times slower than naive simulation
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### Numerical Results

<table>
<thead>
<tr>
<th>Model</th>
<th>$T$</th>
<th>$K$</th>
<th>Price</th>
<th>Error</th>
<th>VRF</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>GH</strong></td>
<td>1</td>
<td>90</td>
<td>12.23708</td>
<td>0.00002</td>
<td>$8.8 \times 10^7$</td>
</tr>
<tr>
<td>($\Delta t = 1/250$)</td>
<td>100</td>
<td>4.91175</td>
<td>0.00003</td>
<td>$2.3 \times 10^7$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>110</td>
<td>1.24135</td>
<td>0.00004</td>
<td>$2.8 \times 10^6$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>90</td>
<td>14.38544</td>
<td>0.00004</td>
<td>$3.0 \times 10^7$</td>
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<tr>
<td></td>
<td>100</td>
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<td>0.00011</td>
<td>$1.4 \times 10^6$</td>
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<tr>
<td><strong>Heston SV</strong></td>
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<td>90</td>
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<td>0.00010</td>
<td>$1.8 \times 10^6$</td>
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<td>($\Delta t = 1/12$)</td>
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<td>$9.9 \times 10^6$</td>
<td></td>
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<td></td>
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<td>$2.8 \times 10^6$</td>
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<td></td>
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<tr>
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<td>110</td>
<td>2.19650</td>
<td>0.00005</td>
<td>$2.5 \times 10^6$</td>
<td></td>
</tr>
<tr>
<td><strong>RS</strong></td>
<td>1</td>
<td>90</td>
<td>12.46569</td>
<td>0.00004</td>
<td>$1.9 \times 10^7$</td>
</tr>
<tr>
<td>($\Delta t = 1/12$)</td>
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<td>4.93411</td>
<td>0.00003</td>
<td>$1.9 \times 10^7$</td>
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<td>1.24898</td>
<td>0.00005</td>
<td>$1.8 \times 10^6$</td>
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<td>110</td>
<td>3.11583</td>
<td>0.00010</td>
<td>$1.7 \times 10^6$</td>
<td></td>
</tr>
</tbody>
</table>

**Table:** Variance reduction factors (VRF) compared to naive simulation. $S(0) = 100, r = 0.05, n = 10^4$
Conclusions

- A new efficient simulation method developed for Asian option pricing under a general model framework
  - GH Lévy process
  - Heston stochastic volatility model
  - Discrete-time regime switching model

- Combination of CV and CMC
  - CV is applicable to all models in which the numerical computation of JCF of the log-return vector is possible
  - CMC is applicable to all models having normal mean-variance mixture representation

- Numerical results show significant variance reduction compared to naive simulation


Thank You